J. Korean Math. Soc. **46** (2009), No. 6, pp. 1293–1307
DOI 10.4134/JKMS.2009.46.6.1293

POLYNOMIAL FACTORIZATION THROUGH $L_r(\mu)$ SPACES

Raffaella Cilia and Joaquín M. Gutiérrez

Abstract. We give conditions so that a polynomial be factorable through an $L_r(\mu)$ space. Among them, we prove that, given a Banach space X and an index m, every absolutely summing operator on X is 1-factorable if and only if every 1-dominated m-homogeneous polynomial on X is right 1-factorable, if and only if every 1-dominated m-homogeneous polynomial on X is left 1-factorable. As a consequence, if X has local unconditional structure, then every 1-dominated homogeneous polynomial on X is right and left 1-factorable.

We give conditions so that a homogeneous polynomial P between Banach spaces be factorable through an $L_r(\mu)$-space, either in the form $P = Q \circ T$, where T is a (linear) operator and Q is a polynomial (right r-factorization), or in the form $P = T \circ Q$ (left r-factorization).

It is shown in particular that, given a Banach space X and an index m, every absolutely summing operator on X is 1-factorable if and only if every 1-dominated m-homogeneous polynomial on X is right 1-factorable, if and only if every 1-dominated m-homogeneous polynomial on X is left 1-factorable. As a consequence, if X has local unconditional structure, then every 1-dominated m-homogeneous polynomial on X is right and left 1-factorable.

Throughout, X, Y, Z denote Banach spaces, X^* is the dual of X, and B_X stands for its closed unit ball. The closed unit ball B_{X^*} will always be endowed with the weak-star topology. By $\mathbb{N}$ we represent the set of all natural numbers, and by $\mathbb{K}$ the scalar field (real or complex). We use the symbol $\mathcal{L}(X,Y)$ for the space of all (linear bounded) operators from X into Y endowed with the operator norm. Given a space Y we shall denote by k_Y the natural embedding of Y into its bidual Y^{**}.

Given $m \in \mathbb{N}$, we denote by $\mathcal{P}(^mX,Y)$ the space of all m-homogeneous (continuous) polynomials from X into Y endowed with the supremum norm. Recall that with each $P \in \mathcal{P}(^mX,Y)$ we can associate a unique symmetric

---

Received April 29, 2008.
2000 *Mathematics Subject Classification.* Primary 46G25; Secondary 47H60.
Key words and phrases. right r-factorable polynomial, left r-factorable polynomial, p-dominated polynomial.
The first named author was supported in part by G.N.A.M.P.A. (Italy).
Both authors were supported in part by Dirección General de Investigación, MTM 2006–03531 (Spain).

m-linear mapping $\widehat{P} : X\times \overset{(m)}{\cdots} \times X \to Y$ so that

$$P(x) = \widehat{P}\left(x, \overset{(m)}{\ldots}, x\right) \qquad (x \in X)$$

and we have

$$\|P\| \le \left\|\widehat{P}\right\| \le \frac{m^m}{m!}\|P\| .$$

Given a polynomial $P \in \mathcal{P}(^m X, Y)$, its derivative is the polynomial

$$dP \in \mathcal{P}(^{m-1}X, \mathcal{L}(X, Y))$$

defined by

$$dP(x)(y) = m\widehat{P}\left(x, \overset{(m-1)}{\ldots}, x, y\right) \qquad (x, y \in X).$$

For the general theory of multilinear mappings and polynomials on Banach spaces, we refer the reader to [14] and [20].

We use the notation $\otimes^m X := X\otimes \overset{(m)}{\cdots} \otimes X$ for the m-fold tensor product of X, and $X\otimes_\pi Y$ (respectively, $X\otimes_\epsilon Y$) for the completed projective (respectively, injective) tensor product of X and Y (see [13] or [11] for the theory of tensor products).

By $\otimes_s^m X := X\otimes_s \overset{(m)}{\cdots} \otimes_s X$ we denote the m-fold symmetric tensor product of X, that is, the set of all elements $u \in \otimes^m X$ of the form

$$u = \sum_{j=1}^n \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \qquad (n \in \mathbb{N},\ \lambda_j \in \mathbb{K},\ x_j \in X,\ 1 \le j \le n).$$

By $\otimes_{\pi,s}^m X$ (respectively, $\otimes_{\epsilon,s}^m X$) we represent the space $\otimes_s^m X$ endowed with the topology induced by that of $\otimes_\pi^m X$ (respectively, $\otimes_\epsilon^m X$).

Given an operator $T \in \mathcal{L}(X, Y)$, we denote by

$$\otimes^m T : \otimes_\pi^m X \longrightarrow \otimes_\pi^m Y$$

the operator defined by

$$\otimes^m T\,(x_1 \otimes \cdots \otimes x_m) := T(x) \otimes \cdots \otimes T(x_m) \qquad (x_1, \ldots, x_m \in X).$$

If $A : X_1 \times \cdots \times X_m \to Y$ is an m-linear mapping, the *linearization* of A is the operator

$$\overline{A} : X_1 \otimes_\pi \cdots \otimes_\pi X_m \longrightarrow Y$$

given by

$$\overline{A}\left(\sum_{j=1}^n x_{1,j} \otimes \cdots \otimes x_{m,j}\right) = \sum_{j=1}^n A(x_{1,j}, \ldots, x_{m,j})$$

for all $x_{k,j} \in X_k$ $(1 \le k \le m,\ 1 \le j \le n)$ [21, p. 24]. Moreover, $\|A\| = \left\|\overline{A}\right\|$ [15, 2.1].

For a polynomial $P \in \mathcal{P}(^m X, Y)$, its *linearization*

$$\overline{P} : \otimes_{\pi,s}^m X \longrightarrow Y$$

is the operator given by

$$\overline{P}\left(\sum_{j=1}^{n}\lambda_j x_j\otimes \overset{(m)}{\cdots} \otimes x_j\right) = \sum_{j=1}^{n}\lambda_j P(x_j)$$

for all $x_j \in X$ and $\lambda_j \in \mathbb{K}$ $(1 \leq j \leq n)$.

By $\delta_m : X \to \otimes_\pi^m X$ we denote the canonical polynomial given by

$$\delta_m(x) := x\otimes \overset{(m)}{\cdots} \otimes x \qquad (x \in X)\,.$$

Given $1 \leq r < \infty$, a polynomial $P \in \mathcal{P}(^mX, Y)$ is *r-dominated* (see, e.g., [19]) if there exists a constant $k > 0$ such that, for all $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subset X$, we have

$$\left(\sum_{i=1}^{n}\|P(x_i)\|^{\frac{r}{m}}\right)^{\frac{m}{r}} \leq k \sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^{n}|x^*(x_i)|^r\right)^{\frac{m}{r}}.$$

The infimum of the constants k that verify this definition is called the *r-dominated quasinorm* of P and will be denoted by $\|P\|_{r\text{-d}}$ (it is a norm if and only if $r \geq m$).

Note that, for $m = 1$, we obtain the ideal (Π_r, π_r) of (absolutely) r-summing operators. If $m = 1$ and $r = 1$, we obtain the class of absolutely summing operators.

A polynomial $P \in \mathcal{P}(^mX, Y)$ is *integral* [1] if there exists a regular countably additive, Y^{**}-valued Borel measure $\mathcal{G}$ of bounded variation on B_{X^*} such that

$$P(x) = \int_{B_{X^*}}[x^*(x)]^m\,d\mathcal{G}(x^*) \qquad (x \in X)\,.$$

A polynomial $P \in \mathcal{P}(^mX, Y)$ is *nuclear* [1] if it can be written in the form

$$P(x) = \sum_{i=1}^{\infty}x_i^*(x)^m y_i \qquad (x \in X),$$

where $(x_i^*) \subset X^*$ and $(y_i) \subset Y$ are bounded sequences such that

$$\sum_{i=1}^{\infty}\|x_i^*\|^m\|y_i\| < \infty\,.$$

It is well known that every nuclear polynomial is integral.

The definition of ideal of polynomials may be seen, for instance, in [5].

For the notion and main properties of $\mathcal{L}_p$-spaces $(1 \leq p \leq \infty)$, we refer the reader to [17].

Definition 1 ([5]). Let $\mathcal{Q}$ be an ideal of polynomials. We say that

(a) $\mathcal{Q}$ is *closed under differentiation* if, for every $m \in \mathbb{N}$, all Banach spaces X and Y, and every polynomial $P \in \mathcal{Q}(^mX, Y)$, we have $dP(a) \in \mathcal{Q}(X, Y)$ for every $a \in X$;

(b) $\mathcal{Q}$ is *closed for scalar multiplication* if, for every $m \in \mathbb{N}$, all Banach spaces X and Y, and every polynomial $P \in \mathcal{Q}(^m X, Y)$, we have $\phi P \in \mathcal{Q}(^{m+1} X, Y)$ for every $\phi \in X^*$.

Definition 2 ([10])**.** Given a polynomial $P \in \mathcal{P}(^m X, Y)$ and $1 \leq r \leq \infty$, we say that P is *left r-factorable* if there exist a positive measure space (Ω, Σ, μ), a polynomial $Q \in \mathcal{P}(^m X, L_r(\mu))$, and an operator $T \in \mathcal{L}(L_r(\mu), Y^{**})$ such that $k_Y \circ P = T \circ Q$.

$$\begin{array}{ccc} X & \xrightarrow{\;P\;} & Y \\ {\scriptstyle Q}\big\downarrow & & \big\downarrow{\scriptstyle k_Y} \\ L_r(\mu) & \xrightarrow[\;T\;]{} & Y^{**} \end{array}$$

In this case we set

$$\gamma_r^{\text{left}}(P) := \inf\{\|Q\|\|T\| \text{ for } Q,\, T \text{ as above}\}\,.$$

We denote by $\mathcal{P}_r^{m,\text{left}}(X, Y)$ the subspace of all $P \in \mathcal{P}(^m X, Y)$ which are left r-factorable.

Definition 3 ([10])**.** Given a polynomial $P \in \mathcal{P}(^m X, Y)$ and $1 \leq r \leq \infty$, we say that P is *right r-factorable* if there exist a positive measure space (Ω, Σ, μ), a polynomial $Q \in \mathcal{P}(^m L_r(\mu), Y^{**})$, and an operator $T \in \mathcal{L}(X, L_r(\mu))$ such that $k_Y \circ P = Q \circ T$.

$$\begin{array}{ccc} X & \xrightarrow{\;P\;} & Y \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle k_Y} \\ L_r(\mu) & \xrightarrow[\;Q\;]{} & Y^{**} \end{array}$$

In this case we set

$$\gamma_r^{\text{right}}(P) := \inf\{\|Q\|\|T\|^m \text{ for } Q,\, T \text{ as above}\}\,.$$

We denote by $\mathcal{P}_r^{m,\text{right}}(X, Y)$ the subspace of all $P \in \mathcal{P}(^m X, Y)$ which are right r-factorable.

Recall [12, Chapter 7] that an operator $T \in \mathcal{L}(X, Y)$ is *r-factorable* if there exist a measure space (Ω, Σ, μ) and operators $b : L_r(\mu) \to Y^{**}$ and $a : X \to L_r(\mu)$ such that $k_Y \circ T = b \circ a$.

In this case, we write

$$\gamma_r(T) := \inf \|a\| \, \|b\|\,,$$

where the infimum extends over all factorizations of T as above; γ_r is a norm on the space $\Gamma_r(X, Y)$ of all r-factorable operators from X into Y.

Proposition 4. *If a polynomial $P \in \mathcal{P}(^m X, Y)$ is right 1-factorable, then it is also left 1-factorable, and*

$$\gamma_1^{\text{left}}(P) \leq \frac{m^m}{m!}\, \gamma_1^{\text{right}}(P)\,.$$

Proof. There exist a positive measure space (Ω, Σ, μ), a polynomial

$$Q \in \mathcal{P}(^m L_1(\mu), Y^{**}),$$

and an operator $T \in \mathcal{L}(X, L_1(\mu))$ such that $k_Y \circ P = Q \circ T$.

$$\begin{array}{ccc} X & \xrightarrow{P} & Y \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle k_Y} \\ L_1(\mu) & \xrightarrow[Q]{} & Y^{**} \end{array}$$

Using the polarization formula [20, Theorem 1.10], we have for $x_1, \ldots, x_m \in X$:

$$\begin{aligned} k_Y \circ \widehat{P}(x_1, \cdots, x_m) &= k_Y \left(\frac{1}{m!2^m} \sum_{\substack{\epsilon_j = \pm 1 \\ 1 \le j \le m}} \epsilon_1 \cdots \epsilon_m P(\epsilon_1 x_1 + \cdots + \epsilon_m x_m) \right) \\ &= \frac{1}{m!2^m} \sum_{\substack{\epsilon_j = \pm 1 \\ 1 \le j \le m}} \epsilon_1 \cdots \epsilon_m k_Y \circ P(\epsilon_1 x_1 + \cdots + \epsilon_m x_m) \\ &= \frac{1}{m!2^m} \sum_{\substack{\epsilon_j = \pm 1 \\ 1 \le j \le m}} \epsilon_1 \cdots \epsilon_m Q \circ T(\epsilon_1 x_1 + \cdots + \epsilon_m x_m) \\ &= \frac{1}{m!2^m} \sum_{\substack{\epsilon_j = \pm 1 \\ 1 \le j \le m}} \epsilon_1 \cdots \epsilon_m Q(\epsilon_1 T(x_1) + \cdots + \epsilon_m T(x_m)) \\ &= \widehat{Q}(T(x_1), \ldots, T(x_m)) \\ &= \widehat{Q}(T, \overset{(m)}{\ldots}, T)(x_1, \ldots, x_m). \end{aligned}$$

It follows that

$$\overline{k_Y \circ \widehat{P}} = \overline{\widehat{Q}} \circ (\otimes^m T).$$

Therefore (see the diagram below)

$$k_Y \circ \overline{P} = \overline{k_Y \circ P} = \overline{k_Y \circ \widehat{P}} \circ i = \overline{\widehat{Q}} \circ (\otimes^m T) \circ i,$$

where i denotes the natural inclusion of $\otimes_{\pi,s}^m X$ into $\otimes_\pi^m X$.

$$\begin{array}{ccc} \otimes_{\pi,s}^m X & \xrightarrow{k_Y \circ \overline{P}} & Y^{**} \\ {\scriptstyle i}\big\downarrow & & \big\uparrow{\scriptstyle \overline{\widehat{Q}}} \\ \otimes_\pi^m X & \xrightarrow[\otimes^m T]{} & \otimes_\pi^m L_1(\mu) \end{array}$$

Since $\otimes_\pi^m L_1(\mu)$ is an $L_1(\mu')$ space [22, Exercise 2.8], $\overline{P}$ is 1-factorable. Then $P = \overline{P} \circ \delta_m$ is left 1-factorable. Moreover, from the equality

$$k_Y \circ P = k_Y \circ \overline{P} \circ \delta_m = \overline{\widehat{Q}} \circ (\otimes^m T) \circ i \circ \delta_m,$$

we have

$$\begin{aligned}\gamma_1^{\text{left}}(P) &\leq \|\delta_m\| \left\| \overline{\widehat{Q}} \circ (\otimes^m T) \circ i \right\| \\ &\leq \left\| \overline{\widehat{Q}} \right\| \|T\|^m \qquad [11,\text{ Ex } 3.2] \\ &= \left\| \widehat{Q} \right\| \|T\|^m \\ &\leq \frac{m^m}{m!} \|Q\| \|T\|^m .\end{aligned}$$

Since the factorization $k_Y \circ P = Q \circ T$ is arbitrary, we get

$$\gamma_1^{\text{left}}(P) \leq \frac{m^m}{m!} \gamma_1^{\text{right}}(P) ,$$

and the proof is finished. □

Remark 5.

(a) There are many left 1-factorable polynomials that are not right 1-factorable. Indeed, it is proved in [10, Theorem 2.3] that every integral polynomial is left 1-factorable so, in particular, a nuclear polynomial is left 1-factorable; however, there are many nuclear polynomials that are not right 1-factorable [10, Propositions 5.8 and 5.9].

(b) The polynomial $Q \in \mathcal{P}(^2\ell_2, \ell_1)$ defined by $Q(x) := (x_k^2)_k$ is obviously left 1-factorable, but it is not integral (otherwise, it would be compact). Moreover, it is right 2-factorable (and then right r-factorable for every $r > 1$ [12, Corollary 9.2]), but it is not left 2-factorable: indeed, this would imply that Q is compact.

Given an operator ideal $\mathcal{A}$, a polynomial $P \in \mathcal{P}(^mX, Y)$ is said to be of type $\mathcal{P}_{\mathcal{L}[\mathcal{A}]}$ if there exist a Banach space Z, an operator $T \in \mathcal{A}(X, Z)$, and a polynomial $Q \in \mathcal{P}(^mZ, Y)$ such that $P = Q \circ T$ [4].

Theorem 6. *Let $\mathcal{A}$ be an operator ideal, and let $1 \leq r \leq \infty$. Let X be a Banach space. Consider the following statements*:

(a) *for every Banach space Y, $\mathcal{A}(X, Y) \subseteq \Gamma_r(X, Y)$;*

(b) *for every Banach space Y and for every index $m \geq 2$, $\mathcal{P}_{\mathcal{L}[\mathcal{A}]}(^mX, Y) \subseteq \mathcal{P}_r^{m,\text{right}}(X, Y)$;*

(c) *there exists an index $m \geq 2$ such that, for every Banach space Y,*

$$\mathcal{P}_{\mathcal{L}[\mathcal{A}]}(^mX, Y) \subseteq \mathcal{P}_r^{m,\text{right}}(X, Y);$$

(d) *for every Banach space Y and for every index $m \geq 2$,*

$$\mathcal{P}_{\mathcal{L}[\mathcal{A}]}(^mX, Y) \subseteq \mathcal{P}_r^{m,\text{left}}(X, Y);$$

(e) *there exists an index $m \geq 2$ such that, for every Banach space Y,*

$$\mathcal{P}_{\mathcal{L}[\mathcal{A}]}(^mX, Y) \subseteq \mathcal{P}_r^{m,\text{left}}(X, Y).$$

Then, (d) ⇒ (e) ⇒ (a) ⇔ (b) ⇔ (c). *Moreover, if $r = 1$, all the statements are equivalent.*

Proof. (a) $\Rightarrow$ (b). Given $m \in \mathbb{N}$ ($m \geq 2$) and a Banach space Y, let $P \in \mathcal{P}_{\mathcal{L}[\mathcal{A}]}(^m X, Y)$. By [5, Proposition 1], $P \in \mathcal{P}_{\mathcal{L}[\Gamma_r]}(^m X, Y)$. Then there exist a Banach space Z, an operator $T \in \Gamma_r(X, Z)$, and a polynomial $Q \in \mathcal{P}(^m Z, Y)$ such that $P = Q \circ T$. Since T is r-factorable, there exist a measure space (Ω, Σ, μ) and operators $A \in \mathcal{L}(X, L_r(\mu))$, $B \in \mathcal{L}(L_r(\mu), Z^{**})$ such that $k_Z \circ T = B \circ A$. Moreover, $\gamma_r(T) \leq \|B\|\|A\|$. Let $\widetilde{Q}$ be the Aron-Berner extension of Q [2] (see also [16]). We have (see Figure 1)

$$k_Y \circ P = k_Y \circ Q \circ T = \widetilde{Q} \circ k_Z \circ T = \widetilde{Q} \circ B \circ A\,.$$

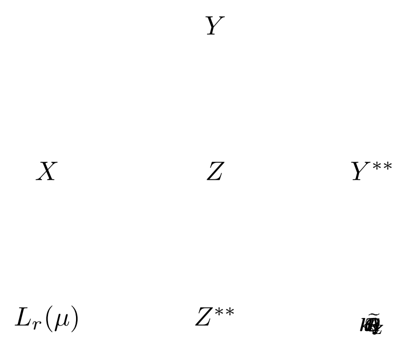

FIGURE 1. Factorization of $k_Y \circ P$

So P is right r-factorable.

(b) $\Rightarrow$ (c). It is obvious.

(c) $\Rightarrow$ (a). We show that the ideal $\mathcal{P}_r^{m,\text{right}}$ is closed under differentiation. Indeed, let $P \in \mathcal{P}_r^{m,\text{right}}(X, Y)$. Fix $a \in X$. We have to prove that $dP(a)$ is r-factorable. Clearly, $k_Y \circ dP(a) = d(k_Y \circ P)(a)$. By our hypothesis, there exist a positive measure space (Ω, Σ, μ), a polynomial $Q \in \mathcal{P}(^m L_r(\mu), Y^{**})$, and an operator $T \in \mathcal{L}(X, L_r(\mu))$ such that $k_Y \circ P = Q \circ T$.

$$\begin{array}{ccc} X & \xrightarrow{\;P\;} & Y \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle k_Y} \\ L_r(\mu) & \xrightarrow[\;Q\;]{} & Y^{**} \end{array}$$

Define the operator $A : L_r(\mu) \to Y^{**}$ by

$$A(z) = \widehat{Q}(T(a), \overset{(m-1)}{\ldots}, T(a), z) \qquad (z \in L_r(\mu))\,.$$

In particular, for every $x \in X$,

$$\begin{aligned}A(T(x)) &= \widehat{Q}(T(a), \overset{(m-1)}{\dots}, T(a), T(x)) \\ &= \widehat{k_Y \circ P}(a, \overset{(m-1)}{\dots}, a, x) \\ &= \frac{1}{m}\, d(k_Y \circ P)(a)(x) \\ &= \frac{1}{m}\, k_Y \circ dP(a)(x)\,.\end{aligned}$$

Hence

$$\frac{1}{m}\, k_Y \circ dP(a) = A \circ T\,,$$

and $dP(a)$ is r-factorable. Since the ideal $\mathcal{P}_{\mathcal{L}[\mathcal{A}]}$ is closed for scalar multiplication [5, Lemma 1], (a) follows from [5, Proposition 2].

(d) $\Rightarrow$ (e). It is obvious.

(e) $\Rightarrow$ (a). Let $T \in \mathcal{A}(X, Y)$. Let $x_0 \in X$ and $y_0 = T(x_0) \neq 0$. Choose $y^* \in Y^*$ such that $y^*(y_0) = 1$. Let $x^* = T^*(y^*)$. So $x^*(x_0) = 1$. For every $1 \le j \le m-1$, we introduce the operators $\pi_j : \otimes_{\pi,s}^{j+1} X \to \otimes_{\pi,s}^{j} X$ and $\pi'_j : \otimes_{\pi,s}^{j+1} Y \to \otimes_{\pi,s}^{j} Y$, given in [3] by

$$\begin{aligned}&\pi_j\left(\sum_{i=1}^{r} \lambda_i x_i \otimes \overset{(j+1)}{\cdots} \otimes x_i\right) \\ &= \sum_{i=1}^{r} \lambda_i x^*(x_i) x_i \otimes \overset{(j)}{\cdots} \otimes x_i \quad (\lambda_i \in \mathbb{K},\ x_i \in X,\ 1 \le i \le r)\end{aligned}$$

and

$$\begin{aligned}&\pi'_j\left(\sum_{i=1}^{r} \lambda_i y_i \otimes \overset{(j+1)}{\cdots} \otimes y_i\right) \\ &= \sum_{i=1}^{r} \lambda_i y^*(y_i) y_i \otimes \overset{(j)}{\cdots} \otimes y_i \quad (\lambda_i \in \mathbb{K},\ y_i \in Y,\ 1 \le i \le r)\,.\end{aligned}$$

Consider the polynomials

$$P := T \circ \pi_1 \circ \cdots \circ \pi_{m-1} \circ \delta_m \in \mathcal{P}(^m X, Y)$$

and

$$Q := \pi'_1 \circ \cdots \circ \pi'_{m-1} \circ \delta'_m \circ T \in \mathcal{P}(^m X, Y)\,,$$

where $\delta_m : X \to \otimes^m_{\pi,s} X$ and $\delta'_m : Y \to \otimes^m_{\pi,s} Y$ are the canonical polynomials. We have $P = Q$. Indeed, for $x \in X$,

$$\begin{aligned} P(x) &= T \circ \pi_1 \circ \cdots \circ \pi_{m-1} \circ \delta_m(x) \\ &= T \circ \pi_1 \circ \cdots \circ \pi_{m-1} \left(x \otimes \overset{(m)}{\cdots} \otimes x \right) \\ &= [x^*(x)] T \circ \pi_1 \circ \cdots \circ \pi_{m-2} \left(x \otimes \overset{(m-1)}{\cdots} \otimes x \right) \\ &= \cdots \\ &= T(x)[x^*(x)]^{m-1} . \end{aligned}$$

On the other hand,

$$\begin{aligned} Q(x) &= \pi'_1 \circ \cdots \circ \pi'_{m-1} \circ \delta'_m \circ T(x) \\ &= \pi'_1 \circ \cdots \circ \pi'_{m-1} \left(T(x) \otimes \overset{(m)}{\cdots} \otimes T(x) \right) \\ &= y^*(T(x)) \pi'_1 \circ \cdots \circ \pi'_{m-2} \left(T(x) \otimes \overset{(m-1)}{\cdots} \otimes T(x) \right) \\ &= \cdots \\ &= [y^*(T(x))]^{m-1} T(x) \\ &= [x^*(x)]^{m-1} T(x) . \end{aligned}$$

Hence $P = T \circ \pi_1 \circ \cdots \circ \pi_{m-1} \circ \delta_m \in \mathcal{P}_{\mathcal{L}[\mathcal{A}]}(^m X, Y)$ and then, by our hypothesis, it is left r-factorable. So its linearization $T \circ \pi_1 \circ \cdots \circ \pi_{m-1}$ is also r-factorable. Now, for every $1 \leq p \leq m-1$, let $j_p : \otimes^p_{\pi,s} X \to \otimes^{p+1}_{\pi,s} X$ be the operator [3, page 168] such that $\pi_p \circ j_p$ is the identity on $\otimes^p_{\pi,s} X$. It follows that

$$T = T \circ \pi_1 \circ \cdots \circ \pi_{m-1} \circ j_{m-1} \circ \cdots \circ j_1$$

is r-factorable.

If $r = 1$, the statements are equivalent since (b) $\Rightarrow$ (d) follows from Proposition 4. □

Remark 7. If $r > 1$, the assertions of Theorem 6 are not equivalent. Indeed, the polynomial $Q \in \mathcal{P}(^m \ell_2, \ell_1)$, given by $Q(x) := (x_n^m)_{n=1}^{\infty}$, belongs to $\mathcal{P}_{\mathcal{L}[\Gamma_2]}(^m \ell_2, \ell_1)$. If $\mathcal{A} := \Gamma_2$, Theorem 6(a) is satisfied, but $Q \notin \mathcal{P}_2^{m,\mathrm{left}}(\ell_2, \ell_1)$, by Remark 5(b).

Corollary 8. *Let X be a Banach space, and let $1 \leq r < \infty$. Consider the following assertions*:

(a) *for every Banach space Y, every r-summing operator $T : X \to Y$ is r-factorable*;
(b) *for every Banach space Y, for every $m \in \mathbb{N}$ ($m \geq 2$), every r-dominated polynomial $P \in \mathcal{P}(^m X, Y)$ is right r-factorable*;
(c) *there exists $m \in \mathbb{N}$ ($m \geq 2$) such that, for every Banach space Y, every r-dominated polynomial $P \in \mathcal{P}(^m X, Y)$ is right r-factorable*;

(d) *for every Banach space* Y*, for every* $m \in \mathbb{N}$ $(m \geq 2)$*, every* r*-dominated polynomial* $P \in \mathcal{P}(^m X, Y)$ *is left* r*-factorable*;

(e) *there exists* $m \in \mathbb{N}$ $(m \geq 2)$ *such that, for every Banach space* Y*, every* r*-dominated polynomial* $P \in \mathcal{P}(^m X, Y)$ *is left* r*-factorable.*

Then (d) $\Rightarrow$ (e) $\Rightarrow$ (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)*. If* $r = 1$*, all the assertions are equivalent.*

Proof. The result follows from Theorem 6 since the ideal of r-dominated polynomials coincides with $\mathcal{P}_{\mathcal{L}[\Pi_r]}$ (see, for instance, [8, Theorem 5] or [7, Theorem 5]). □

Given a Banach space X, let $\mathcal{F}_X$ denote the collection of all finite dimensional subspaces of X. We say that X has *local unconditional structure* (*l.u.st.,* for short) [12, Chapter 17] if there is a constant $\Lambda \geq 1$ such that, for all $E \in \mathcal{F}_X$, the canonical embedding $E \hookrightarrow X$ has a factorization $E \xrightarrow{v} Y \xrightarrow{u} X$, through a Banach space Y with unconditional basis; u and v are operators satisfying $\|u\|\,\|v\|\,\mathrm{ub}(Y) \leq \Lambda$, where $\mathrm{ub}(Y)$ is the unconditional basis constant of Y. The smallest of all such Λ's is called the *l.u.st. constant* of X, and is denoted by $\Lambda(X)$.

Every $\mathcal{L}_p$-space $(1 \leq p \leq \infty)$ and every Banach lattice have local unconditional structure [12, Theorem 17.1].

In the following lemma, the factorization result is well known (see [8, Theorem 5] or [7, Theorem 5]). We are interested here in the equality of the norms.

Lemma 9. *A polynomial* $P \in \mathcal{P}(^m X, Y)$ *is* r*-dominated if and only if there are a Banach space* Z*, an* r*-summing operator* $T \in \mathcal{L}(X, Z)$*, and a polynomial* $Q \in \mathcal{P}(^m Z, Y)$ *such that* $P = Q \circ T$*. Moreover,*

$$\|P\|_{r\text{-d}} = \inf\{\|Q\|\,\pi_r(T)^m : \ Q, T \ \textit{as above}\}\,.$$

Proof. Let P be r-dominated. We know that it admits a factorization $P = Q \circ T$ as in the statement. For every $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$, we have

$$\begin{aligned}
\left(\sum_{i=1}^n \|P(x_i)\|^{r/m}\right)^{\frac{m}{r}} &= \left(\sum_{i=1}^n \|QT(x_i)\|^{r/m}\right)^{\frac{m}{r}} \\
&\leq \left[\sum_{i=1}^n (\|Q\|\,\|T(x_i)\|^m)^{\frac{r}{m}}\right]^{\frac{m}{r}} \\
&= \|Q\| \left(\sum_{i=1}^n \|T(x_i)\|^r\right)^{\frac{m}{r}} \\
&\leq \|Q\| \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |x^*(x_i)|^r\right)^{\frac{m}{r}} \pi_r(T)^m\,.
\end{aligned}$$

Hence,

$$\|P\|_{r\text{-d}} \leq \|Q\|\,\pi_r(T)^m\,.$$

Given $\epsilon > 0$, by [18, Proposition 3.1], there are a constant $C_\epsilon > 0$ with

$$C_\epsilon < \|P\|_{r\text{-d}} + \epsilon$$

and a regular Borel probability measure μ_ϵ on B_{X^*} such that

$$\|P(x)\| \leq C_\epsilon \left[\int_{B_{X^*}} |x^*(x)|^r \, d\mu_\epsilon(x^*) \right]^{\frac{m}{r}} \qquad (x \in X).$$

Let $T_0 : X \to L_r\left(B_{X^*}, \mu_\epsilon\right)$ be the operator given by

$$T_0(x)(x^*) := x^*(x) \qquad (x \in X,\ x^* \in B_{X^*}).$$

Since, for $x \in X$,

$$\begin{aligned} \|T_0(x)\| &= \left[\int_{B_{X^*}} |T_0(x)(x^*)|^r \, d\mu_\epsilon(x^*) \right]^{\frac{1}{r}} \\ &= \left[\int_{B_{X^*}} |x^*(x)|^r \, d\mu_\epsilon(x^*) \right]^{\frac{1}{r}} \\ &\leq \|x\| , \end{aligned} \tag{1}$$

T_0 is continuous. Let $Z_\epsilon := \overline{T_0(X)}$, that is, the closure of $T_0(X)$ in $L_r\left(B_{X^*}, \mu_\epsilon\right)$. Let

$$T_\epsilon : X \longrightarrow Z_\epsilon$$

be the operator defined by $T_\epsilon(x) := T_0(x)$ for all $x \in X$. By (1), T_ϵ is r-summing [12, Theorem 2.12], and $\pi_r(T_\epsilon) \leq 1$. Define a polynomial $Q_0 : T_0(X) \to Y$ by

$$Q_0(T_0(x)) := P(x).$$

We have

$$\begin{aligned} \|Q_0 T_0(x)\| &= \|P(x)\| \\ &\leq C_\epsilon \left[\int_{B_{X^*}} |x^*(x)|^r \, d\mu_\epsilon(x^*) \right]^{\frac{m}{r}} \\ &= C_\epsilon \, \|T_0(x)\|^m , \end{aligned}$$

so Q_0 is continuous with $\|Q_0\| \leq C_\epsilon$. Let Q_ϵ be the continuous extension of Q_0 to Z_ϵ with $\|Q_\epsilon\| = \|Q_0\|$. Then $P = Q_\epsilon \circ T_\epsilon$, with T_ϵ r-summing, and

$$\|Q_\epsilon\| \, \pi_r(T_\epsilon)^m \leq \|Q_0\| \leq C_\epsilon < \|P\|_{r\text{-d}} + \epsilon ,$$

and the proof is finished. □

Corollary 10. *Let X be a Banach space with l.u.st. Then, for every Banach space Y and every index $m \geq 2$, every 1-dominated polynomial $P \in \mathcal{P}(^m X, Y)$ is right 1-factorable, with $\gamma_1^{\text{right}}(P) \leq \Lambda(X)^m \|P\|_{1\text{-d}}$.*

Proof. By Lemma 9, there exist a Banach space Z, an absolutely summing operator $T \in \mathcal{L}(X, Z)$ and a polynomial $Q \in \mathcal{P}(^m Z, Y)$ such that $P = Q \circ T$. Since X has l.u.st., T is 1-factorable [12, 17.7], so there exist a measure space (Ω, Σ, μ) and operators $A \in \mathcal{L}(X, L_1(\mu))$, $B \in \mathcal{L}(L_1(\mu), Z^{**})$ such that $k_Z \circ T = B \circ A$. Moreover, $\gamma_1(T) \leq \Lambda(X)\pi_1(T)$ [12, 17.7]. As in the proof of Theorem 6 (see Figure 1, with $r = 1$), using the Aron-Berner extension $\widetilde{Q}$ of Q, we have

$$k_Y \circ P = k_Y \circ Q \circ T = \widetilde{Q} \circ B \circ A\,.$$

So P is right 1-factorable. Moreover, we observe that

$$\gamma_1^{\text{right}}(P) \leq \|A\|^m \left\|\widetilde{Q} \circ B\right\| \leq \|Q\| \|A\|^m \| \|B\|^m\,,$$

where we have used the equality $\left\|\widetilde{Q}\right\| = \|Q\|$ [6, Proposition 1.3]. Taking the infimum over A and B such that $k_Z \circ T = B \circ A$, we have

$$\gamma_1^{\text{right}}(P) \leq \|Q\| \gamma_1(T)^m \leq \|Q\| \Lambda(X)^m \pi_1(T)^m\,.$$

Taking again the infimum over Q and T such that $P = Q \circ T$, by Lemma 9, we obtain

$$\gamma_1^{\text{right}}(P) \leq \Lambda(X)^m \|P\|_{\text{1-d}}\,,$$

and the proof is finished. □

Remark 11. The assertion (a) in Corollary 8 holds in particular:

(a) for every Banach space X when $r = 2$ [12, Corollary 2.16];

(b) if X is an $\mathcal{L}_p$-space, with $1 \leq p \leq 2$, and $1 < r < 2$ since, in this case, every r-summing operator is also r-integral [12, Corollary 6.19], and then r-factorable;

(c) if X is a $C(K)$ space, for every $1 \leq r < \infty$, since in this case, every r-summing operator is also r-integral [12, Corollary 5.8], and then r-factorable.

Remark 12. Every m-homogeneous integral polynomial on a $C(K)$ space is right m-factorable. Indeed, by [9, Lemma 1], P is m-dominated. By Corollary 8 and Remark 11(c), P is right m-factorable.

Corollary 13. *Let X be an $\mathcal{L}_p$-space with $1 \leq p < \infty$ and let $m \in \mathbb{N}$. Every q-dominated m-homogeneous polynomial on X, with*

$$\frac{1}{q} \geq \left|\frac{1}{p} - \frac{1}{2}\right|\,,$$

is right 2-factorable.

Proof. It is enough to apply Theorem 6 since, under our hypothesis, for every Banach space Z, every q-summing operator $T \in \mathcal{L}(X, Z)$ is 2-factorable [12, p. 168]. □

Recall that right 2-factorable implies right r-factorable, for every $1 < r < \infty$, by [12, Corollary 9.2].

Corollary 14. *Let X be a Banach space with l.u.st. and with cotype 2. Then, for every Banach space Y, every integral polynomial $P \in \mathcal{P}(^2X, Y)$ is right 1-factorable.*

Proof. If $P \in \mathcal{P}(^2X, Y)$ is integral, it is also 2-dominated [9, Lemma 1]. So there exist a Banach space Z, a 2-summing operator $T \in \mathcal{L}(X, Z)$ and a polynomial $Q \in \mathcal{P}(^2Z, Y)$ such that $P = Q \circ T$. Since X has cotype 2, T is also absolutely summing [12, Corollary 11.16], and then P is 1-dominated. By Corollary 10, P is right 1-factorable. □

Proposition 15. *Let X be a subspace of an $\mathcal{L}_p$-space ($1 \le p \le 2$). Then, for every Banach space Y, every integral polynomial $P \in \mathcal{P}(^2X, Y)$ is right 1-factorable.*

Proof. Let G be an $\mathcal{L}_p$-space ($1 \le p \le 2$), let X be a subspace of G, and suppose that $P \in \mathcal{P}(^2X, Y)$ is an integral polynomial. As above, there exist a Banach space Z, a 2-summing operator $T \in \mathcal{L}(X, Z)$ and a polynomial $Q \in \mathcal{P}(^2Z, Y)$ such that $P = Q \circ T$. The operator T admits a 2-summing extension $\widetilde{T} \in \mathcal{L}(G, Z)$ [12, Theorem 4.15]. Since G has cotype 2, $\widetilde{T}$ is absolutely summing [12, Corollary 11.16]. Then the polynomial $\widetilde{P} := Q \circ \widetilde{T} \in \mathcal{P}(^2G, Y)$ is 1-dominated. By Corollary 10, P is right 1-factorable. So there exist a measure space (Ω, Σ, μ), an operator $A \in \mathcal{L}(G, L_1(\mu))$ and a polynomial $R \in \mathcal{P}(^2L_1(\mu), Y^{**})$ such that $k_Y \circ \widetilde{P} = R \circ A$. Then

$$k_Y \circ P = k_Y \circ Q \circ T = k_Y \circ Q \circ \widetilde{T} \circ i = k_Y \circ \widetilde{P} \circ i = R \circ A \circ i\,,$$

where i denotes the natural embedding of X into G. This finishes the proof. □

Remark 16. There are subspaces of $L_p[0,1]$ ($1 \le p < 2$) without l.u.st. [12, page 364], so the last result does not follow from Corollary 14.

In the following theorem, $(e_n)_{n=1}^{\infty}$ denotes the unit vector basis of ℓ_1 (or ℓ_2).

Theorem 17. *Let X be a Banach space containing a copy of ℓ_1. Then for every index $m \ge 2$, there exists a polynomial $P \in \mathcal{P}(^mX, \ell_1)$ that is not left r-factorable for any choice of $1 < r \le \infty$.*

Proof. Since X contains a copy of ℓ_1, there exists a surjective 2-summing operator $q \in \mathcal{L}(X, \ell_2)$ [12, Corollary 4.16]. Let (x_n) be a bounded sequence in X such that $q(x_n) = e_n$ for all $n \in \mathbb{N}$. Let $Q \in \mathcal{P}(^m\ell_2, \ell_1)$ be the polynomial defined by

$$Q(x) := (x_k^m)_{k=1}^{\infty} \quad \text{for } x = (x_k)_{k=1}^{\infty} \in \ell_2\,.$$

Consider the polynomial $P := Q \circ q \in \mathcal{P}(^mX, \ell_1)$. If P were left r-factorable for some $1 < r \le \infty$, then there would exist a positive measure space (Ω, Σ, μ), a polynomial $R \in \mathcal{P}(^mX, L_r(\mu))$ and an operator $T \in \mathcal{L}(L_r(\mu), \ell_\infty^*)$ such that $k_{\ell_1} \circ P = T \circ R$. Let $H \in \mathcal{L}(\ell_\infty^*, \ell_1)$ be a projection such that $H \circ k_{\ell_1}$ is the identity map on ℓ_1. Then $P = H \circ k_{\ell_1} \circ P = H \circ T \circ R$. If $1 < r < \infty$, $H \circ T \in \mathcal{L}(L_r(\mu), \ell_1)$ is a compact operator. The same happens if $r = \infty$ since, in this case, $H \circ T$ is 2-summing (and hence weakly compact) with values in ℓ_1 [12, Theorem 11.14]. In both cases, P would be a compact polynomial, in contradiction with the fact that $P(x_n) = e_n$ for all $n \in \mathbb{N}$. □

Remark 18. (a) The polynomial constructed in Theorem 17 is right 2-factorable, and so right r-factorable, for $1 < r < \infty$, by [12, Corollary 9.2].

(b) Recall that X is said to be a *GL-space* if every absolutely summing operator from X into ℓ_2 is 1-factorable. Suppose that at least one of the spaces X and Y is a GL-space, and that X^* and Y have cotype 2. Then $\mathcal{L}(X, Y) = \Gamma_2(X, Y)$ [12, Theorem 17.12]. The same conditions on the spaces X and Y do not imply the equality $\mathcal{P}(^m X, Y) = \mathcal{P}_2^{m,\text{left}}(X, Y)$. Indeed, it is enough to apply Theorem 17 when $X = \ell_\infty$ and $Y = \ell_1$.

(c) In [10, Proposition 5.8] there are examples of nuclear m-homogeneous polynomials from ℓ_∞ into ℓ_1 that are not right 2-factorable, so we have $\mathcal{P}(^m \ell_\infty, \ell_1) \neq \mathcal{P}_2^{m,\text{right}}(\ell_\infty, \ell_1)$ in spite of the equality $\mathcal{L}(\ell_\infty, \ell_1) = \Gamma_2(\ell_\infty, \ell_1)$ [12, Theorem 17.12].

References

[1] R. Alencar, *On reflexivity and basis for $P(^m E)$*, Proc. Roy. Irish Acad. Sect. A **85** (1985), no. 2, 131–138.

[2] R. M. Aron and P. D. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France **106** (1978), no. 1, 3–24.

[3] F. Blasco, *Complementation in spaces of symmetric tensor products and polynomials*, Studia Math. **123** (1997), no. 2, 165–173.

[4] G. Botelho, *Ideals of polynomials generated by weakly compact operators*, Note Mat. **25** (2005/06), no. 1, 69–102.

[5] G. Botelho and D. M. Pellegrino, *Two new properties of ideals of polynomials and applications*, Indag. Math. (N.S.) **16** (2005), no. 2, 157–169.

[6] D. Carando, *Extendible polynomials on Banach spaces*, J. Math. Anal. Appl. **233** (1999), no. 1, 359–372.

[7] R. Cilia, M. D'Anna, and J. M. Gutiérrez, *Polynomials on Banach spaces whose duals are isomorphic to $l_1(\Gamma)$*, Bull. Austral. Math. Soc. **70** (2004), no. 1, 117–124.

[8] R. Cilia and J. M. Gutiérrez, *Polynomial characterization of Asplund spaces*, Bull. Belg. Math. Soc. Simon Stevin **12** (2005), no. 3, 393–400.

[9] ———, *Dominated, diagonal polynomials on l_p spaces*, Arch. Math. (Basel) **84** (2005), no. 5, 421–431.

[10] ———, *Ideals of integral and r-factorable polynomials*, to appear in Bol. Soc. Mat. Mexicana.

[11] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies, 176. North-Holland Publishing Co., Amsterdam, 1993.

[12] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995.

[13] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.

[14] S. Dineen, *Complex Analysis on Infinite-Dimensional Spaces*, Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 1999.

[15] K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Note Mat. **17** (1997), 153–188.

[16] J. M. Gutiérrez and I. Villanueva, *Extensions of multilinear operators and Banach space properties*, Proc. Roy. Soc. Edinburgh Sect. A **133** (2003), no. 3, 549–566.
[17] J. Lindenstrauss and H. P. Rosenthal, *The $\mathcal{L}_p$ spaces*, Israel J. Math. **7** (1969), 325–349.
[18] M. C. Matos, *Absolutely summing holomorphic mappings*, An. Acad. Brasil. Ciênc. **68** (1996), no. 1, 1–13.
[19] Y. Meléndez and A. Tonge, *Polynomials and the Pietsch domination theorem*, Math. Proc. R. Ir. Acad. **99A** (1999), no. 2, 195–212.
[20] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland Publishing Co., Amsterdam, 1986.
[21] R. A. Ryan, *Applications of topological tensor products to infinite dimensional holomorphy*, Ph. D. Thesis, Trinity College, Dublin 1980.
[22] ———, *Introduction to Tensor Products of Banach Spaces*, Springer-Verlag London, Ltd., London, 2002.

Raffaella Cilia
Dipartimento di Matematica
Facoltà di Scienze
Università di Catania
Viale Andrea Doria 6
95125 Catania, Italy
E-mail address: `cilia@dmi.unict.it`

Joaquín M. Gutiérrez
Departamento de Matemática Aplicada
ETS de Ingenieros Industriales
Universidad Politécnica de Madrid
C. José Gutiérrez Abascal 2
28006 Madrid, Spain
E-mail address: `jgutierrez@etsii.upm.es`