J. Korean Math. Soc. **46** (2009), No. 6, pp. 1207–1218
DOI 10.4134/JKMS.2009.46.6.1207

ON ENERGY ESTIMATES FOR A LANDAU-LIFSCHITZ TYPE FUNCTIONAL IN HIGHER DIMENSIONS

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Abstract. The authors study the asymptotic behavior of radial minimizers of an energy functional associated with ferromagnets and antiferromagnets in higher dimensions. The location of the zeros of the radial minimizer is discussed. Moreover, several uniform estimates for the radial minimizer are presented. Based on these estimates, the authors establish global convergence of radial minimizers.

1. Introduction

Let $B_r = \{x \in \mathbb{R}^n; |x| < r\}(n \geq 2)$. Denote

$$S^n = \{x \in \mathbb{R}^{n+1}; (x^1)^2 + (x^2)^2 + \cdots + (x^{n+1})^2 = 1\}.$$

Consider the minimizers u_ε of the energy functional

$$E_\varepsilon(u, B_1) = \frac{1}{n}\int_{B_1} |\nabla u|^n dx + \frac{1}{2\varepsilon^n}\int_{B_1} (u^{n+1})^2 dx,\ (\varepsilon > 0),$$

on the function class $W = \{u(x) = (\sin f(r)\frac{x}{|x|}, \cos f(r)) \in W^{1,n}(B_1, S^n); f(1) = \frac{\pi}{2}, r = |x|\}$. Sometimes we write

$$u_\varepsilon = (u_\varepsilon^1, u_\varepsilon^2, \ldots, u_\varepsilon^n, u_\varepsilon^{n+1}) = (u'_\varepsilon, u_\varepsilon^{n+1}).$$

In the case of $n = 2$, the functional $E_\varepsilon(u, B)$ was the Landau-Lifschitz type introduced in the study of some simplified model of high-energy physics, which controls the statics of planar ferromagnets and antiferromagnets (see [3, 7]). The asymptotic behavior of minimizers of $E_\varepsilon(u, B_1)$ has been considered in [3]. In particular, they discussed the asymptotic behavior of the radial minimizer of $E_\varepsilon(u, B)$ in §5. When the penalization term $\frac{1}{2\varepsilon^2}\int_{B_1}(u^3)^2 dx$ is replaced by $\frac{1}{4\varepsilon^2}\int_{B_1}(1-|u|^2)^2 dx$ and S^2 is replaced by $\mathbb{R}^2$, the functional becomes the well-known Ginzburg-Landau energy introduced in the theory of superconductors

Received February 24, 2008; Revised November 24, 2008.
2000 *Mathematics Subject Classification.* 35B25, 35J70.
Key words and phrases. radial minimizer, energy functional of Landau-Lifschitz type, planar ferromagnets and antiferromagnets, higher dimensions, location of zeros.
The research was supported partly by NSF (No.10871097) of China and Natural Science Foundation of Jiangsu Higher Education Institutions (No.08KJB110005).

(cf. [1] and the references therein). 19 open problems were proposed in [1]. P. Mironescu studied problem 7 in [2, 6]. Afterwards, the results were extended to the higher dimensions (cf. [5, Theorem 1.2]). In this paper, we will discuss this problem for the radial minimizer of the energy functional $E_\varepsilon(u, B_1)$ when the dimension $n \geq 2$.

Similar to the argument of Remark 3 in [5, §3], we also have $f \in C[0,1]$ and $f(0) = 0$ as long as $u \in W$. Observing the functional $E_\varepsilon(u, B_1)$, we can assume $f \in [0, \frac{\pi}{2}]$ for simplicity. First we will verify in §2 that the zeros of u'_ε are located near the origin. Next, as in [5, §3], we will also consider whether

$$\begin{aligned} A_\varepsilon &= \int_{B_1} (1 - |u'_\varepsilon|)^\alpha |\nabla u'_\varepsilon|^n dx, \\ B_\varepsilon &= \int_{B_1} (1 - |u'_\varepsilon|)^\alpha |u'_\varepsilon|^\alpha |\nabla \frac{u'_\varepsilon}{|u'_\varepsilon|}|^n dx, \\ C_\varepsilon &= \int_{B_1} |\det(\nabla u'_\varepsilon)| dx \end{aligned}$$

have uniform upper estimates for any $\alpha > 0$. We will prove in §3 the following:

Theorem 1.1. *Assume u_ε is a radial minimizer of $E_\varepsilon(u, B_1)$. Then for any $\alpha > 0$, there exists a constant $C > 0$ which is independent of ε, such that $A_\varepsilon \leq C$ when $\varepsilon \to 0$.*

Theorem 1.2. *Assume u_ε is a radial minimizer of $E_\varepsilon(u, B_1)$. Then for any $\alpha \geq 1$, there exists a constant $C > 0$ which is independent of ε, such that $B_\varepsilon \leq C$ when $\varepsilon \to 0$.*

Theorem 1.3. *Assume u_ε is a radial minimizer of $E_\varepsilon(u, B_1)$. Then there exists a constant $C > 0$ which is independent of ε, such that $C_\varepsilon \leq C$ when $\varepsilon \to 0$.*

According to Theorem 1.2 in [4], it is not difficult to obtain the local convergence:

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = \Big(\frac{x}{|x|}, 0\Big) \quad \text{in} \quad W^{1,n}_{loc}(B_1 \setminus \{0\}).$$

We will set up the global convergence based on the uniform estimates in Theorems 1.1, 1.2 and 1.3.

Theorem 1.4. *Assume u_ε is a radial minimizer of $E_\varepsilon(u, B_1)$. Then we can find positive constants L_1, L_2, L_3 which are independent of ε, such that as $\varepsilon \to 0$,*

$$(1 - |u'_\varepsilon|)^\alpha |\nabla u'_\varepsilon|^n \to L_1 \delta_o, \quad \textit{weakly star in} \quad C(\overline{B_1}), \quad \forall \alpha > 0, \tag{1.1}$$

$$(1 - |u'_\varepsilon|)^\alpha |u'_\varepsilon|^\alpha \Big|\nabla \frac{u'_\varepsilon}{|u'_\varepsilon|}\Big|^n \to L_2 \delta_o, \quad \textit{weakly star in} \quad C(\overline{B_1}), \quad \forall \alpha \geq 1, \tag{1.2}$$

$$|\det(\nabla u_\varepsilon)| \to L_3 \delta_o, \quad \textit{weakly star in} \quad C(\overline{B_1}), \tag{1.3}$$

where δ_o is the Dirac mass at the origin 0.

2. Location of zeros

From the direct method in the calculus of variations, it is easy to get:

Lemma 2.1. *The radial minimizer* $u_\varepsilon \in W$ *satisfies*

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u) = u|\nabla u|^n + \frac{1}{\varepsilon^n}[u(u^{n+1})^2 - u^{n+1}e_{n+1}] \text{ in } B_1, \tag{2.1}$$

in the weak sense, where $e_{n+1} = (0, 0, \ldots, 0, 1)$.

Lemma 2.2. *Assume* u_ε *is a radial minimizer of* $E_\varepsilon(u, B_1)$. *Then there exists a constant* $C > 0$ *which is independent of* ε, *such that*

$$E_\varepsilon(u, B_1) \leq \frac{1}{n}(n-1)^{\frac{n}{2}}|S^{n-1}||\ln\varepsilon| + C. \tag{2.2}$$

Proof. Set

$$I(\varepsilon, R) = \min\left\{\int_{B_R}\left[\frac{1}{n}|\nabla u|^n + \frac{1}{2\varepsilon^n}(u^{n+1})^2\right]dx; u_\varepsilon \in W^{1,n}(B_R, S^n)\right\}.$$

Then,

$$\begin{aligned} I(\varepsilon, 1) &= E_\varepsilon(u_\varepsilon, B_1) = \frac{1}{n}\int_{B_1}|\nabla u_\varepsilon|^n dx + \frac{1}{2\varepsilon^n}\int_{B_1}(u_\varepsilon^{n+1})^2 dx \\ &= \frac{1}{n}\int_{B_{\varepsilon^{-1}}}|\nabla u_\varepsilon|^n dy + \frac{1}{2}\int_{B_{\varepsilon^{-1}}}(u_\varepsilon^{n+1})^2 dy = I(1, \varepsilon^{-1}). \end{aligned}$$

Assume u_1 is a solution to $I(1, 1)$. Define

$$u_2 = u_1,\ x \in B_1;\ u_2 = \frac{x}{|x|},\ x \in B_{\varepsilon^{-1}} \setminus B_1.$$

Since u_ε is a minimizer, we have $E_\varepsilon(u_\varepsilon, B_1) \leq E_\varepsilon(u_2, B_1)$. Then

$$\begin{aligned} I(1, \varepsilon^{-1}) &\leq \frac{1}{n}\int_{B_{\varepsilon^{-1}}}|\nabla u_2|^n dx + \frac{1}{2}\int_{B_{\varepsilon^{-1}}}(u_2^{n+1})^2 dx \\ &= \frac{1}{n}\int_{B_1}|\nabla u_1|^n dx + \frac{1}{n}\int_{B_{\varepsilon^{-1}}\setminus B_1}\left|\nabla\frac{x}{|x|}\right|^n dx + \frac{1}{2}\int_{B_1}(u_1^{n+1})^2 dx \\ &= I(1,1) + \frac{1}{n}(n-1)^{\frac{n}{2}}|S^{n-1}|\int_1^{\varepsilon^{-1}}\frac{1}{r}dr \\ &= I(1,1) + \frac{1}{n}(n-1)^{\frac{n}{2}}|S^{n-1}||\ln\varepsilon| \\ &\leq \frac{1}{n}(n-1)^{\frac{n}{2}}|S^{n-1}||\ln\varepsilon| + C. \end{aligned}$$

Substituting this into $I(\varepsilon, 1) = I(1, \varepsilon^{-1})$, we have (2.2). □

Lemma 2.3. *Assume* $\varepsilon = \varepsilon_k$ *is a subsequence which converges to 0, and* u_ε *is a radial minimizer of* $E_\varepsilon(u, B_1)$. *Then there exist a positive constant* C *independent of* $\varepsilon \in (0, 1)$, *and a natural number* k_0, *such that*

$$\frac{1}{\varepsilon_k^n}\int_{B_1}(u^{n+1})^2 dx \leq C, \tag{2.3}$$

when $k > k_0$.

Proof. We use the idea in [8]. Denote $V(\varepsilon) = \inf E_\varepsilon(u, B_1), u \in W$. For fixed $u \in W$, the map $\varepsilon \to E_\varepsilon(u, B_1)$ is not increasing, and

$$\frac{\partial}{\partial \varepsilon} E_\varepsilon(u, B_1) = -\frac{n}{2\varepsilon^{n+1}} \int_{B_1} (u^{n+1})^2 dx.$$

Since $V(\varepsilon+\delta) \le E_{\varepsilon+\delta}(u, B_1) \le E_\varepsilon(u, B_1) = V(\varepsilon)$ for the minimizer $u = u_\varepsilon$ of $E_\varepsilon(u, B_1)$,

$$\begin{aligned} & \frac{n}{2\varepsilon^{n+1}} \int_{B_1} (u^{n+1})^2 dx \\ = & \lim_{\delta \to 0} \frac{E_\varepsilon(u, B_1) - E_{\varepsilon+\delta}(u, B_1)}{\delta} \\ \le & \overline{\lim}_{\delta \to 0} \frac{V(\varepsilon) - V(\varepsilon+\delta)}{\delta} \\ = & -V'(\varepsilon). \end{aligned}$$

We claim that there exists a subsequence of ε_k, which is still written as ε_k, such that

$$-\varepsilon_k V'(\varepsilon_k) \le M \ (\varepsilon_k \to 0),$$

where $M \ge \frac{1}{n}(n-1)^{\frac{n}{2}} |S^{n-1}|$. Otherwise, we suppose that there exists $\varepsilon_0 > 0$ such that $-V'(\varepsilon) > \frac{M}{\varepsilon}$. Integrating over $(\varepsilon, \varepsilon_0)$, we obtain

$$V(\varepsilon) \ge V(\varepsilon_0) - \int_\varepsilon^{\varepsilon_0} V'(\varepsilon) dx > M |\ln \varepsilon| - C,$$

which contradicts (2.2) as long as ε is small enough. □

Lemma 2.4. *Assume* u_ε *is a radial minimizer of* $E_\varepsilon(u, B_1)$. *Then there exist positive constants* λ, μ *which are independent of* $\varepsilon \in (0,1)$, *such that if*

$$\frac{1}{\varepsilon^n} \int_{B_1 \cap B_{2l\varepsilon}} (u_\varepsilon^{n+1})^2 dx \le \mu, \tag{2.4}$$

where $B_{2l\varepsilon}$ *is some ball of radius* $2l\varepsilon$ *with* $l > \lambda$, *then*

$$|u_\varepsilon'(x)| \ge \frac{1}{2}, \ \forall x \in B_1 \cap B_{l\varepsilon}.$$

Proof. We use the idea in [1]. First, we observe that there exists a constant $C_2 > 0$, such that for any $x \in B_1$, $|B_1 \cap B(x,r)| \ge C_2 r^n$. To prove the conclusion, we will use a consequence in [4, §2]. Choose $\lambda = \frac{1}{4C_1}$, $\mu = \frac{C_2}{16}\lambda^n$, where C_1 is the constant in (2.11) of [4]. Suppose that there is a point $x_0 \in B_1 \cap B_{l\varepsilon}$, such that $|u_\varepsilon'(x_0)| < \frac{1}{2}$. By (2.11) in [4], $\forall x \in B(x_0, \lambda\varepsilon)$,

$$|u_\varepsilon'(x) - u_\varepsilon'(x_0)| \le C_1 \varepsilon^{-1} |x - x_0| \le C_1 \varepsilon^{-1}(\lambda\varepsilon) = C_1 \lambda = \frac{1}{4}.$$

Hence $(u_\varepsilon^{n+1})^2 = 1-(u_\varepsilon^{'})^2 > \frac{1}{16} \forall x \in B(x_0,\lambda\varepsilon)$, and

$$\int_{B_1\cap B(x_0,\lambda\varepsilon)} (u_\varepsilon^{n+1})^2 dx > \frac{1}{16}|B_1 \cap B(x_0,\lambda\varepsilon)| > C_2 \times \frac{1}{16}\times(\lambda\varepsilon)^n = \mu\varepsilon^n. \tag{2.5}$$

Since $x_0 \in B_1 \cap B_{l\varepsilon}$ and $(B_1 \cap B(x_0,\lambda\varepsilon)) \subset (B_1 \cap B_{2l\varepsilon})$, (2.5) implies

$$\frac{1}{\varepsilon^n}\int_{B_1\cap B_{2l\varepsilon}} (u_\varepsilon^{n+1})^2 dx > \mu,$$

which contradicts (2.4) and thus Lemma 2.4 is proved. □

Let u_ε be a radial minimizer of $E_\varepsilon(u,B_1)$. α and μ are the constants in Lemma 2.4. If

$$\frac{1}{\varepsilon^n}\int_{B_1\cap B_{2\lambda\varepsilon}} (u_\varepsilon^{n+1})^2 dx \le \mu,$$

then $B(x^\varepsilon,\lambda\varepsilon)$ is called a good ball. Otherwise $B(x^\varepsilon,\lambda\varepsilon)$ is called a bad ball.

Now suppose that $B(x_i^\varepsilon,\lambda\varepsilon); i\in I$ is a family of balls satisfying

$$\begin{gathered}\text{(i) } x_i^\varepsilon \in B_1, i\in I; \quad \text{(ii) } B_1 \subset \cup_{i\in I} B(x_i^\varepsilon,\lambda\varepsilon);\\ \text{(iii) } B(x_i^\varepsilon,\frac{\lambda}{4}\varepsilon)\cap B(x_j^\varepsilon,\frac{\lambda}{4}\varepsilon)=\emptyset,\ i\ne j.\end{gathered} \tag{2.6}$$

Denote $J_\varepsilon = \{i \in I; B(x_i^\varepsilon,\lambda\varepsilon) \text{ is a bad ball}\}$.

Lemma 2.5. *There exists a positive integer N independent of $\varepsilon \in (0,1)$, such that the number of bad balls* $\mathrm{Card}J_\varepsilon \le N$.

Proof. In fact, (2.6) implies that every point in B_1 can be covered by finite, say m (independent of ε) balls. From (2.3) and the definition of bad balls, we have

$$\begin{aligned}\mu\varepsilon^n \mathrm{Card}J_\varepsilon &\le \sum_{i\in J_\varepsilon}\int_{B_1\cap B(x_i^\varepsilon,2\lambda\varepsilon)} (u_\varepsilon^{n+1})^2 dx\\ &\le m\int_{\cup_{i\in J_\varepsilon} B_1\cap B(x_i^\varepsilon,2\lambda\varepsilon)} (u_\varepsilon^{n+1})^2 dx\\ &\le m\int_{B_1} (u_\varepsilon^{n+1})^2 dx\\ &\le mC\varepsilon^n\end{aligned}$$

and hence $\mathrm{Card}J_\varepsilon \le \frac{mC}{\mu} \le N$. □

Similar to the argument of [1, Theorem IV.1], based on Lemma 2.5 we also have the following lemma:

Lemma 2.6. *There exist a subset $J \subset J_\varepsilon$ and a constant $h \ge \lambda$, such that*

$$\bigcup_{i\in J_\varepsilon} B(x_i^\varepsilon,\lambda\varepsilon) \subset \bigcup_{i\in J} B(x_j^\varepsilon,h\varepsilon)\ |x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon,\ i,j\in J,\ i\ne j.$$

Applying Lemma 2.6, we may modify the family of bad balls such that the new one, denoted by $B(x_i^\varepsilon, h\varepsilon); i \in J$, satisfies

$$\bigcup_{i\in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \subset \bigcup_{i\in J} B(x_j^\varepsilon, h\varepsilon),$$
$$\mathrm{Card}J \leq \mathrm{Card}J_\varepsilon,$$
$$|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon,\ i, j \in J,\ i \neq j.$$

The last condition implies that every two balls in the new family are not intersected.

Theorem 2.7. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B_1)$. Then there exists a $h > 0$ independent of $\varepsilon \in (0, 1)$, such that*

$$Z_\varepsilon = \{x \in B_1; |u_\varepsilon'(x)| < \frac{1}{2}\} \subset B_{h\varepsilon}.$$

Proof. Suppose there exists a point $x_0 \in Z_\varepsilon$ such that $x_0 \notin B_{h\varepsilon}$. Then all points on the circle $S_0 = \{x \in B_1; |x| = |x_0|\}$ satisfy $|u_\varepsilon'(x)| < \frac{1}{2}$, and hence by virtue of Lemma 2.4 and (2.6), all points on S_0 are contained in bad balls. On the other hand, since $|x_0| \geq h\varepsilon$, S_0 cannot be covered by a single bad ball, i.e., S_0 is covered by at least two bad balls (which are not intersected). However, this is impossible. Theorem 2.7 is proved. □

This theorem means that the zeros of u_ε are contained in $B_{h\varepsilon}$. When $\varepsilon \to 0$, the zeros converge to the origin 0.

3. Proof of theorems

Lemma 3.1. *Let $R \in (\frac{1}{3}, \frac{1}{2})$. Then there exists a constant $C > 0$ independent of ε, such that*

$$\int_{B_R\setminus B_{h\varepsilon}} \left|\nabla \frac{x}{|x|}\right|^n dx \geq (n-1)^{\frac{n}{2}} |S^{n-1}||\ln\varepsilon| - C, \tag{3.1}$$

when ε is sufficiently small.

Proof. Clearly,

$$\int_{B_R\setminus B_{h\varepsilon}} \left|\nabla \frac{x}{|x|}\right|^n dx = (n-1)^{\frac{n}{2}} |S^{n-1}| \int_{h\varepsilon}^R \frac{1}{r} dr \geq (n-1)^{\frac{n}{2}} |S^{n-1}||\ln\varepsilon| - C.$$ □

Lemma 3.2. *For any $\alpha > 0$, there exists a constant $C > 0$ independent of ε, such that*

$$\int_{B_R\setminus B_{h\varepsilon}} (1-\sin f)^\alpha \left|\nabla \frac{x}{|x|}\right|^n dx \leq C. \tag{3.2}$$

Proof. For $\forall\alpha > 0$, we choose $q = \frac{1}{\alpha}$. Applying (2.3), we obtain

$$\frac{1}{\varepsilon^n}\int_0^1 (1-\sin^2 f)^{q\alpha} r^{n-1} dr \leq \frac{1}{\varepsilon^n}\int_0^1 (\cos^2 f) r^{n-1} dr \leq C.$$

When $\alpha \in (0,1)$, we can deduce from (2.3) that

$$\begin{aligned}
&\int_{B_R\setminus B_{h\varepsilon}} (1-\sin f)^\alpha \Big|\nabla \frac{x}{|x|}\Big|^n dx \\
&\le C\int_{h\varepsilon}^R (1-\sin f)^\alpha \frac{r^{n-1}}{r^n} dr \\
&\le C\Big[\frac{1}{\varepsilon^n}\int_0^1 (1-\sin^2 f)^{q\alpha} r^{n-1} dr\Big]^{\frac{1}{q}} \varepsilon^{\frac{n}{q}} \Big[\int_{h\varepsilon}^R r^{\frac{q+n-1}{1-q}} dr\Big]^{1-\frac{1}{q}} \\
&\le C\varepsilon^{\frac{n}{q}} [\varepsilon^{\frac{n}{1-q}} - C(R)]^{1-\frac{1}{q}} \\
&\le C\varepsilon^{\frac{n}{q}} \varepsilon^{-\frac{n}{q}} = C.
\end{aligned}$$

When $\alpha \ge 1$, it is easy to deduce (3.2) by the argument above. □

Lemma 3.3. *There exists a constant $C > 0$ which is independent of ε, such that*

(3.3)

$$\int_{B_1} \cos^n f|\nabla f|^n dx + \int_{B_{h\varepsilon}} \sin^n f\Big|\nabla \frac{x}{|x|}\Big|^n dx + \int_{B_1\setminus B_R} \sin^n f\Big|\nabla \frac{x}{|x|}\Big|^n dx \le C.$$

Proof. From Lemma 2.2, we have

$$\int_{B_1} |\nabla u_\varepsilon'|^n dx \le (n-1)^{\frac{n}{2}} |S^{n-1}||\ln\varepsilon| + C.$$

Noting

$$\nabla u_\varepsilon' = \nabla(\sin f\frac{x}{|x|}) = \nabla(\sin f)\frac{x}{|x|} + \sin f\nabla\frac{x}{|x|} = \cos f\nabla f\frac{x}{|x|} + \sin f\nabla\frac{x}{|x|}.$$

We obtain from Jensen's inequality that

(3.4)

$$\begin{aligned}
&\int_{B_1} \cos^n f|\nabla f|^n dx + \int_{B_{h\varepsilon}} \sin^n f\Big|\nabla \frac{x}{|x|}\Big|^n dx + \int_{B_1\setminus B_{h\varepsilon}} \sin^n f\Big|\nabla \frac{x}{|x|}\Big|^n dx \\
&\le (n-1)^{\frac{n}{2}} |S^{n-1}||\ln\varepsilon| + C.
\end{aligned}$$

(3.4) subtracts (3.1). Then we can deduce that

$$\begin{aligned}
\int_{B_1} \cos^n f|\nabla f|^n dx &+ \int_{B_{h\varepsilon}} \sin^n f\Big|\nabla \frac{x}{|x|}\Big|^n dx + \int_{B_1\setminus B_R} \sin^n f\Big|\nabla \frac{x}{|x|}\Big|^n dx \\
&- \int_{B_R\setminus B_{h\varepsilon}} (1-\sin^n f)\Big|\nabla \frac{x}{|x|}\Big|^n dx \le C.
\end{aligned}$$

Using Lemma 3.2, we get

$$\int_{B_1} \cos^n f|\nabla f|^n dx + \int_{B_{h\varepsilon}} \sin^n f\Big|\nabla \frac{x}{|x|}\Big|^n dx + \int_{B_1\setminus B_R} \sin^n f\Big|\nabla \frac{x}{|x|}\Big|^n dx \le C.$$

□

Proof of Theorem 1.1. From Lemma 3.3, we deduce that

$$
\begin{aligned}
&\int_{(B_1\setminus B_R)\cup B_{h\varepsilon}} (1-|u'_\varepsilon|)^\alpha |\nabla u'_\varepsilon|^n dx \\
&= \int_{(B_1\setminus B_R)\cup B_{h\varepsilon}} (1-\sin f)^\alpha \Big|\nabla(\sin f \frac{x}{|x|})\Big|^n dx \\
&\le \int_{(B_1\setminus B_R)\cup B_{h\varepsilon}} \Big|\nabla(\sin f)\frac{x}{|x|} + \sin f \nabla \frac{x}{|x|}\Big|^n dx \\
&\le C\int_{(B_1\setminus B_R)\cup B_{h\varepsilon}} (\cos^n f|\nabla f|^n + \sin^n f\Big|\nabla\frac{x}{|x|}\Big|^n) dx \\
&\le C(\int_{B_1} \cos^n f|\nabla f|^n dx + \int_{B_{h\varepsilon}} \sin^n f\Big|\nabla\frac{x}{|x|}\Big|^n dx + \int_{B_1\setminus B_R} \sin^n f\Big|\nabla\frac{x}{|x|}\Big|^n dx) \\
&\le C.
\end{aligned} \tag{3.5}
$$

By Lemma 3.2 and Lemma 3.3, we obtain

$$
\begin{aligned}
&\int_{B_R\setminus B_{h\varepsilon}} (1-|u'_\varepsilon|)^\alpha |\nabla u'_\varepsilon|^n dx \\
&= \int_{B_R\setminus B_{h\varepsilon}} (1-\sin f)^\alpha \Big|\nabla(\sin f\frac{x}{|x|})\Big|^n dx \\
&\le C(\int_{B_R\setminus B_{h\varepsilon}} \cos^n f|\nabla f|^n dx + \int_{B_R\setminus B_{h\varepsilon}} (1-\sin f)^\alpha \Big|\nabla\frac{x}{|x|}\Big|^n dx) \le C.
\end{aligned} \tag{3.6}
$$

Combining (3.5) with (3.6) yields

$$
A_\varepsilon = \int_{B_1} (1-|u'_\varepsilon|)^\alpha |\nabla u'_\varepsilon|^n dx \le C.
$$

Theorem 1.1 is complete. □

Proof of Theorem 1.2. Obviously,

$$
\begin{aligned}
B_\varepsilon &= \int_{B_1} (1-|u'_\varepsilon|)^\alpha |u'_\varepsilon|^\alpha \Big|\nabla\frac{u'_\varepsilon}{|u'_\varepsilon|}\Big|^2 dx = \int_{B_1} (1-\sin f)^\alpha \sin^\alpha f\Big|\nabla\frac{x}{|x|}\Big|^n dx \\
&\le \int_{B_1} (1-\sin^2 f)^\alpha \sin^\alpha f\Big|\nabla\frac{x}{|x|}\Big|^n dx = \int_{B_1} \cos^{2\alpha} f\sin^\alpha f\Big|\nabla\frac{x}{|x|}\Big|^n dx \\
&= (n-1)^{\frac{n}{2}}|S^{n-1}|\int_0^1 \cos^{2\alpha} f\sin^\alpha f\frac{1}{r}dr \\
&= (n-1)^{\frac{n}{2}}|S^{n-1}|\int_0^\delta \cos^{2\alpha} f\sin^\alpha f\frac{1}{r}dr + (n-1)^{\frac{n}{2}}|S^{n-1}|\int_\delta^1 \cos^{2\alpha} f\sin^\alpha f\frac{1}{r}dr \\
&\le (n-1)^{\frac{n}{2}}|S^{n-1}|\int_0^\delta (\cos^{2\alpha} f) f^\alpha\frac{1}{r}dr + C(\delta).
\end{aligned}
$$

When $\alpha \in (1, n]$, by the mean value theorem and $f(0) = 0$, there exists $\xi \in (0, 1)$ such that

$$\frac{1}{r}\sin^\alpha f \leq |\cos f(\xi r)|^\alpha |f'(\xi r)|^\alpha r^{\alpha-1} \leq C|\cos f(\xi r)|^\alpha |\nabla f(\xi r)|^\alpha r^{\alpha-1}.$$

Thus, we have

$$\begin{aligned} B_\varepsilon &\leq (n-1)^{\frac{n}{2}}|S^{n-1}| \int_0^\delta \cos^{2\alpha} f|\nabla f|^\alpha r^{\alpha-1} dr + C \\ &\leq C\Big[\int_0^\delta \cos^n f(\xi r)|\nabla f(\xi r)|^n r^{n-1} dr\Big]^{\frac{\alpha}{n}} \Big[\int_0^\delta r^{\frac{\alpha-n}{n-1}} dr\Big]^{\frac{n-1}{n}} + C \\ &\leq C\xi^{\frac{\alpha(1-n)}{n}} \Big[\int_0^{\xi\delta} \cos^n f|\nabla f|^n s^{n-1} ds\Big]^{\frac{\alpha}{n}} \Big(\frac{n-1}{\alpha-1} r^{\frac{\alpha-1}{n-1}}|_0^\delta\Big)^{\frac{n-1}{n}} + C \\ &\leq C\int_{B_1} \cos^n f|\nabla f|^n dx + C. \end{aligned}$$

From Lemma 3.3, we obtain $B_\varepsilon \leq C$.

When $\alpha = 1$, the conclusion is easy to be obtained by the argument above. When $\alpha > n$, we can set $\alpha = n + \beta$. Thus by the mean value theorem and $f(0) = 0$, there exists $\xi \in (0, 1)$ such that

$$\frac{1}{r}\sin^n f \leq |\cos f(\xi r)|^n |f'(\xi r)|^n r^{n-1}.$$

Hence, we can deduce that, by using Lemma 3.3,

$$\begin{aligned} &\int_0^\delta (\cos^{2\alpha} f) f^\alpha \frac{1}{r} dr \\ \leq &\int_0^\delta (\cos^{2\alpha} f) f^\beta |\cos f(\xi r)|^n |f'(\xi r)|^n r^{n-1} dr \\ \leq &\int_0^\delta |\cos f(\xi r)|^n |f'(\xi r)|^n r^{n-1} dr \leq C. \end{aligned}$$

The rest proof is easy to be completed. □

The proof of Theorem 1.3.

$$\det(\nabla u'_\varepsilon) = \begin{vmatrix} (\sin f\frac{x_1}{|x|})_{x_1} & (\sin f\frac{x_2}{|x|})_{x_1} & \cdots & (\sin f\frac{x_n}{|x|})_{x_1} \\ (\sin f\frac{x_1}{|x|})_{x_2} & (\sin f\frac{x_2}{|x|})_{x_2} & \cdots & (\sin f\frac{x_n}{|x|})_{x_2} \\ \cdots & \cdots & \cdots & \cdots \\ (\sin f\frac{x_1}{|x|})_{x_n} & (\sin f\frac{x_2}{|x|})_{x_n} & \cdots & (\sin f\frac{x_n}{|x|})_{x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \cos f f_{x_1}\frac{x_1}{|x|} + \sin f\frac{|x|^2 - x_1^2}{|x|^3} & \cos f f_{x_1}\frac{x_2}{|x|} - \sin f\frac{x_1 x_2}{|x|^3} & \cdots & \cos f f_{x_1}\frac{x_n}{|x|} - \sin f\frac{x_1 x_n}{|x|^3} \\ \cos f f_{x_2}\frac{x_1}{|x|} - \sin f\frac{x_1 x_2}{|x|^3} & \cos f f_{x_2}\frac{x_2}{|x|} + \sin f\frac{|x|^2 - x_2^2}{|x|^3} & \cdots & \cos f f_{x_2}\frac{x_n}{|x|} - \sin f\frac{x_2 x_n}{|x|^3} \\ \cdots & \cdots & \cdots & \cdots \\ \cos f f_{x_n}\frac{x_1}{|x|} - \sin f\frac{x_1 x_n}{|x|^3} & \cos f f_{x_n}\frac{x_2}{|x|} - \sin f\frac{x_2 x_n}{|x|^3} & \cdots & \cos f f_{x_n}\frac{x_n}{|x|} + \sin f\frac{|x|^2 - x_n^2}{|x|^3} \end{vmatrix}.$$

It is easy to derive that

(3.7)

$$\begin{aligned}
&|\det(\nabla u_\varepsilon')| \\
\le\ & \cos^n f|\nabla f|^n + \cos^{n-1} f|\nabla f|^{n-1}\sin f\frac{1}{|x|} + \cos^{n-2} f|\nabla f|^{n-2}\sin^2 f\frac{1}{|x|^2} \\
&+\cdots+\cos f|\nabla f|\sin^{n-1} f\frac{1}{|x|^{n-1}} + \sin^n f\frac{1}{|x|^n} \\
\le\ & \cos^n f|\nabla f|^n + \cos^{n-1} f|\nabla f|^{n-1}\frac{f}{r} + \cos^{n-2} f|\nabla f|^{n-2}\frac{f^2}{r^2} \\
&+\cdots+\cos f|\nabla f|\frac{f^{n-1}}{r^{n-1}} + \frac{f^n}{r^n}.
\end{aligned}$$

Similar to the derivation of Theorem 1.2, using the mean value theorem, we derive

$$\begin{aligned}
|\det(\nabla u_\varepsilon')| &\le \cos^n f|\nabla f|^n + \cos^{n-1} f|\nabla f|^n + \cos^{n-2} f|\nabla f|^n + \cdots + \\
&\quad + \cos f|\nabla f|^n + |\nabla f|^n.
\end{aligned}$$

Noting $\cos f(0) = \cos 0 = 1$ and $\cos f$ is continuous near the zero, we know that there exists $\delta > 0$, such that when $r \in (0,\delta)$,

$$\begin{aligned}
|\cos f(r) - \cos f(0)| \le \tfrac{1}{2} \quad &\Rightarrow \cos f(r) \ge \cos f(0) - \tfrac{1}{2} = \tfrac{1}{2} \\
&\Rightarrow \cos^i f(r) \le C\cos^n f(r), \quad i = 1,2,\ldots,n-1.
\end{aligned}$$

From this result, (3.7) and Hölder's inequality, we can deduce that

$$\begin{aligned}
C_\varepsilon &= \int_{B_1} |\det(\nabla u_\varepsilon')|dx \\
&\le C(n,\delta)\int_{B_1\setminus B_\delta} \cos^n f|\nabla f|^n dx + C\int_{B_\delta} \cos^n f|\nabla f|^n dx \\
&\le C + C\int_{B_1} \cos^n f|\nabla f|^n dx \\
&\le C.
\end{aligned}$$

Theorem 1.3 is complete. □

Proof of Theorem 1.4. Theorem 1.1 means that the $L^1(B_1)$-norm of

$$(1-|u_\varepsilon'|)^\alpha|\nabla u_\varepsilon'|^n$$

is bounded. Therefore, there exists a Radon measure μ_1 such that

$$\lim_{\varepsilon\to 0}(1-|u_\varepsilon'|)^\alpha|\nabla u_\varepsilon'|^n = \mu_1 \quad \text{weakly star in} \quad C(\overline{B_1}).$$

Similar to the derivation (3.3) in [5], from (3.3) and (2.3) it also follows

$$\lim_{\varepsilon\to 0}\int_{B_1\setminus B_R}(1-|u_\varepsilon'|)^\alpha|\nabla u_\varepsilon'|^n dx = 0$$

for any $R > 0$, and hence $\mathrm{supp}(\mu_1) = \{0\}$. Thus, we can find $L_1 \in [0, \infty)$ such that

$$\mu_1 = L_1 \delta_o.$$

We claim $L_1 > 0$. In fact, by virtue of $f(r) \in C[0,1]$ and $f(0) = 0$, $f(h\varepsilon) \geq 1/2$ which can be seen by Theorem 2.7, there must exists $r_\varepsilon \in (0, h\varepsilon)$ such that $f(r_\varepsilon) = 1/4$. Using (2.11) in [4], we can find a sufficiently small positive constant δ which is independent of ε, such that

$$\frac{1}{8} \leq f(x) \leq \frac{3}{8}, \quad r \in (r_\varepsilon - \delta\varepsilon, r_\varepsilon + \delta\varepsilon).$$

Therefore,

$$\int_{B(0,r_\varepsilon+\delta\varepsilon)\setminus B(0,r_\varepsilon+\delta\varepsilon)} (1 - \sin f)^\alpha |\nabla u_\varepsilon|^n dx$$

$$\geq |S^{n-1}|(\sin \frac{1}{8})^2 (1 - \sin \frac{3}{8})^\alpha \int_{r_\varepsilon - \delta\varepsilon}^{r_\varepsilon+\delta\varepsilon} \frac{dr}{r} > 0.$$

This implies $L_1 > 0$. Eq.(1.1) is proved.

By the same argument of above, (1.2) can also be verified.

Next, Theorem 1.3 means that the $L^1(B_1)$-norm of $|\det(\nabla u'_\varepsilon)|$ is bounded. Therefore, there exists a Radon measure μ_3 such that

$$\lim_{\varepsilon\to 0} |\det(\nabla u'_\varepsilon)| = \mu_3 \quad \text{weakly star in} \quad C(\overline{B_1}).$$

By an analogous argument of Remarks 2 and 3 in [5, pp. 131–133], we can also find $L_3 \in (0, \infty)$ such that

$$\mu_3 = L_3 \delta_o.$$

Then (1.3) is proved and Theorem 1.4 is complete. □

Acknowledgements. The authors would like to thank the referee(s) for the helpful comments which improve the presentation of this paper.

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