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ROMAN k-DOMINATION IN GRAPHS

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Abstract. Let k be a positive integer, and let G be a simple graph with vertex set $V(G)$. A Roman k-dominating function on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least k vertices $v_1, v_2, \ldots, v_k$ with $f(v_i) = 2$ for $i = 1, 2, \ldots, k$. The weight of a Roman k-dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman k-dominating function on a graph G is called the Roman k-domination number $\gamma_{kR}(G)$ of G. Note that the Roman 1-domination number $\gamma_{1R}(G)$ is the usual *Roman domination number* $\gamma_R(G)$. In this paper, we investigate the properties of the Roman k-domination number. Some of our results extend these one given by Cockayne, Dreyer Jr., S. M. Hedetniemi, and S. T. Hedetniemi [2] in 2004 for the Roman domination number.

1. Terminology and introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$.

The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N(v) \cup \{v\}$. The maximum degree of a graph G is denoted by $\Delta(G) = \Delta$. For a subset $S \subseteq V(G)$, we define $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$, $N[S] = N_G[S] = N(S) \cup S$, and $G[S]$ is the subgraph induced by S. The complement of a graph G is denoted by $\overline{G}$. If $\omega(G)$ is the number of components of G and $m(G) = |E(G)|$, then

$$c(G) = m(G) - n(G) + \omega(G)$$

is the well-known *cyclomatic number* of G. A graph is a *cactus graph* if all its cycles are edge-disjoint.

We write K_n for the *complete graph* of order n, and $K_{p,q}$ for the *complete bipartite graph* with bipartition X, Y such that $|X| = p$ and $|Y| = q$.

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Let k be a positive integer. A subset $D \subseteq V(G)$ is a *k-dominating set* of the graph G, if $|N_G(v) \cap D| \geq k$ for every $v \in V(G) - D$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the classical *domination number* $\gamma(G)$. A k-dominating set of minimum cardinality of a graph G is called a *$\gamma_k(G)$-set*.

In this paper, we study an extension of the *Roman dominating function* which is suggested by an article in Scientific American by Ian Steward, entitled "Defend the Roman Empire!" [9]. According to [2], Constantine the Great (Emperor of Rome) issued a decree in the 4th century A.D. for the defense of his cities. He decreed that any city without a legion stationed to secure it must neighbor another city having two stationed legions. If the first were attacked, then the second could deploy a legion to protect it without becoming vulnerable itself. The objective, of course, is to minimize the total number of legions needed. However, the Roman Empire has had a lot of enemies, and if a number of k enemies attack k cities without a legion, then these cities are secured in the above sense if they are neighbored to at least k cities having two stationed legions. This leads in a natural way to the following generalization of the Roman dominating function.

A *Roman k-dominating function* on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least k vertices $v_1, v_2, \ldots, v_k$ with $f(v_i) = 2$ for $i = 1, 2, \ldots, k$. The *weight* of a Roman k-dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman k-dominating function on a graph G is called the *Roman k-domination number* $\gamma_{kR}(G)$ of G. Note that the Roman 1-domination number $\gamma_{1R}(G)$ is the usual *Roman domination number* $\gamma_R(G)$. A Roman k-dominating function of minimum weight is called a *γ_{kR}-function.* If $f : V(G) \to \{0, 1, 2\}$ is a Roman k-dominating function, then let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 0, 1, 2$. Note that there is a 1-1 correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. Thus we will write $f = (V_0, V_1, V_2)$.

In [4], [5], Fink and Jacobson introduced the concept of k-domination, and the definition of the Roman dominating function was given implicitly by Steward [9] and ReVelle and Rosing [8]. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [6], [7].

2. Main results

Our first observation is an extension of a corresponding inequality chain in [2] for $k = 1$.

Proposition 2.1. *For any graph G*

$$\gamma_k(G) \leq \gamma_{kR}(G) \leq 2\gamma_k(G).$$

Proof. If $f = (V_0, V_1, V_2)$ is a γ_{kR}-function of G, then $V_1 \cup V_2$ is a k-dominating set of G and thus $\gamma_k(G) \le |V_1| + |V_2| \le |V_1| + 2|V_2| = \gamma_{kR}(G)$.

If D is a γ_k-set of G, then $f = (V(G) - D, \emptyset, D)$ is a Roman k-dominating set of G and thus $\gamma_{kR}(G) \le 2|D| = 2\gamma_k(G)$. □

Following Cockayne, Dreyer Jr., S. M. Hedetniemi, and S. T. Hedetniemi [2], we will say that a graph G is a *k-Roman graph* if $\gamma_{kR}(G) = 2\gamma_k(G)$.

Proposition 2.2. *A graph G is a k-Roman graph if and only if it has a γ_{kR}-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$.*

Proof. Let G be a k-Roman graph, and let D be a γ_k-set of G. Then $f = (V(G) - D, \emptyset, D)$ is a Roman k-dominating function of G such that

$$f(V(G)) = 2|D| = 2\gamma_k(D) = \gamma_{kR}(G),$$

and therefore f is a γ_{kR}-function with $V_1 = \emptyset$.

Conversely, let $f = (V_0, V_1, V_2)$ be a γ_{kR}-function with $V_1 = \emptyset$ and thus $\gamma_{kR}(G) = 2|V_2|$. Then V_2 is also a k-dominating set of G, and hence it follows that $2\gamma_k(G) \le 2|V_2| = \gamma_{kR}(G)$. Applying Proposition 2.1, we obtain the identity $\gamma_{kR}(G) = 2\gamma_k(G)$, i.e., G is a k-Roman graph. □

Corollary 2.3 ([2])**.** *A graph G is a 1-Roman graph if and only if it has a γ_R-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$.*

Proposition 2.4. *If G is a graph of order n, then the following conditions are equivalent:*

(i) $\gamma_k(G) = \gamma_{kR}(G)$,
(ii) $\gamma_k(G) = n$,
(iii) $\Delta(G) \le k - 1$.

Proof. Assume that $\gamma_k(G) = \gamma_{kR}(G)$. If $f = (V_0, V_1, V_2)$ is a γ_{kR}-function of G, then the assumption implies that we have equality in $\gamma_k(G) \le |V_1| + |V_2| \le |V_1| + 2|V_2| = \gamma_{kR}(G)$. This implies that $|V_2| = 0$ and hence we deduce that $|V_0| = 0$. Therefore $\gamma_k(G) = \gamma_{kR}(G) = |V_1| = |V(G)| = n$.

Clearly, if $\gamma_k(G) = n$, then $\Delta(G) \le k - 1$.

If $\Delta(G) \le k - 1$, then $\gamma_k(G) = n$ is immediate and thus Proposition 2.1 shows that $\gamma_k(G) = \gamma_{kR}(G)$. □

Corollary 2.5 ([2])**.** *Let G be a graph of order n. Then $\gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K_n}$.*

Proposition 2.6. *If G is a graph of order n, then*

$$\gamma_{kR}(G) \ge \min\{n, \gamma_k(G) + k\}.$$

Proof. If $\gamma_{kR}(G) = n$, then we are done. Assume now that $\gamma_{kR}(G) < n$, and suppose on the contrary that $\gamma_{kR}(G) \le \gamma_k(G) + k - 1$. If $f = (V_0, V_1, V_2)$ is a

γ_{kR}-function of G, then $V_1 \cup V_2$ is a k-dominating set of G and thus

$$\begin{aligned}\gamma_k(G) &\leq |V_1| + |V_2| \leq |V_1| + 2|V_2| \\ &= \gamma_{kR}(G) \leq \gamma_k(G) + k - 1 \\ &\leq |V_1| + |V_2| + k - 1.\end{aligned}$$

This implies $|V_2| \leq k - 1$ and hence we conclude that $|V_0| = 0$. This leads to $|V_2| = 0$ and therefore we arrive at the contradiction $\gamma_{kR}(G) = |V_1| = n$. □

Proposition 2.7. *Let G be a graph of order n.*

(i) *If $n \leq 2k$, then $\gamma_{kR}(G) = n$.*
(ii) *If $n \geq 2k + 1$, then $\gamma_{kR}(G) \geq 2k$.*
(iii) *If $n \geq 2k + 1$ and $\gamma_k(G) = k$, then $\gamma_{kR}(G) = \gamma_k(G) + k = 2k$.*

Proof. (i) Assume that $n \leq 2k$, and suppose on the contrary that $\gamma_{kR}(G) < n$. This implies $|V_0| \geq 1$ and thus $|V_2| \geq k$ for every γ_{kR}-function $f = (V_0, V_1, V_2)$. However, this leads to the contradiction $\gamma_{kR}(G) = |V_1| + 2|V_2| \geq 2|V_2| \geq 2k \geq n$.

(ii) Assume that $n \geq 2k + 1$. If $\gamma_{kR}(G) = n$, then we are done. If $\gamma_{kR}(G) < n$, then $|V_0| \geq 1$ and thus $|V_2| \geq k$ for every γ_{kR}-function $f = (V_0, V_1, V_2)$. Therefore we obtain the desired bound $\gamma_{kR}(G) = |V_1| + 2|V_2| \geq 2|V_2| \geq 2k$.

(iii) Assume that $n \geq 2k + 1$ and $\gamma_k(G) = k$. If D is a γ_k-set of G, then $(V(G) - D, \emptyset, D)$ is a Roman k-dominating set of G and thus $\gamma_{kR}(G) \leq 2|D| = 2k$. Using (ii), we arrive at the desired identity $\gamma_{kR}(G) = 2k = \gamma_k(G) + k$. □

Theorem 2.8. *If G is a graph of order n, then*

$$\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) \geq \min\{2n, 4k + 1\}. \tag{1}$$

Furthermore, equality holds in (1) *if and only if $n \leq 2k$ or $k \geq 2$ and $n = 2k+1$ or $k = 1$ and G or $\overline{G}$ has a vertex of degree $n - 1$ and its complement has a vertex of degree $n - 2$.*

Proof. Assume that $n \leq 2k$. Then Proposition 2.7 (i) shows that

$$\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) = 2n = \min\{2n, 4k + 1\}.$$

Assume now that $n \geq 2k+1$. In addition, assume, without loss of generality, that $\gamma_{kR}(\overline{G}) \geq \gamma_{kR}(G)$. If $\gamma_{kR}(G) \geq 2k + 1$, then we deduce that $\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) \geq 4k + 2$. Therefore (1) is proved, and we notice that equality in (1) is impossible in this case.

In view of Proposition 2.7 (ii), there remains the case that $\gamma_{kR}(G) = 2k < n$. It follows that $|V_0| \geq 1$ and thus $|V_2| = k$ and $|V_1| = 0$ for every γ_{kR}-function $f = (V_0, V_1, V_2)$. Since $|V_2| = k$, every vertex of V_0 is adjacent to every vertex of V_2 in G. Consequently, there is no edge between V_0 and V_2 in $\overline{G}$. Applying Proposition 2.7 again, we see that

$$\begin{aligned}\gamma_{kR}(\overline{G}) &= \gamma_{kR}(\overline{G}[V_2]) + \gamma_{kR}(\overline{G}[V_0]) \\ &\geq k + \min\{n - k, 2k\} \\ &= \min\{n, 3k\}.\end{aligned} \tag{2}$$

Combining this with the assumption $\gamma_{kR}(G) = 2k$, we obtain (1).

Clearly, if $k = 1$ and G or $\overline{G}$ has a vertex of degree $n-1$ and its complement has a vertex of degree $n-2$, then $\gamma_{kR}(G)+\gamma_{kR}(\overline{G}) = 4k+1 = 5$. If $k \geq 2$ and $n = 2k+1$, then $\gamma_{kR}(G) = 2k$ and, according to (2), $\gamma_{kR}(G)+\gamma_{kR}(\overline{G}) = 4k+1$.

Conversely, assume that $\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) = 4k + 1$. Combining this with (2), we arrive at

$$2k + 1 = \gamma_{kR}(\overline{G}) = k + \gamma_{kR}(\overline{G}[V_0]) = \min\{n, 3k\}.$$

In the case $k \geq 2$, we conclude that $n = 2k + 1$. If $k = 1$, then we have seen above that $|V_2| = 1$, $|V_0| = n - 1$ and there is no edge between V_0 and V_2 in $\overline{G}$. Thus G has a vertex of degree $n - 1$ and, because of $\gamma_{kR}(\overline{G}[V_0]) = 2$, $\overline{G}$ has a vertex of degree $n - 2$. □

Corollary 2.9 ([1]). *If G is a graph of order $n \geq 3$, then $\gamma_R(G) + \gamma_R(\overline{G}) \geq 5$ with equality if and only if G or $\overline{G}$ has a vertex of degree $n - 1$ and its complement has a vertex of degree $n - 2$.*

Next we derive some properties of γ_{kR}-functions, which extend these one by Cockayne, Dreyer Jr., S. M. Hedetniemi, and S. T. Hedetniemi [2].

Proposition 2.10. *Let $f = (V_0, V_1, V_2)$ be any γ_{kR}-function of a graph G. Then*

(a) *The complete bipartite graph $K_{k,k+1}$ is not a subgraph of $G[V_1]$.*

(b) *If $w \in V_1$, then $|N_G(w) \cap V_2| \leq k - 1$.*

(c) *If $A = \{u_1, u_2, \ldots, u_k\} \subseteq V_0$, then $|V_1 \cap N_G(u_1) \cap N_G(u_2) \cap \cdots \cap N_G(u_k)| \leq 2k$.*

(d) *V_2 is a γ_k-set of the induced subgraph $G[V_0 \cup V_2]$.*

(e) *Let $H = G[V_0 \cup V_2]$, and let $v \in V_2$. Then there exists a vertex $u_1 \in N_H(v) \cap V_0$ such that u_1 has exactly $k - 1$ neighbors in $V_2 - \{v\}$. In addition, there exists either a second vertex $u_2 \in N_H(v) \cap V_0$ such that u_2 has exactly $k - 1$ neighbors in $V_2 - \{v\}$ or v has at most $k - 1$ neighbors in $V_2 - \{v\}$.*

(f) *Let $v \in V_2$ such that $d_{G[V_2]}(v) = k-1$ and v has precisely one neighbor in V_0, say w, with the property that w has exactly $k - 1$ neighbors in $V_2 - \{v\}$. If $S_1 \subseteq V_1$ is a set such that each vertex of S_1 has precisely $k - 1$ neighbors in $V_2 - \{v\}$, then $N_G(w) \cap S_1 = \emptyset$.*

(g) *Let $S_2 \subseteq V_2$ be the set of vertices of degree at least k in $G[V_2]$, and let $C = \{x \in V_0 \mid |N_G(x) \cap V_2| \geq k + 1\}$. Then*

$$|V_0| \geq \max\left\{|V_2| + \frac{|V_2| + |S_2|}{k} + |C|\right\}.$$

Proof. (a) Suppose on the contrary that $K_{k,k+1}$ is a subgraph of $G[V_1]$, and let $A = \{x_1, x_2, \ldots, x_k\}$ and $B = \{y_1, y_2, \ldots, y_{k+1}\}$ be a bipartition of $K_{k,k+1}$. Then we observe that $f' = (V_0 \cup B, V_1 - (A \cup B), V_2 \cup A)$ is also a Roman

k-dominating function of G with the weight

$$\begin{aligned} f'(V(G)) &= |V_1 - (A \cup B)| + 2|V_2 \cup A| \\ &= |V_1| + 2|V_2| + |A| - |B| \\ &= |V_1| + 2|V_2| - 1 \\ &= f(V(G)) - 1. \end{aligned}$$

This is a contradiction to the hypothesis that f is a γ_{kR}-function of the graph G and (a) is proved.

(b) Suppose on the contrary that $|N_G(w) \cap V_2| \geq k$. Then $f' = (V_0 \cup \{w\}, V_1 - \{w\}, V_2)$ is also a Roman k-dominating function of G with $f'(V(G)) = f(V(G)) - 1$, a contradiction.

(c) Suppose on the contrary that $|V_1 \cap N_G(u_1) \cap N_G(u_2) \cap \cdots \cap N_G(u_k)| \geq 2k+1$. Let $B = \{w_1, w_2, \ldots, w_{2k+1}\} \subseteq V_1 \cap N_G(u_1) \cap N_G(u_2) \cap \cdots \cap N_G(u_k)$. Then $f' = ((V_0 - A) \cup B, V_1 - B, V_2 \cup A)$ is also a Roman k-dominating function of G, and we arrive at the contradiction

$$\begin{aligned} f'(V(G)) &= |V_1 - B| + 2|V_2 \cup A| \\ &= |V_1| + 2|V_2| + 2|A| - |B| \\ &= |V_1| + 2|V_2| - 1 \\ &= f(V(G)) - 1. \end{aligned}$$

(d) is immediate by the definition of the γ_{kR}-function of a graph G.

(e) First we note that v has a neighbor in V_0. Because otherwise, $f' = (V_0, V_1 \cup \{v\}, V_2 - \{v\})$ is also a Roman k-dominating function of G, and we arrive at the contradiction $f'(V(G)) = f(V(G)) - 1$.

Let $\{u_1, u_2, \ldots, u_s\} = N_H(v) \cap V_0$. If u_i has at least k neighbors in $V_2 - \{v\}$ for each $i = 1, 2, \ldots, s$, then $f' = (V_0, V_1 \cup \{v\}, V_2 - \{v\})$ is also a Roman k-dominating function of G, and we arrive at the contradiction $f'(V(G)) = f(V(G)) - 1$. Hence there exists at least one vertex, say u_1, in $N_H(v) \cap V_0$ such that u_1 has exactly $k-1$ neighbors in $V_2 - \{v\}$.

If there is a second vertex $w \in N_H(v) \cap V_0$ such that w has exactly $k-1$ neighbors in $V_2 - \{v\}$, then we are done. If not, then we suppose on the contrary that v has at least k neighbors in $V_2 - \{v\}$. Since each vertex in $\{u_2, u_3, \ldots, u_s, v\}$ has at least k neighbors in $V_2 - \{v\}$, we conclude that $f' = ((V_0 - \{u_1\}) \cup \{v\}, V_1 \cup \{u_1\}, V_2 - \{v\})$ is also a Roman k-dominating function of G. However, this leads to the contradiction $f'(V(G)) = f(V(G)) - 1$.

(f) Suppose on the contrary that $N_G(w) \cap S_1 \neq \emptyset$, and let $u \in N_G(w) \cap S_1$. Then $f' = ((V_0 - \{w\}) \cup \{u, v\}, V_1 - \{u\}, (V_2 - \{v\}) \cup \{w\})$ is also a Roman k-dominating function of G, and we arrive at the contradiction $f'(V(G)) = f(V(G)) - 1$.

(g) If we suppose that $|V_2| > |V_0|$, then we arrive at the contradiction $\gamma_{kR}(G) = |V_1| + 2|V_2| = |V_1| + |V_2| + |V_2| > |V_0| + |V_1| + |V_2| = n$. This implies that $|V_0| \geq |V_2|$.

In view of (e), every vertex $v \in V_2$ has a neighbor $u \in V_0$ such that u has exactly $k-1$ neighbors in $V_2 - \{v\}$, and every vertex $v \in S_2$ even has at least two neighbors in V_0 with this property. If $V_0' \subseteq V_0$ consists of all these neighbors, then it follows that $k|V_0'| \geq 2|S_2| + (|V_2| - |S_2|) = |V_2| + |S_2|$. Since all the vertices of V_0' have precisely k neighbors in V_2 they are different from these one in $C \subseteq V_0$, and thus we deduce that $|V_0| \geq (|V_2| + |S_2|)/k + |C|$. Combining this with $|V_0| \geq |V_2|$, we obtain the desired bound. □

Corollary 2.11 ([2]). *Let $f = (V_0, V_1, V_2)$ be any γ_R-function of a graph G. Then*

(a) *The induced subgraph $G[V_1]$ has maximum degree 1.*
(b) *No edge of G joins V_1 and V_2.*
(c) *Each vertex of V_0 is adjacent to at most two vertices of V_1.*
(d) *V_2 is a γ-set of the induced subgraph $G[V_0 \cup V_2]$.*
(e) *Let $H = G[V_0 \cup V_2]$. Then each vertex $v \in V_2$ has at least two private neighbors relative to V_2 in the graph H.*
(f) *If v is isolated in $G[V_2]$ and has precisely one neighbor in V_0, say w, with the property that w has no neighbor in $V_2 - \{v\}$, then $N_G(w) \cap V_1 = \emptyset$.*
(g) *Let $S_2 \subseteq V_2$ be the set of non-isolated vertices in $G[V_2]$, and let $C = \{x \in V_0 \,|\, |N_G(x) \cap V_2| \geq 2\}$. Then $|V_0| \geq |V_2| + |S_2| + |C|$.*

The special case $k = 1$ of the following lower bound on the Roman k-domination number can be find in the article [3].

Theorem 2.12. *If G is a graph of order n and maximum degree $\Delta \geq k$, then*

$$\gamma_{kR}(G) \geq \frac{2n}{\frac{\Delta}{k} + 1}.$$

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{kR}-function of G. Since each vertex $v \in V_0$ is adjacent to at least k vertices of V_2, we deduce that

$$k|V_0| \leq \Delta|V_2|.$$

This inequality and the hypothesis $\Delta \geq k$ imply the desired bound as follows:

$$\begin{aligned}
\left(\frac{\Delta}{k} + 1\right)\gamma_{kR}(G) &= \left(\frac{\Delta}{k} + 1\right)(|V_1| + 2|V_2|) \\
&= \left(\frac{\Delta}{k} + 1\right)|V_1| + 2\left(\frac{\Delta}{k} + 1\right)|V_2| \\
&\geq \left(\frac{\Delta}{k} + 1\right)|V_1| + 2|V_2| + 2|V_0| \\
&\geq 2|V_1| + 2|V_2| + 2|V_0| \\
&= 2n.
\end{aligned}$$

□

Corollary 2.13. *If G is a graph of order n and maximum degree $\Delta = k$, then $\gamma_{kR}(G) = n$.*

Next we derive a slight extension of Corollary 2.13 for $k \geq 2$.

Proposition 2.14. *Let G be a graph of order n. If $\gamma_{kR}(G) < n$, then $\Delta(G) \geq k+2$ or there exist at least k vertices $u_1, u_2, \ldots, u_k$ such that $d_G(u_i) = k+1$ for $i = 1, 2, \ldots, k$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{kR}-function of G. The hypothesis $|V_0|+|V_1|+|V_2| = n > \gamma_{kR}(G) = |V_1| + 2|V_2|$ implies $|V_0| \geq |V_2| + 1$. Since each vertex $w \in V_0$ is adjacent to at least k vertices of V_2, we deduce that

$$\sum_{u \in V_2} d_G(u) \geq k|V_0| \geq k(|V_2| + 1).$$

If we suppose on the contrary that $\Delta(G) \leq k+1$ and there are at most $k-1$ vertices of degree at most $k+1$, then we arrive at the contradiction

$$k|V_2| + k - 1 \geq \sum_{u \in V_2} d_G(u) \geq k(|V_2| + 1) = k|V_2| + k.$$

□

Now we present a characterization of the graphs G with $\gamma_{kR}(G) < n(G)$.

Theorem 2.15. *Let G be a graph of order n. Then $\gamma_{kR}(G) < n$ if and only if G contains a bipartite subgraph H with bipartition X, Y such that $|X| > |Y| \geq k$ and $d_H(v) \geq k$ for each $v \in X$.*

Proof. Assume first that G contains a bipartite subgraph H with the bipartition X, Y such that $|X| > |Y| \geq k$ and $d_H(v) \geq k$ for each $v \in X$. Then $f = (X, V(G) - (X \cup Y), Y)$ is a Roman k-domination function of weight

$$f(V(G)) = |V(G) - (X \cup Y)| + 2|Y| = n - |X| + |Y| < n.$$

Conversely, assume that $\gamma_{kR}(G) < n$, and let (V_0, V_1, V_2) be a γ_{kR}-function. It follows that $|V_0|+|V_1|+|V_2| = n > \gamma_{kR}(G) = |V_1|+2|V_2|$ and thus $|V_0| > |V_2|$. Since $|V_0| > 0$, we deduce that $|V_2| \geq k$. Now define H as the bipartite graph consisting of the bipartition V_0 and V_2 together with all edges of G connecting a vertex of V_0 with a vertex of V_2. As $d_H(v) \geq k$ for each vertex $v \in V_0$, the subgraph H has the desired properties, and the proof is complete. □

Finally, we give two applications of Theorem 2.15. It is well-known that a graph G is a forest if and only if its cyclomatic number $c(G) = 0$, and that G is a unicyclic graph if and only if $c(G) = 1$ (see for example Volkmann [10], pp. 29–31).

Theorem 2.16. *Let G be a graph of order n. If $k \geq 2$, then*

$$\gamma_{kR}(G) \geq \min\{n, n+1-c(G)\}. \tag{3}$$

Proof. Clearly, it is enough to show that inequality (3) is valid for $k = 2$. For $k = 2$ we proceed by induction on $c(G)$.

First assume that $c(G) \leq 1$. Suppose on the contrary that $\gamma_{2R}(G) < n$. According to Theorem 2.15, G contains a bipartite subgraph H with bipartition X, Y such that $|X| > |Y| \geq 2$ and $d_H(v) \geq 2$ for each $v \in X$. It follows that

$c(H) = m(H) - n(H) + \omega(H) \geq 2|X| - |X| - |Y| + 1 \geq 2$. Hence H and so G contains at least two cycles, a contradiction to the hypothesis that $c(G) \leq 1$.

Assume next that $c(G) \geq 2$. Then G contains a cycle C. Let $e = uv$ be an edge of the cycle C, and define the subgraph $H = G - e$. Then $c(H) = c(G) - 1 \geq 1$, and therefore we deduce from the induction hypothesis that

$$\gamma_{2R}(H) \geq n + 1 - c(H). \tag{4}$$

Now let $f = (V_0, V_1, V_2)$ be any γ_{2R}-function of G. If $f(u) = 0$ and $f(v) = 2$, then $f' = (V_0 - \{u\}, V_1 \cup \{u\}, V_2)$ is a Roman 2-dominating function of H. Therefore (4) implies the desired bound (3) as follows:

$$\begin{aligned} \gamma_{2R}(G) &= |V_1| + 2|V_2| = |V_1 \cup \{u\}| + 2|V_2| - 1 \\ &\geq \gamma_{2R}(H) - 1 \geq n - c(H) = n + 1 - c(G) \end{aligned}$$

Since all the remaining cases are similar to the case $f(u) = 0$ and $f(v) = 2$, the proof of Theorem 2.16 is complete. □

Corollary 2.17. *If G is a graph of order n with at most one cycle, then $\gamma_{kR}(G) = n$ when $k \geq 2$.*

The graph G of order 7 consisting of two cycles $x_1x_2x_3x_4x_1$ and $y_1y_2y_3y_4y_1$ with $x_1 = y_1$ and the Roman 2-dominating function f such that $f(x_1) = f(x_3) = f(y_3) = 2$ and $f(x_2) = f(x_4) = f(y_2) = f(y_4) = 0$ shows that Corollary 2.17 is no longer true if the graph contains more than one cycle.

Applying this example, it is easy to see that the Roman 2-domination number $\gamma_{2R}(G_{i,j}) < ij$ for each $i \times j$ grid $G_{i,j}$ when $i, j \geq 3$. In addition, it is a simple matter to prove that $\gamma_{3R}(G_{i,j}) < ij$ when $i \geq 5$ and $j \geq 9$, and Proposition 2.14 implies that $\gamma_{kR}(G_{i,j}) = ij$ when $k \geq 4$.

For the next result, we use the following lemma, which can be found in [10] on p. 30.

Lemma 2.18. *If G is a cactus graph, then $2m(G) \leq 3n(G) - 3$.*

Proposition 2.19. *If G is a cactus graph of order n, then $\gamma_{kR}(G) = n$ when $k \geq 3$.*

Proof. Clearly, it is enough to show that $\gamma_{3R}(G) = n$. Suppose on the contrary that $\gamma_{3R}(G) < n$. According to Theorem 2.15, G contains a bipartite subgraph H with bipartition X, Y such that $|X| > |Y| \geq 3$ and $d_H(v) \geq 3$ for each $v \in X$. It follows that $2m(H) \geq 6|X| > 3|X| + 3|Y| > 3n(H) - 3$. Applying Lemma 2.18, we arrive at the contradiction that H and so G is not a cactus graph. □

Let W_n be a wheel of order n. We finally notice that $\gamma_{kR}(W_n) = n$ for $k \geq 3$, $\gamma_R(W_n) = 2$ and $\gamma_{2R}(W_n) = \lceil \frac{2(n-1)}{3} \rceil + 2$ when $n \geq 4$.

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