J. Korean Math. Soc. **46** (2009), No. 6, pp. 1219–1232
DOI 10.4134/JKMS.2009.46.6.1219

GENERALIZED COMPOSITION OPERATORS FROM GENERALIZED WEIGHTED BERGMAN SPACES TO BLOCH TYPE SPACES

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Abstract. Let $H(B)$ denote the space of all holomorphic functions on the unit ball B of $\mathbb{C}^n$. Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of B and $g \in H(B)$ with $g(0) = 0$. In this paper we study the boundedness and compactness of the generalized composition operator

$$C_\varphi^g f(z) = \int_0^1 \Re f(\varphi(tz)) g(tz) \frac{dt}{t}$$

from generalized weighted Bergman spaces into Bloch type spaces.

1. Introduction

Let B be the unit ball of $\mathbb{C}^n$. We denote by $H(B)$ the space of all holomorphic functions on B. Let dv be the normalized Lebesgue measure of B, i.e., $v(B) = 1$. Let $dv_\alpha(z) = c_\alpha(1-|z|)^\alpha dv(z)$, where $c_\alpha = (\Gamma(n+\alpha+1))/(n!\Gamma(\alpha+1))$. For $f \in H(B)$, let

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

represent the radial derivative of $f \in H(B)$. We write $\Re^m f = \Re(\Re^{m-1} f)$.

For $\alpha > 0$, recall that the Bloch type space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(B)$, is the space of all $f \in H(B)$ for which (see [12])

$$b_\alpha(f) = \sup_{z \in B} (1-|z|^2)^\alpha |\Re f(z)| < \infty. \tag{1}$$

The little Bloch type space $\mathcal{B}_0^\alpha$, comprises all $f \in H(B)$ such that

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |\Re f(z)| = 0.$$

Received March 8, 2008.
2000 *Mathematics Subject Classification.* Primary 47B35, Secondary 30H05.
Key words and phrases. generalized composition operator, generalized weighted Bergman space, Bloch type space.

Under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$, $\mathcal{B}^\alpha$ is a Banach space. When $\alpha = 1$, we get the classical Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$. For more information of the Bloch space and the Bloch type space (see, e.g. [12]).

For any $p > 0$ and $\alpha \in \mathbb{R}$, let N be the smallest nonnegative integer such that $pN + \alpha > -1$. The generalized weighted Bergman space, which is introduced by Zhao and Zhu (see, e.g. [10]), is the space of all $f \in H(B)$ such that

$$\|f\|_{A^p_\alpha} = |f(0)| + \Big[\int_B |\Re^N f(z)|^p (1-|z|^2)^{pN} dv_\alpha(z)\Big]^{1/p} < \infty.$$

This space covers the classical Bergman space, the Besov space, the Hardy space H^2 and the so-called Arveson space. For example, when $\alpha = 0$, the space $A^p_0(B) = A^p(B)$ is the classical Bergman space; when $\alpha = -n$ and $p = 2$, the space A^p_α is the so-called Arveson space; when $\alpha = -(n+1)$, the space A^p_α is the Besov space. See [10, 12] for some basic facts on the Bergman space.

Denote by D the unit disk and $H(D)$ the space of all analytic functions on D. Let ϕ be an analytic self-map of D. For $f \in H(D)$, the composition operator is defined by

$$(C_\phi f)(z) = (f \circ \phi)(z).$$

It is interesting to provide a function theoretic characterization when ϕ induce a bounded or compact composition operator on various spaces. The books [1, 9, 11] contain much information on this topic.

Let $h \in H(D)$ and ϕ be an analytic self-map of D. In [7], the authors defined and studied the generalized composition operator as following

$$C^h_\phi f(z) = \int_0^z f'(\phi(\xi))h(\xi)d\xi, \ f \in H(D), \ z \in D.$$

Note that the difference $C^{\phi'}_\phi - C_\phi$ is a constant. The boundedness and compactness of the generalized composition operator on Zygmund spaces and Bloch type spaces were investigated in [7].

It is natural to generalize the operator C^h_ϕ to the unit ball. Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of B and $g \in H(B)$ with $g(0) = 0$. We consider the higher-dimensional version of the generalized composition operator via

$$C^g_\varphi f(z) = \int_0^1 \Re f(\varphi(tz))g(tz)\frac{dt}{t}, \qquad f \in H(B), \ z \in B. \tag{2}$$

The operator C^g_φ is still called the generalized composition operator. Note that when $\varphi(z) = z$, then C^g_φ is the Riemann-Stieltjes operator

$$L_g f(z) = \int_0^1 \Re f(tz)g(tz)\frac{dt}{t}, \qquad f \in H(B), \ z \in B,$$

which was first studied in [3]. To the best of our knowledge, the operator C^g_φ on the unit ball is introduced in the present article for the first time.

In this paper we study the boundedness and compactness of generalized composition operators C_φ^g from the generalized weighted Bergman space into the Bloch type space and little Bloch type space. As some corollaries, we obtain characterizations of the composition operator C_ϕ and the Riemann-Stieltjes operator L_g from the generalized weighted Bergman space into the Bloch type space and the little Bloch type space. These results are new even for C_ϕ and L_g.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Main results and proofs

In this section we give our main results and proofs. We distinguish three cases: $n+1+\alpha+p>0$, $n+1+\alpha+p=0$ and $n+1+\alpha+p<0$. Before we formulate our main results, we state several auxiliary results which will be used in the proofs. The following result can be found in [10].

Lemma 1. (i) *Suppose $p>0$ and $\alpha+n+1>0$. Then there exists a constant $C>0$ such that*

$$|f(z)| \le \frac{C\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{n+\alpha+1}{p}}}$$

for all $f \in A^p_\alpha$ and $z \in B$.

(ii) *Suppose $p>0$ and $\alpha+n+1<0$ or $0<p\le 1$ and $\alpha+n+1=0$. Then every function in A^p_α is continuous on the closed unit ball and so is bounded.*

(iii) *Suppose $p>1$, $1/p+1/q=1$ and $\alpha+n+1=0$. Then there exists a constant $C>0$ such that*

$$|f(z)| \le C\Big[\ln\frac{2}{1-|z|^2}\Big]^{1/q}$$

for all $f \in A^p_\alpha$ and $z \in B$.

Lemma 2. *A closed set K in $\mathcal{B}^\alpha_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)^\alpha|\Re f(z)| = 0.$$

Proof. The proof is similar to the proof of Lemma 1 in [8]. We omit the details. □

The following criterion for compactness follows from standard arguments similar to those outlined in Proposition 3.11 of [1].

Lemma 3. *Suppose $p>0$ and α is a real number. Let φ be a holomorphic self-map of B, $g\in H(B)$ with $g(0)=0$ and $0<\beta<\infty$. Then $C_\varphi^g : A^p_\alpha \to \mathcal{B}^\beta$ is compact if and only if $C_\varphi^g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in A^p_α which converges to zero uniformly on compact subset of B as $k\to\infty$, we have $\|C_\varphi^g f_k\|_{\mathcal{B}^\beta} \to 0$ as $k\to\infty$.*

2.1. Case of $n+1+\alpha+p>0$

Theorem 1. *Suppose $p>0$ and $n+1+\alpha+p>0$. Let φ be a holomorphic self-map of B, $g\in H(B)$ with $g(0)=0$ and $0<\beta<\infty$. Then $C_\varphi^g: A_\alpha^p\to\mathcal{B}^\beta$ is bounded if and only if*

$$M_1=\sup_{z\in B}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}}<\infty. \tag{3}$$

Proof. Sufficiency: Suppose that (3) holds. From the assumption, we see that $(C_\varphi^g f)(0)=0$. From the definition of radical derivative we can easily show that (see, e.g. [2])

$$\Re[C_\varphi^g(f)](z)=\Re f(\varphi(z))g(z). \tag{4}$$

Then for arbitrary $z\in B$ and $f\in A_\alpha^p$, since $\Re f\in A_{\alpha+p}^p$ and $\|\Re f\|_{A_{\alpha+p}^p}\le C\|f\|_{A_\alpha^p}$(see [10]), by (4) and Lemma 1 we have

$$\begin{aligned}(1-|z|^2)^\beta|\Re(C_\varphi^g f)(z)| &= (1-|z|^2)^\beta|\Re f(\varphi(z))||g(z)|\\ &\le C\|\Re f\|_{A_{\alpha+p}^p}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}}\\ &\le C\|f\|_{A_\alpha^p}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}}.\end{aligned} \tag{5}$$

From this and (3), we see that $C_\varphi^g: A_\alpha^p\to\mathcal{B}^\beta$ is bounded.

Necessity: Suppose that $C_\varphi^g: A_\alpha^p\to\mathcal{B}^\beta$ is bounded. Taking $f_a(z)=\frac{\langle z,a\rangle}{|a|^2}$, $a\ne0$, then by the boundedness of $C_\varphi^g: A_\alpha^p\to\mathcal{B}^\beta$ we get

$$\sup_{z\in B}(1-|z|^2)^\beta|g(z)|<\infty. \tag{6}$$

Assume that

$$t>n\max\left(1,\frac{1}{p}\right)+\frac{\alpha+1}{p}. \tag{7}$$

For $a\in B$, set

$$f_a(z)=\frac{(1-|a|^2)^{t-\frac{n+1+\alpha}{p}}}{(1-\langle z,a\rangle)^t}.$$

Then

$$\Re f_a(z)=t\frac{(1-|a|^2)^{t-\frac{n+1+\alpha}{p}}\langle z,a\rangle}{(1-\langle z,a\rangle)^{t+1}}.$$

From Theorem 32 of [12] we see that $f_a\in A_\alpha^p$ and $C=\sup_{a\in B}\|f_a\|_{A_\alpha^p}<\infty$. Therefore

$$\begin{aligned}C\|C_\varphi^g\|_{A_\alpha^p\to\mathcal{B}^\beta} &\ge \|C_\varphi^g f_{\varphi(b)}\|_{\mathcal{B}^\beta}=\sup_{z\in B}(1-|z|^2)^\beta|\Re(C_\varphi^g f_{\varphi(b)})(z)|\\ &\ge t\frac{(1-|b|^2)^\beta|g(b)||\varphi(b)|^2}{(1-|\varphi(b)|^2)^{\frac{n+1+\alpha+p}{p}}}.\end{aligned} \tag{8}$$

From (6) we obtain

$$\frac{(1-|b|^2)^\beta|g(b)|}{(1-|\varphi(b)|^2)^{\frac{n+1+\alpha+p}{p}}} \le \Big(\frac{4}{3}\Big)^{\frac{n+1+\alpha+p}{p}}(1-|b|^2)^\beta|g(b)|<\infty \tag{9}$$

for $b\in B$ such that $|\varphi(b)|\le 1/2$. It follows from (8) that

$$\frac{(1-|b|^2)^\beta|g(b)|}{(1-|\varphi(b)|^2)^{\frac{n+1+\alpha+p}{p}}} \le 4\frac{(1-|b|^2)^\beta|g(b)||\varphi(b)|^2}{(1-|\varphi(b)|^2)^{\frac{n+1+\alpha+p}{p}}} \le C\|C_\varphi^g\|_{A_\alpha^p\to\mathcal{B}^\beta}<\infty \tag{10}$$

for $b\in B$ such that $1/2<|\varphi(b)|<1$. Combining (9) with (10) we get (3). This completes the proof of Theorem 1. □

Theorem 2. *Suppose $p>0$ and $n+1+\alpha+p>0$. Let φ be a holomorphic self-map of B, $g\in H(B)$ with $g(0)=0$ and $0<\beta<\infty$. Then $C_\varphi^g: A_\alpha^p\to\mathcal{B}^\beta$ is compact if and only if*

$$M_2=\sup_{z\in B}|g(z)|(1-|z|^2)^\beta<\infty \tag{11}$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}}=0. \tag{12}$$

Proof. Sufficiency: Suppose that (11) and (12) hold. Combining (11) with (12) we get (3). Hence $C_\varphi^g: A_\alpha^p\to\mathcal{B}^\beta$ is bounded. From (12) we have that if given $\varepsilon>0$, there is a constant $\delta\in(0,1)$, such that when $\delta<|\varphi(z)|<1$ implies

$$\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}}<\varepsilon. \tag{13}$$

Suppose that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A_α^p such that $f_k\to 0$ uniformly on compact subsets of B as $k\to\infty$. It follows from Cauchy's estimate that the sequence $\Re f_k$ converges to zero on compact subsets of B as $k\to\infty$. Then for the above $\varepsilon>0$, there exists a k_0 such that for $|\varphi(z)|\le r$ and $k\ge k_0$, we get $|\Re f_k(\varphi(z))|\le\varepsilon$. Thus for $|\varphi(z)|\le r$ and $k\ge k_0$, we obtain

$$(1-|z|^2)^\beta|\Re f_k(\varphi(z))g(z)|\le\varepsilon\sup_{z\in B}(1-|z|^2)^\beta|g(z)|. \tag{14}$$

Now for $|\varphi(z)|>r$ and all k, by (13) we get

$$(1-|z|^2)^\beta|\Re f_k(\varphi(z))g(z)|\le C\|f\|_{A_\alpha^p}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}}<C\|f\|_{A_\alpha^p}\varepsilon. \tag{15}$$

Combining (14) with (15) we get

$$\lim_{k\to\infty}\|C_\varphi^g f_k\|_{\mathcal{B}^\beta}=0.$$

Employing Lemma 3, the implication follows.

Necessity: Suppose that $C_\varphi^g: A_\alpha^p\to\mathcal{B}^\beta$ is compact. Then $C_\varphi^g: A_\alpha^p\to\mathcal{B}^\beta$ is bounded. It follows from the proof of Theorem 1 that (11) holds. Let $(z_k)_{k\in\mathbb{N}}$

be a sequence in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. We choose test functions $(f_k)_{k\in\mathbb{N}}$ defined by

$$f_k(z) = \frac{(1-|z_k|^2)^{t-\frac{n+\alpha+1}{p}}}{(1-\langle z, z_k\rangle)^t}, \ k \in \mathbb{N},$$

where t satisfying (7). It is easy to check that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A^p_α and $f_k \to 0$ uniformly on compact subsects of B. By Lemma 3 we have

$$\|C^g_\varphi f_k\|_{\mathcal{B}^\beta} \to 0, \quad \text{as } k \to \infty.$$

Since

$$\begin{aligned}\|C^g_\varphi f_k\|_{\mathcal{B}^\beta} &= \sup_{z\in B}(1-|z|^2)^\beta |\Re(C^g_\varphi f_k)(z)| \\ &\geq t\frac{(1-|z_k|^2)^\beta |g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha+p}{p}}},\end{aligned}$$

we obtain

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |g(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha+p}{p}}} = \lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha+p}{p}}} = 0,$$

from which we get the desired result. The proof is completed. □

Theorem 3. *Suppose $p > 0$ and $n + 1 + \alpha + p > 0$. Let φ be a holomorphic self-map of B, $g \in H(B)$ with $g(0) = 0$ and $0 < \beta < \infty$. Then $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded if and only if $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g(z)| = 0. \tag{16}$$

Proof. Necessity: Suppose that $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded. Then $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Taking $f(z) = \langle z, a\rangle/|a|^2$, $|a| > 1/2$, and employing the boundedness of $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$, (16) follows.

Sufficiency: Suppose that $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded and (16) holds. Suppose that $f \in A^p_\alpha$ with $\|f\|_{A^p_\alpha} \leq Q$, using polynomial approximations we see that (see, e.g. [10])

$$\lim_{|z|\to 1}(1-|z|^2)^{\frac{n+1+\alpha}{p}} |f(z)| = 0$$

and hence

$$\lim_{|z|\to 1}(1-|z|^2)^{\frac{n+1+\alpha+p}{p}} |\Re f(z)| = 0.$$

Hence for every $\varepsilon > 0$, there exists a $\delta \in (0, 1)$ such that when $\delta < |z| < 1$,

$$(1-|z|^2)^{\frac{n+1+\alpha+p}{p}} |\Re f(z)| < \varepsilon/M_1 \tag{17}$$

and

$$(1-|z|^2)^\beta |g(z)| < \frac{\varepsilon(1-\delta^2)^{\frac{n+1+\alpha+p}{p}}}{Q}. \tag{18}$$

Therefore if $\delta < |z| < 1$ and $\delta < |\varphi(z)| < 1$, from (3) and (17) we have

$$\begin{aligned}
&(1-|z|^2)^\beta|\Re(C_\varphi^g f)(z)| \\
&= \frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}}(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}|\Re f(\varphi(z))| \\
&\le M_1(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}|\Re f(\varphi(z))| < \varepsilon.
\end{aligned} \tag{19}$$

If $\delta < |z| < 1$ and $|\varphi(z)| \le \delta$, using Lemma 1 and (18) we have

$$\begin{aligned}
&(1-|z|^2)^\beta|\Re(C_\varphi^g f)(z)| \\
&= \frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}}(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}|\Re f(\varphi(z))| \\
&\le C\|f\|_{A_\alpha^p}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}} \\
&\le C\|f\|_{A_\alpha^p}\frac{1}{(1-\delta^2)^{\frac{n+1+\alpha+p}{p}}}(1-|z|^2)^\beta|g(z)| < \varepsilon.
\end{aligned} \tag{20}$$

Combining (19) with (20) we get that $C_\varphi^g f \in \mathcal{B}_0^\beta$. By the arbitrary of f we see that $C_\varphi^g(A_\alpha^p) \subset \mathcal{B}_0^\beta$, which together with the boundedness of $C_\varphi^g : A_\alpha^p \to \mathcal{B}^\beta$, we get the desired result. This completes the proof of the theorem. $\square$

Theorem 4. *Suppose $p > 0$ and $n+1+\alpha+p > 0$. Let φ be a holomorphic self-map of B, $g \in H(B)$ with $g(0) = 0$ and $0 < \beta < \infty$. Then $C_\varphi^g : A_\alpha^p \to \mathcal{B}_0^\beta$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}} = 0. \tag{21}$$

Proof. Necessity: Assume that $C_\varphi^g : A_\alpha^p \to \mathcal{B}_0^\beta$ is compact. Then $C_\varphi^g : A_\alpha^p \to \mathcal{B}_0^\beta$ is bounded and $C_\varphi^g : A_\alpha^p \to \mathcal{B}^\beta$ is compact. By Theorems 2 and 3 we get

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}} = 0 \tag{22}$$

and

$$\lim_{|z|\to 1}(1-|z|^2)^\beta|g(z)| = 0. \tag{23}$$

Combing (22) with (23) and similarly to the proof of Theorem 3 of [7], we get (21), as desired.

Sufficiency: Suppose that (21) holds. It follows from Lemma 2 that $C_\varphi^g : A_\alpha^p \to \mathcal{B}_0^\beta$ is compact if and only if

$$\lim_{|z|\to 1}\sup_{\|f\|_{A_\alpha^p}\le 1}(1-|z|^2)^\beta|\Re(C_\varphi^g f)(z)| = 0. \tag{24}$$

Since for any $f \in A^p_\alpha$ with $\|f\|_{A^p_\alpha} \le 1$, by (5) we have

$$(1-|z|^2)^\beta |\Re(C^g_\varphi f)(z)| \le C\|f\|_{A^p_\alpha} \frac{(1-|z|^2)^\beta |g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}}. \tag{25}$$

Using (21) we get

$$\begin{aligned} &\lim_{|z|\to 1} \sup_{\|f\|_{A^p_\alpha}\le 1} (1-|z|^2)^\beta |\Re(C^g_\varphi f)(z)| \\ \le &\lim_{|z|\to 1} \sup_{\|f\|_{A^p_\alpha}\le 1} \frac{(1-|z|^2)^\beta |g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha+p}{p}}} = 0, \end{aligned}$$

as desired. This completes the proof of the theorem. □

Let $\varphi(z) = z$. From Theorems 1-4, we have the following corollary. Partial result can be found in [3, 6].

Corollary 1. *Suppose $p > 0$ and $n+1+\alpha+p > 0$. Let $g \in H(B)$ and $0 < \beta < \infty$. Then the following statements hold.*

(i) *$L_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^{\beta - \frac{n+1+\alpha+p}{p}} |g(z)| < \infty;$$

(ii) *$L_g : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded if and only if $L_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g(z)| = 0;$$

(iii) *$L_g : A^p_\alpha \to \mathcal{B}^\beta$ is compact if and only if $L_g : A^p_\alpha \to \mathcal{B}^\beta_0$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\beta - \frac{n+1+\alpha+p}{p}} |g(z)| = 0.$$

Let $n = 1$. From Theorems 1-4, we get the characterization of the composition operator from the weighted Bergman space to the Bloch type space (see, e.g. [4, 5] for the case of $\alpha > -1$).

Corollary 2. *Suppose $p > 0$ and $2+\alpha+p > 0$. Let φ be a holomorphic self-map of D and $0 < \beta < \infty$. Then the following statements hold.*

(i) *$C_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z\in D} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}} < \infty;$$

(ii) *$C_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is compact if and only if $\varphi \in \mathcal{B}^\beta$ and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}} = 0;$$

(iii) *$C_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded if and only if $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded and $\varphi \in \mathcal{B}^\beta_0$;*

(iv) $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}} = 0.$$

2.2. Case $n+1+\alpha+p=0$

Theorem 5. *Suppose $p>1$, $1/p+1/q=1$ and $n+1+\alpha+p=0$. Let φ be a holomorphic self-map of B, $g \in H(B)$ with $g(0)=0$ and $0<\beta<\infty$. Then $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$M_3 = \sup_{z\in B}(1-|z|^2)^\beta |g(z)| \Big(\ln \frac{2}{1-|\varphi(z)|^2}\Big)^{1/q} < \infty. \tag{26}$$

Proof. Sufficiency: Suppose that (26) holds. Then for arbitrary $z \in B$ and $f \in A^p_\alpha$, similarly to the proof of Theorem 1, by Lemma 1 we have

$$\begin{aligned}(1-|z|^2)^\beta |\Re(C^g_\varphi f)(z)| &= (1-|z|^2)^\beta |\Re f(\varphi(z))||g(z)| \\ &\le C\|f\|_{A^p_\alpha}(1-|z|^2)^\beta |g(z)|\Big(\ln \frac{2}{1-|\varphi(z)|^2}\Big)^{1/q}, \end{aligned}\tag{27}$$

from which and (26), we see that $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded.

Necessity: Suppose that $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Similarly to the proof of Theorem 1 we have

$$\sup_{z\in B}(1-|z|^2)^\beta |g(z)| < \infty. \tag{28}$$

For $a \in B$, set

$$f_a(z) = \int_0^1 \Big(\ln \frac{2}{1-|a|^2}\Big)^{-1/p}\Big(\ln \frac{2}{1-\langle tz, a\rangle}\Big)\frac{dt}{t}. \tag{29}$$

Then

$$\Re f_a(z) = \Big(\ln \frac{2}{1-|a|^2}\Big)^{-1/p}\Big(\ln \frac{2}{1-\langle z, a\rangle}\Big) \in A^p_{-(n+1)},$$

the Besov space on the unit ball. From [10] we known that $f_a \in A^p_{-(p+n+1)}$. Therefore

$$\begin{aligned} C\|C^g_\varphi\|_{A^p_\alpha \to \mathcal{B}^\beta} \ge \|C^g_\varphi f_{\varphi(b)}\|_{\mathcal{B}^\beta} &= \sup_{z\in B}(1-|z|^2)^\beta |\Re(C^g_\varphi f_{\varphi(b)})(z)| \\ &\ge (1-|b|^2)^\beta |g(b)|\Big(\ln \frac{2}{1-|\varphi(b)|^2}\Big)^{1/q}. \end{aligned}\tag{30}$$

From the last inequality we get the desired result. The proof is completed. □

Theorem 6. *Suppose $p>1$ and $n+1+\alpha+p=0$. Let φ be a holomorphic self-map of B, $g \in H(B)$ with $g(0)=0$ and $0<\beta<\infty$. Then $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is compact if and only if*

$$M_4 = \sup_{z\in B}|g(z)|(1-|z|^2)^\beta < \infty \tag{31}$$

and

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\beta |g(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1/q}=0. \tag{32}$$

Proof. Sufficiency: The proof is similar to the proof of Theorem 2, we omit the details.

Necessity: Suppose that $C_\varphi^g : A_\alpha^p \to \mathcal{B}^\beta$ is compact. Then $C_\varphi^g : A_\alpha^p \to \mathcal{B}^\beta$ is bounded. It follows from the proof of Theorem 5 that (31) holds. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. Set

$$f_k(z)=\Big(\ln\frac{2}{1-|\varphi(z_k)|^2}\Big)^{-1/p}\int_0^1\Big(\ln\frac{2}{1-\langle tz,\varphi(z_k)\rangle}\Big)\frac{dt}{t},\ k\in\mathbb{N}. \tag{33}$$

Analogous to the proof of Theorem 5 we see that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A_α^p. Moreover, $f_k\to 0$ uniformly on compact subsects of B. In view of Lemma 3 it follows that

$$\|C_\varphi^g f_k\|_{\mathcal{B}^\beta}\to 0,\quad \text{as } k\to\infty.$$

Because

$$\begin{aligned}\|C_\varphi^g f_k\|_{\mathcal{B}^\beta} &= \sup_{z\in B}(1-|z|^2)^\beta|\Re(C_\varphi^g f_k)(z)|\\ &\geq (1-|z_k|^2)^\beta|g(z_k)|\Big(\ln\frac{2}{1-|\varphi(z_k)|^2}\Big)^{1/q},\end{aligned}$$

we obtain

$$\lim_{k\to\infty}(1-|z_k|^2)^\beta|g(z_k)|\Big(\ln\frac{2}{1-|\varphi(z_k)|^2}\Big)^{1/q}=0,$$

from which we get the desired result. The proof is completed. □

Theorem 7. *Suppose $p>1$ and $n+1+\alpha+p=0$. Let φ be a holomorphic self-map of B, $g\in H(B)$ with $g(0)=0$ and $0<\beta<\infty$. Then $C_\varphi^g : A_\alpha^p\to\mathcal{B}_0^\beta$ is bounded if and only if $C_\varphi^g : A_\alpha^p\to\mathcal{B}^\beta$ is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^\beta|g(z)|=0. \tag{34}$$

Proof. Necessity: Suppose that $C_\varphi^g : A_\alpha^p\to\mathcal{B}_0^\beta$ is bounded. Then $C_\varphi^g : A_\alpha^p\to\mathcal{B}^\beta$ is bounded. Taking $f(z)=\langle z,a\rangle/|a|^2$, $|a|>1/2$, and employing the boundedness of $C_\varphi^g : A_\alpha^p\to\mathcal{B}_0^\beta$, (34) follows.

Sufficiency: Suppose that $C_\varphi^g : A_\alpha^p\to\mathcal{B}^\beta$ is bounded and (34) holds. Then for each polynomial $p(z)$, we have

$$(1-|z|^2)^\beta|\Re(C_\varphi^g p)(z)|=(1-|z|^2)^\beta|\Re p(\varphi(z))||g(z)|\leq\|\Re p\|_\infty(1-|z|^2)^\beta|g(z)|.$$

From the above inequality, it follows that for each polynomial p, $C_\varphi^g(p)\in\mathcal{B}_0^\beta$. The set of all polynomials is dense in A_α^p, thus for every $f\in A_\alpha^p$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|p_k - f\|_{A^p_\alpha} \to 0$ as $k \to \infty$. From the boundedness of $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$, we have that

$$\|C^g_\varphi p_k - C^g_\varphi f\|_{\mathcal{B}^\beta} \le \|C^g_\varphi\|_{A^p_\alpha \to \mathcal{B}^\beta}\, \|p_k - f\|_{A^p_\alpha} \to 0, \quad \text{as } k \to \infty. \tag{35}$$

From this and since $\mathcal{B}^\beta_0$ is a closed subset of $\mathcal{B}^\beta$, we obtain

$$C^g_\varphi f = \lim_{k\to\infty} C^g_\varphi p_k \in \mathcal{B}^\beta_0. \tag{36}$$

From the arbitrary of f, we see that $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded. The proof is completed. □

Using Theorems 6 and 7, similarly to the proof of Theorem 4, we obtain the following result. We omit the proof.

Theorem 8. *Suppose $p > 1$ and $n + 1 + \alpha + p = 0$. Let φ be a holomorphic self-map of B, $g \in H(B)$ with $g(0) = 0$ and $0 < \beta < \infty$. Then $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ is compact if and only if*

$$\lim_{|z|\to 1}(1 - |z|^2)^\beta |g(z)| \Big(\ln \frac{2}{1 - |\varphi(z)|^2}\Big)^{1/q} = 0. \tag{37}$$

Analogues to Lemmas 1 and 2, from Theorems 5-8 we have the following two corollaries. These results are new even for the operators L_g and C_φ.

Corollary 3. *Suppose $p > 1$, $1/p + 1/q = 1$ and $n + 1 + \alpha + p = 0$. Let $g \in H(B)$ and $0 < \beta < \infty$. Then the following statements hold.*

(i) *$L_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z\in B}(1 - |z|^2)^\beta |g(z)| \Big(\ln \frac{2}{1 - |z|^2}\Big)^{1/q} < \infty;$$

(ii) *$L_g : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded if and only if $L_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded and*

$$\lim_{|z|\to 1}(1 - |z|^2)^\beta |g(z)| = 0;$$

(iii) *$L_g : A^p_\alpha \to \mathcal{B}^\beta$ is compact if and only if $L_g : A^p_\alpha \to \mathcal{B}^\beta_0$ is compact if and only if*

$$\lim_{|z|\to 1}(1 - |z|^2)^\beta |g(z)| \Big(\ln \frac{2}{1 - |z|^2}\Big)^{1/q} = 0.$$

Corollary 4. *Suppose $p > 1$, $1/p + 1/q = 1$ and $2 + \alpha + p = 0$. Let φ be a holomorphic self-map of D and $0 < \beta < \infty$. Then the following statements hold.*

(i) *$C_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z\in D}(1 - |z|^2)^\beta |\varphi'(z)| \Big(\ln \frac{2}{1 - |\varphi(z)|^2}\Big)^{1/q} < \infty;$$

(ii) $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ *is compact if and only if* $\varphi \in \mathcal{B}^\beta$ *and*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\beta|\varphi'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1/q}=0;$$

(iii) $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ *is bounded if and only if* $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded and* $\varphi \in \mathcal{B}^\beta_0$;

(iv) $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ *is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^\beta|\varphi'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1/q}=0.$$

2.3. Case $n+1+\alpha+p<0$

Theorem 9. *Suppose* $p>0$ *and* $n+1+\alpha+p<0$. *Let* φ *be a holomorphic self-map of* B, $g\in H(B)$ *with* $g(0)=0$ *and* $0<\beta<\infty$. *Then the following statements are equivalent.*

(i) $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded*;

(ii) $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ *is compact*;

(iii)

$$\sup_{z\in B}(1-|z|^2)^\beta|g(z)|<\infty. \tag{38}$$

Proof. (ii) $\Rightarrow$ (i). It is obvious.

(i) $\Rightarrow$ (iii). Suppose that $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Taking $f_a(z)=\frac{\langle z,a\rangle}{|a|^2}$, $a\neq 0$, then by the boundedness of $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ we get (38), as desired.

(iii) $\Rightarrow$ (ii). Suppose that (38) holds. When $n+1+\alpha+p<0$ or $n+1+\alpha+p=0$ and $0<p\le 1$. From Lemma 1, we see that $\Re f$ is continuous on the closed unit ball and so is bounded in B. Then for arbitrary $z\in B$ and $f\in A^p_\alpha$, similarly to the proof of Theorem 1 we have

$$\begin{aligned}(1-|z|^2)^\beta|\Re(C^g_\varphi f)(z)| &= (1-|z|^2)^\beta|\Re f(\varphi(z))||g(z)|\\ &\le C\|\Re f\|_{A^p_{\alpha+p}}(1-|z|^2)^\beta|g(z)|\le C\|f\|_{A^p_\alpha}(1-|z|^2)^\beta|g(z)|.\end{aligned} \tag{39}$$

From the above inequality we see that $C^g_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in A^p_α such that $f_k\to 0$ uniformly on compact subsets of B as $k\to\infty$. By Lemma 1, it can be easily prove that $\lim_{k\to\infty}\sup_{z\in B}|\Re f_k(w)|=0$. Hence we have

$$\begin{aligned}\lim_{k\to\infty}\|C^g_\varphi f_k\|_{\mathcal{B}^\beta} &= \lim_{k\to\infty}\sup_{z\in B}(1-|z|^2)^\beta|\Re f_k(\varphi(z))g(z)|\\ &\le \sup_{z\in B}(1-|z|^2)^\beta|g(z)|\lim_{k\to\infty}\sup_{z\in B}|\Re f_k(\varphi(z))|=0.\end{aligned}$$

Then the result follows from Lemma 3. The proof is completed. □

Theorem 10. *Suppose* $p>0$ *and* $n+1+\alpha+p<0$. *Let* φ *be a holomorphic self-map of* B, $g\in H(B)$ *with* $g(0)=0$ *and* $0<\beta<\infty$. *Then the following statements are equivalent.*

(i) $C_\varphi^g : A_\alpha^p \to \mathcal{B}_0^\beta$ *is bounded*;

(ii) $C_\varphi^g : A_\alpha^p \to \mathcal{B}_0^\beta$ *is compact*;

(iii)

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g(z)| = 0. \tag{40}$$

Proof. (ii) $\Rightarrow$ (i). This implication is obvious.

(i) $\Rightarrow$ (iii). Suppose that $C_\varphi^g : A_\alpha^p \to \mathcal{B}_0^\beta$ is bounded. Taking $f(z) = \langle z, a\rangle/|a|^2$, $|a| > 1/2$, and employing the boundedness of $C_\varphi^g : A_\alpha^p \to \mathcal{B}_0^\beta$ we see that (40) holds.

(iii) $\Rightarrow$ (ii). Suppose that (40) holds. For any $f \in A_\alpha^p$ with $\|f\|_{A_\alpha^p} \le 1$, by (39) we have

$$(1-|z|^2)^\beta |\Re(C_\varphi^g f)(z)| \le C\|f\|_{A_\alpha^p}(1-|z|^2)^\beta |g(z)|, \tag{41}$$

from which we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{A_\alpha^p}\le 1}(1-|z|^2)^\beta |\Re(C_\varphi^g f)(z)| \le C \lim_{|z|\to 1} \sup_{\|f\|_{A_\alpha^p}\le 1}(1-|z|^2)^\beta |g(z)| = 0. \tag{42}$$

By Lemma 2 we see that $C_\varphi^g : A_\alpha^p \to \mathcal{B}_0^\beta$ is compact. This completes the proof of the theorem. $\square$

From Theorems 9 and 10 we obtain the following two corollaries.

Corollary 5. *Suppose $p > 0$ and $n + 1 + \alpha + p < 0$. Let $g \in H(B)$ and $0 < \beta < \infty$. Then the following statements hold.*

(i) *$L_g : A_\alpha^p \to \mathcal{B}^\beta$ is bounded if and only if $L_g : A_\alpha^p \to \mathcal{B}^\beta$ is compact if and only if*

$$\sup_{z\in B}(1-|z|^2)^\beta |g(z)| < \infty;$$

(ii) *$L_g : A_\alpha^p \to \mathcal{B}_0^\beta$ is bounded if and only if $L_g : A_\alpha^p \to \mathcal{B}_0^\beta$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g(z)| = 0.$$

Corollary 6. *Suppose $p > 0$ and $2 + \alpha + p < 0$. Let φ be a holomorphic self-map of D and $0 < \beta < \infty$. Then the following statements hold.*

(i) *$C_\varphi : A_\alpha^p \to \mathcal{B}^\beta$ is bounded if and only if $C_\varphi : A_\alpha^p \to \mathcal{B}^\beta$ is compact if and only if $\varphi \in \mathcal{B}^\beta$*;

(ii) *$C_\varphi : A_\alpha^p \to \mathcal{B}_0^\beta$ is bounded if and only if $C_\varphi : A_\alpha^p \to \mathcal{B}_0^\beta$ is compact if and only if $\varphi \in \mathcal{B}_0^\beta$.*

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