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FINITE GROUPS WHICH HAVE MANY NORMAL SUBGROUPS

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Abstract. In this paper we classify finite groups whose nonnormal subgroups are of order p or pq, where p, q are primes. As a by-product, we also classify the finite groups in which all nonnormal subgroups are cyclic.

1. Introduction

As is well known, normal subgroups of a group play an important role in determining the structure of a group. Two classical classes of groups are Dedekind groups and finite simple groups. The classification of the above two kinds of finite groups have been completed, see [2] and [3]. It is easy to see that the number of nontrivial normal subgroups of a finite group has great influence on its structure. This motivates us to study finite groups which have "many" normal subgroups or "few" normal subgroups. Along one of the two lines, Zhang and Cao [9] determined finite groups which have an unique nontrivial normal subgroups. Along another of the two lines, it is natural to ask: what can be said about finite groups which have "many" normal subgroups? In this paper, a finite group G which has "many" normal subgroups means that a finite group whose nonnormal subgroups are of order p or pq, where p, q are primes (not necessarily distinct). Passman gave a classification of finite p-groups all of whose nonnormal subgroups are of order p (see [6, Proposition 2.4]) and are cyclic (see [6, Proposition 2.9]). This paper can be regards as a continuation of Passman's work.

The notation and terminology we use are standard; see [4] for instance. But we use C_n, D_{2^n}, Q_{2^n} and C_n^m to denote a cyclic group of order n, a dihedral group of order 2^n, a generalized quaternion group of order 2^n and the direct product of m cyclic groups of order n, respectively. If A and B are subgroups

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of G with $G = AB$ and $[A, B] = 1$, we call G a *central product* of A and B, denoted by $G = A * B$. Clearly, $A \cap B \le Z(G)$. In this paper, we always assume $A \cap B \ne 1$.

2. Preliminaries

A group G is said to be a *minimal non-nilpotent group* if G is not nilpotent but all proper sections of G are nilpotent; G is said to be *inner abelian* if G is nonabelian but all of its proper subgroups are abelian.

Theorem 2.1 (Rédei). *Assume that G is an inner abelian p-group. Then G is one of the following groups*:

(1) $M_{(m,n)} = \langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}}\rangle$, $m \ge 2, n \ge 1$, (*metacyclic*).

(2) $M_{(m,n,1)} = \langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$, *if* $p = 2$, $m + n \ge 3$ (*non-metacyclic*).

(3) Q_8.

Lemma 2.2 ([6, Lemma 2.1]). *Suppose G is not a Dedekindian p-group. Then there exists a $K \trianglelefteq G$ with $|G' : K| = p$ such that G/K is not Dedekindian.*

Theorem 2.3 ([6, Proposition 2.4]). *Suppose that all nonnormal subgroups of a non Dedekindian p-group G have order p. Then one of the following holds*:

(1) $G \cong M_{(m,1)}$;

(2) $G \cong D_8 * C_{2^n}$, $n \ge 2$;

(3) $G \cong M_{(1,1,1)} * C_{p^n}$;

(4) $G \cong D_8 * Q_8$.

We can get the following result from the proof of [1, Lemma 5.2].

Lemma 2.4. *Let E be an inner abelian subgroup of a p-group G. If $[G, E] = E'$, then $G = E * C_G(E)$.*

Remark 2.5. [1, Lemma 5.3] is a stronger result than Lemma 2.4. The authors do not assume that E is inner abelian and get the same result as in Lemma 2.4. However, there is a minor error in the proof of [1, Lemma 5.3] and we cannot fix it. In fact, we find a counterexample for that lemma. Let $G = Q_8 * Q_8$ and $E = Q_8 * C_4$. Then $C_G(E) \le E$ and $E * C_G(E) \le E$. Thus, $G \ne E * C_G(E)$.

Proposition 2.6. *Let G be a finite p-group with $|G'| = p$. If $H \le G$ and $H \nleq Z(G)$. Then $H \trianglelefteq G$ if and only if $G' \le H$.*

Proof. It is straightforward. □

Proposition 2.7. *Let $G = M * C$, where $M \cong M_{(m,1)}$ and $C \cong C_{p^n}$. Assume that $M \cap C \cong C_{p^i}$, $1 \le i \le n - 1$.*

(1) *If $m > n$, then $G \cong M_{(m,1)} \times C_{p^{n-i}}$.*

(2) *If $m \le n$, then $G \cong M_{(m-i,1,1)} * C_{p^n}$ with $M_{(m-i,1,1)} \cap C_{p^n} = M'_{(m-i,1,1)}$.*

Proof. Assume that $M = \langle a, b \mid a^{p^m} = b^p = 1, a^b = a^{1+p^{m-1}} \rangle$. Then $Z(M) = \langle a^p \rangle$. Since $M \cap C \cong C_{p^i}$, letting $C = \langle c \rangle$, we may assume $c^{p^{n-i}} = a^{p^{m-i}}$. If $m > n$, letting $c_1 = ca^{-p^{m-n}}$, then $o(c_1) = p^{n-i}$ and $G = M \times \langle c_1 \rangle$, (1) holds. If $m \leq n$, letting $a_1 = ac^{-p^{n-m}}$, then $o(a_1) = p^{m-i}$ and $M_1 = \langle a_1, b \rangle \cong M_{(m-i,1,1)}$. We have $G = M_1 * C$ and $M_1 \cap C = \langle a^{p^{m-1}} \rangle = M_1'$, the derived subgroup of $M_{(m-i,1,1)}$, (2) holds. □

Proposition 2.8. *$Q_8 * C_{2^m} \cong D_8 * C_{2^m} \leq Q_8 * M_{(m,n)}$, where $Q_8 \cap M_{(m,n)} = M'_{(m,n)}$, $m \geq 2$.*

Proof. This follows from the well-known fact that $Q_8 * C_4 \cong D_8 * C_4$. □

3. Finite non-p-groups having "many" normal subgroups

Assume G is a finite group, and $|G| = \Pi_{i=1}^s p_i^{k_i}$, where p_i is a prime, $i = 1, 2, \ldots, s$, and $p_i \neq p_j$ when $i \neq j$. For convenience we write $w(G) = \Sigma_{i=1}^s k_i$.

Theorem 3.1. *Assume that G is a nilpotent group but not a group of prime power order. If each subgroup H of G with $w(H) \geq 3$ is normal, then G is Dedekindian or $G = P \times H$, where $P \in \mathrm{Syl}_p(G)$ and P is one of the groups listed in the Theorem 2.3, $H \cong C_q$, where $q(\neq p)$ is a prime.*

Proof. If G is not Dedekindian, by hypothesis, there exists $P \in \mathrm{Syl}_p(G)$ such that P is not Dedekindian. Thus there exists $L < P$ and $L \ntrianglelefteq P$. Let $G = P \times H$. It follows that $L \times H \ntrianglelefteq G$. By hypothesis again we get $|H| = q$, where $q(\neq p)$ is a prime. It follows that all nonnormal subgroups of P have order p. The conclusion follows from Theorem 2.3. □

Lemma 3.2. *Assume G is a finite group. If each nonnormal subgroup H of G with $w(H) \geq 3$ is abelian, then G has a normal Sylow subgroup which has a nonnormal complement in G. Moreover, G is solvable.*

Proof. We prove by induction on $|G|$. First we prove that G has a normal Sylow subgroup. Assume that p is the smallest prime divisor of $|G|$, $P \in \mathrm{Syl}_p(G)$ and $P \ntrianglelefteq G$. Then $N_G(P) \ntrianglelefteq G$. If $|P| = p$, then it is easy to see $N_G(P) = C_G(P)$. If $|P| \geq p^2$, then P and $N_G(P)$ is abelian by hypothesis, and we also get $N_G(P) = C_G(P)$. Hence P has a normal complement H by the Burnside's p-nilpotency criterion. Since $|H| < |G|$, H has a normal Sylow subgroup which is also normal in G.

Let P be a normal Sylow subgroup of G and H be a complement of P in G. If H is nonnormal, the conclusion holds. Assume H is normal. Then H is non-nilpotent. Since $|H| < |G|$, H has a normal Sylow subgroup Q which has a nonnormal complement K in H. Then $Q \trianglelefteq G$ and the complement $P \times K$ of Q is not normal in G. The proof is completed. □

Lemma 3.3. *Assume G is a non-nilpotent group whose nonnormal subgroups are cyclic. Then $G \cong P \rtimes H$, where P is a Sylow subgroup and H is cyclic. Moreover, $d(P) \leq 2$, $[\Phi(P), H] = 1$ and $\Phi(P)$ is cyclic.*

Proof. By Lemma 3.2, we may take $P \in \mathrm{Syl}_p(G)$ and $H \leq G$ such that $P \lhd G$, $H \ntrianglelefteq G$ and $G \cong P \rtimes H$. By assumption, H is cyclic.

If $H\Phi(P) \trianglelefteq G$, then $[P,H] \leq P \cap H\Phi(P) = \Phi(P)$. By [5, 8.2.7 (a)], $P = C_P(H)[P,H] = C_P(H)$. Then G is nilpotent, a contradiction. So $H\Phi(P) \ntrianglelefteq G$ and hence $H\Phi(P)$ is cyclic. We have $[\Phi(P),H] = 1$ and $\Phi(P)$ is cyclic.

Assume $d(P) \geq 3$. Write $\overline{G} = G/\Phi(P)$, $\overline{P} = P/\Phi(P)$ and $\overline{H} = H\Phi(P)/\Phi(P)$. Let M be a maximal subgroup of $\overline{P}$. Since $d(\overline{P}) = d(P) \geq 3$, we have $d(M) \geq 2$ and M is not cyclic. Since the hypothesis of the lemma is inherited to factor groups, we have $M \trianglelefteq \overline{G}$. It follows that $M\overline{H}$ is a non-cyclic subgroup of $\overline{G}$ and hence $M\overline{H} \trianglelefteq \overline{G}$. Therefore $\bigcap_{M \lessdot \overline{P}} M\overline{H} = \overline{H} \trianglelefteq \overline{G}$ and hence $H\Phi(P) \trianglelefteq G$, a contradiction. So we have $d(P) \leq 2$. □

Theorem 3.4. *Assume G is a non-nilpotent group whose nonnormal subgroups are cyclic. Then G is one of the following groups*:

(i) $C_p \rtimes C_n$, *where p is a prime*;

(ii) $C_p^2 \rtimes C_n$, *where p is a prime, $(p,n) = 1$ and C_n acts irreducibly on C_p^2*;

(iii) $(Q_8 \rtimes C_{3^m}) \times C_n$, *where $(2,n) = (3,n) = 1$.*

Proof. It is easy to check that all groups listed in Theorem 3.4 satisfy the hypothesis. It suffices to show the converse.

By Lemma 3.3, we have $G = P \rtimes H$, where $d(P) \leq 2$, $[\Phi(P),H] = 1$ and $\Phi(P)$ is cyclic.

If P is abelian, then $P = C_P(H) \times [P,H]$ by [5, 8.4.2]. If $1 < N \leq [P,H]$ and $N \trianglelefteq G$, then NH is non-abelian, and hence $NH \trianglelefteq G$. It follows that $[P,H] \leq [P,H] \cap NH = N$. Thus $[P,H]$ is a minimal normal subgroup of G. Since G is solvable and non-cyclic subgroups of G are normal, $[P,H] \cong C_p$ or C_p^2. We get a group of type (i) or (ii).

Assume that P is non-abelian. Write $\overline{G} = G/\mho_1(P')$, $\overline{P} = P/\mho_1(P')$ and $\overline{H} = H\mho_1(P')/\mho_1(P')$. We have $|\overline{P}'| = p$. Noting that $d(\overline{P}) = 2$, $\overline{P}$ is inner abelian. Since $\Phi(P)$ is cyclic, $\overline{P} \cong M_{(m,1)}$, $M_{(1,1,1)}$ or Q_8 by Theorem 2.1. Now we claim that $[\overline{P},\overline{H}] \neq 1$. Assume the contrary, $[P,H] \leq \mho_1(P)$. By [5, 8.2.7(a)], $P = C_P(H)[P,H] = C_P(H)$. Thus G is nilpotent, a contradiction.

If $\overline{P} \cong D_8$, then $\overline{H}/C_{\overline{H}}(\overline{P}) \lesssim \mathrm{Aut}(D_8) \cong D_8$. Hence $\overline{H} = C_{\overline{H}}(\overline{P})$, contradicting the fact that $[\overline{P},\overline{H}] \neq 1$. If $\overline{P} \cong M_{(m,1)}$, where $m \geq 3$ when $p = 2$, then $\Omega_1(\overline{P}) \cong C_p^2$ and hence $\Omega_1(\overline{P}) \trianglelefteq \overline{G}$. It follows that $\Omega_1(\overline{P})\overline{H} \trianglelefteq \overline{G}$. Hence $[\overline{P},\overline{H}] \leq \overline{P} \cap \Omega_1(\overline{P})\overline{H} = \Omega_1(\overline{P})$. Since $[\Phi(P),H] = 1$, we have $|[\Omega_1(\overline{P}),\overline{H}]| \leq p$ by [5, 8.4.2]. By [5, 8.2.7(b)], $[\overline{P},\overline{H}] = [\overline{P},\overline{H},\overline{H}]$, and hence $|[\overline{P},\overline{H}]| \leq |[\Omega_1(\overline{P}),\overline{H}]| \leq p$. We may take $a,b \in \overline{P}$ such that $a \in C_{\overline{P}}(\overline{H})$, b generates $[\overline{P},\overline{H}]$ and $\overline{P} = \langle a,b \mid a^{p^m} = b^p = 1, a^b = a^{1+p^{m-1}} \rangle$. Let $\overline{H} = \langle c \rangle$ and $b^c = b^k$, where $0 \leq k < p$. Then $[a,b] = [a,b]^c = [a,b^k] = [a,b]^k$. Hence $k = 1$. This is contrary to $[\langle b \rangle, \overline{H}] = \langle b \rangle$. If $\overline{P} \cong M_{(1,1,1)}$, noting that all maximal subgroup of $M_{(1,1,1)}$ is not cyclic, we may get a contradiction by an argument similar to that in the proof of Lemma 3.3.

Thus we have that $\overline{P} \cong Q_8$ and hence $|P : P'| = 4$. It follows that P is a 2-group of maximal class by [4, III, Satz 11.9]. Note that Q_8 cannot be a proper quotient group of any 2-group of maximal class. We have $P = \overline{P} \cong Q_8$ and we get a group of type (iii). □

Corollary 3.5. *Assume G is a non-nilpotent group whose nonnormal subgroups are of prime order. Then G is a minimal non-nilpotent group of order pq or p^2q.*

Theorem 3.6. *Assume that G is a non-nilpotent group and p, q, r are primes* (*not necessarily distinct*)*. If each subgroup H of G with $w(H) \geq 3$ is normal, then G is one of the following groups*:

(i) *a non-nilpotent group G with $w(G) \leq 3$;*

(ii) *$H \times C_p$, where H is a minimal non-nilpotent group of order p^2q;*

(iii) *a minimal non-nilpotent group of order p^3q;*

(iv) *$P \rtimes C_q$, where P is a group of order p^3 and $G/\Omega_1(\Phi(P))$ is a minimal non-nilpotent group of order p^2q;*

(v) *$C_p^2 \rtimes H$, where $|H| = qr$ and H acts irreducibly on C_p^2;*

(vi) *$C_p^3 \rtimes H$, where $|H| = qr$ and each nonidentity element acts irreducibly on C_p^3.*

Proof. It is easy to check that all groups listed in Theorem 3.6 satisfy the hypothesis. It suffices to show the converse.

If $w(G) \leq 3$, we get the groups of type (i). From now on we always assume $w(G) \geq 4$. By Lemma 3.2, G has a normal Sylow subgroup P having a nonnormal complement H in G. By the hypothesis, $w(H) \leq 2$ and $w(P) \geq 2$.

If $w(H) = 1$, then $|P| \geq p^3$. We claim that $|P| = p^3$. Since $w(G) \geq 4$, G has a nonnormal subgroup K with $w(K) = 2$ by Corollary 3.5. If there is a normal subgroup N of G satisfying $N \leq P$ and $|N| = p$, then G/N satisfy the hypothesis of Corollary 3.5, and hence $|P| = p^3$. So we may assume that there is no normal subgroup N of G with the above conditions. If $P' \neq 1$, then $|P'| \geq p^2$. By hypothesis G/P' is Dedekindian. Thus $[H, P] \leq P'$. By [5, 8.2.7 (a)], $P = C_P(H)[P, H] = C_P(H)$. Hence G is nilpotent, a contradiction. So P is abelian, and hence $C_P(H) \leq Z(G) \cap P = 1$. Thus $P = [P, H]$ by [5, 8.2.7 (a)]. Let $N \trianglelefteq G$ with $N \leq P$ and $|N| \geq p^2$. Then $NH \trianglelefteq G$. It follows that $P = [P, H] \leq NH \cap P = N$. Therefore P is a minimal normal subgroup of G and hence $|P| = p^3$ by the assumption of the theorem.

If $\Omega_1(\Phi(P)) \neq 1$, then $G/\Omega_1(\Phi(P))$ satisfy the hypothesis of Corollary 3.5. We get the groups of type (iv). Assume now that $\Omega_1(\Phi(P)) = 1$. Then P is elementary abelian, and $P = C_P(H) \times [P, H]$ by [5, 8.4.2]. It follows that $G = C_P(H) \times ([P, H] \rtimes H)$. We claim that $C_P(H)H \ntrianglelefteq G$ (If $C_P(H)H \trianglelefteq G$ then $[P, H] \leq P \cap C_P(H)H = C_P(H)$, implying that G is nilpotent). Hence $|C_P(H)| \leq p$. By an argument similar to that in the proof of Theorem 3.4, $[P, H]$ is a minimal normal subgroup and we get the groups of type (ii) and (iii).

Assume $w(H) = 2$. Since $C_P(H)H$ is nonnormal in G, $C_P(H) = 1$ and hence $P = [P, H]$ by [5, 8.2.7 (a)]. If there is a normal subgroup N of P satisfying $1 < N < P$, then $w(NH) \geq 3$ and $NH \trianglelefteq G$. Therefore $P = [P, H] \leq NH \cap P = N$, a contradiction. So P is a minimal normal subgroup of G and hence $P \cong C_p^2$ or C_p^3. If $P \cong C_p^2$, we get the groups of type (v). If $P \cong C_p^3$, let K be a minimal subgroup of H. We claim that K acts irreducibly on P. Otherwise, we may assume N is a K-invariant subgroup of order p^2 by [5, 8.4.5]. Hence $w(NK) = 3$ and $NK \trianglelefteq G$. Thus $N = NH \cap P \trianglelefteq G$, this contradicts the minimality of P. We get the groups of type (vi) in this case. □

4. Finite p-groups having "many" normal subgroups

In this section we deal with the p-group case. We always assume the following.

Assumption: *G is a finite p-group all of whose subgroups of order $\geq p^3$ are normal, and that G has at least one nonnormal subgroup of order p^2.*

Lemma 4.1. *$|G'| \leq p^2$.*

Proof. Assume $|G'| > p^2$. By Lemma 2.2, G has a normal subgroup K with $|G' : K| = p$ such that G/K is not Dedekindian. Hence G/K has a nonnormal subgroup H/K. It follows that $H \ntrianglelefteq G$. Since $|K| \geq p^2$, we get $|H| \geq p^3$, a contradiction. □

By Lemma 4.1, we divide our analysis into two cases: (1) $|G'| = p$, and (2) $|G'| = p^2$.

Lemma 4.2. *Assume $|G'| = p$, $H \leq G$, H is not a Dedekind group and $Z(H)$ is cyclic. Then all subgroups of $C_G(H)$ of order $\geq p^2$ are normal in G.*

Proof. By hypothesis, it is easy to see that $G' = H'$ is a unique subgroup of $Z(H)$ of order p. Assume that $N \leq C_G(H)$, $|N| = p^2$ and $N \ntrianglelefteq G$. Let $M \ntrianglelefteq H$. Then $M \nleq C_G(H)$, and hence $M \nleq N$. It follows that $|MN| \geq p^3$ and $MN \trianglelefteq G$. Since $MN \nleq Z(G)$, $G' \leq MN$ by Proposition 2.6. Hence $G' \leq MN \cap H = M(N \cap H)$. Since $N \ntrianglelefteq G$, $G' \nleq N \cap H \leq Z(H)$. It follows that $N \cap H = 1$. Hence $G' \leq M$ and $M \trianglelefteq G$, a contradiction. □

Theorem 4.3. *Assume $|G'| = p$. Then G is one of the following non-isomorphic groups*:

(i) $G \cong M_{(1,1,1)} * M_{(m,1)}$;
(ii) $G \cong M_{(1,1,1)} * M_{(1,1,1)} * C_{p^n}$;
(iii) $G \cong D_8 * M_{(m,1)}$, $m \geq 3$;
(iv) $G \cong D_8 * D_8 * C_{2^n}$;
(v) $G \cong D_8 * D_8 * Q_8$;
(vi) $G \cong (D_8 * Q_8) \times C_2$;
(vii) $G \cong (M_{(1,1,1)} * C_{p^n}) \times C_p$;
(viii) $G \cong (D_8 * C_{2^n}) \times C_2$;

(ix) $G \cong M_{(m,1)} \times C_p$, *where* $m \geq 3$ *when* $p = 2$;
(x) $G \cong M_{(2,1,1)} * C_{p^n}$, *where* $M_{(2,1,1)} \cap C_{p^n} = M'_{(2,1,1)}$;
(xi) $G \cong M_{(2,1,1)} * Q_8$, *where* $M_{(2,1,1)} \cap Q_8 = M'_{(2,1,1)}$;
(xii) $G \cong Q_8 \times C_4$;
(xiii) $G \cong M_{(m,2)}$.

Proof. ($\Rightarrow$) By Theorem 2.1, it is easy to check that inner abelian groups whose nonnormal subgroups have order at most p^2 are $M_{(m,2)}$, $M_{(m,1)}$, $M_{(2,1,1)}$, $M_{(1,1,1)}$ and Q_8. Since G is nonabelian, G has inner abelian subgroups. We distinguish the following five cases:

Case 1. G has a subgroup H which isomorphic to $M_{(1,1,1)}$ or D_8.

By Lemma 2.4 and Lemma 4.2, $G = H * C_G(H)$, where $C_G(H)$ is a Dedekind group or a group listed in Theorem 2.3.

If $C_G(H)$ is a group listed in Theorem 2.3, it is easy to see G is a group of type (i), (ii), (iii), (iv) or (v).

Assume $C_G(H)$ is a Dedekind group. If $C_G(H) \cong Q_8 \times C_2^k$, then $G \cong (D_8 * Q_8) \times C_2^k$ and it is easy to see that $k = 1$, we get the group of type (vi).

Assume $C_G(H)$ is abelian. If $C_G(H)$ is elementary abelian, then $G \cong H \times C_p^k = (H * C_p) \times C_p^k$. If $C_G(H)$ is not elementary abelian, taking $N \leq H$ and $N \ntrianglelefteq H$, then for any $x \in C_G(H)$ with $o(x) > p$, we have $|N\langle x\rangle| \geq p^3$, hence $N\langle x\rangle \trianglelefteq G$. By Proposition 2.6, $G' \leq N\langle x\rangle \cap C_G(H) = \langle x\rangle$. Thus $C_G(H) \cong C_{p^n} \times C_p^k$ and $G \cong (H * C_{p^n}) \times C_p^k$, where $n > 1$. In either case, we have $G \cong (H * C_{p^n}) \times C_p^k$, where $n \geq 1$. By hypothesis, it is easy to see $k = 1$. we get a group of type (vii) or (viii).

Case 2. G has no subgroup isomorphic to $M_{(1,1,1)}$ or D_8 but has a subgroup H isomorphic to $M_{(m,1)}$, where $m \geq 3$ when $p = 2$.

By Lemma 2.4 and Lemma 4.2, we still have $G = H * C_G(H)$, where $C_G(H)$ is a Dedekind group or a group listed in Theorem 2.3. If the latter happens, we may assume $G = M_1 * M_2$, $M_i = \langle a_i, b_i \mid a_i^{p^{m_i}} = b_i^p = 1, a_i^b = a_i^{1+p^{m_i-1}}\rangle$, $i = 1, 2$, $m_1 \leq m_2$, where $M_1 = H$ and $M_2 = C_G(H)$. Suppose $a_1^p = a_2^{sp}$. Letting $a_3 = a_1 a_2^{-s}$, we have $a_3^p = 1$, $\langle a_3, b_1\rangle \cong M_{(1,1,1)}$, a contradiction. Thus $C_G(H)$ is a Dedekind group. If $C_G(H) \cong Q_8 \times C_2^k$, then G has a subgroup isomorphic to $M_{(m,1)} * Q_8$. By Proposition 2.8, G has a subgroup isomorphic to D_8, a contradiction. Therefore $C_G(H)$ is abelian. By an argument similar to that of Case 1, we have $G \cong (M_{(m,1)} * C_{p^n}) \times C_p^k$, where $n \geq 1$ and $k \leq 1$.

If $C_{p^n} \leq M_{(m,1)}$, we get the group of type (ix).

If $C_{p^n} \nleq M_{(m,1)}$, then $M_{(m,1)} * C_{p^n}$ has a nonnormal subgroup of order $\geq p^2$ by Theorem 2.3. Hence $k = 0$ and $G \cong M_{(m,1)} * C_{p^n}$. By Proposition 2.7, we get a group of type (ix) when $m > n$ and get a group of type (x) when $m \leq n$.

Conversely, the groups of type (ix) and type (x) for $n \geq 2$ satisfy the hypothesis of Case 2.

Case 3. G has no subgroup isomorphic to $M_{(1,1,1)}$ or $M_{(m,1)}$ but has a subgroup H isomorphic to $M_{(2,1,1)}$.

By Lemma 2.4, $G = H * C_G(H)$. Assume $H = \langle a, b \,|\, a^{p^2} = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$ and $N \le C_G(H)$. If $|N| \ge p^2$, then $|\langle b\rangle N| \ge p^3$, and hence $\langle b\rangle N \trianglelefteq G$. By the Proposition 2.6, $G' \le \langle b\rangle N \cap C_G(H) = N$ and hence $N \trianglelefteq G$. If $|N| = p$, we claim that $N \le Z(G)$. Otherwise, $|\langle a\rangle N| \ge p^3$, and hence $\langle a\rangle N \trianglelefteq G$. By the Proposition 2.6, $G' \le \langle a\rangle N \cap C_G(H) = \langle a^p\rangle N$. Since $|N| = p$ and $G' \cap \langle a\rangle = 1$, we have $N \le \langle a^p\rangle G' \le Z(G)$, a contradiction. Hence $C_G(H)$ is a Dedekind group.

If $C_G(H)$ is abelian, then for any element x of order p in $C_G(H)$, $x \in Z(G)$ by the same argument as that in the above paragraph. Since $M_{(2,1,1)} \times C_p$ has a nonnormal subgroup of order p^3, we have $x \in H$. Hence for any element y in $C_G(H) \setminus H$, we have $o(y) > p$ and $G' \le \langle y\rangle$ by the argument in the above paragraph. Thus we have $G = H * C_{p^n}$. If $n \ge 2$, then G has a subgroup isomorphic to $M_{(m,1)}$ by Proposition 2.7. This contradicts our hypothesis. Thus we get a group of type (x), where $n = 1$.

If $C_G(H) \cong Q_8 \times C_2^k$, it is easy to see $C_2^k \le H$ and we get a group of type (xi).

Conversely, the groups of type (x) for $n = 1$ and type (xi) satisfy the hypothesis of Case 3.

Case 4. G has no subgroup isomorphic to $M_{(m,1)}$ or $M_{(2,1,1)}$ but has a subgroup H isomorphic to Q_8.

By Lemma 2.4, $G = H * C_G(H)$. Since G is not Dedekindian, we get $\exp(C_G(H)) \ge 4$. Thus for any element x of order ≥ 4 in $C_G(H) \setminus H$, we have $H \cap \langle x\rangle = 1$ by Proposition 2.8. Hence $G \cong Q_8 \times A$, where A is abelian. It is easy to see A must be C_4, we get a group of type (xii).

Conversely, the group of type (xii) satisfy the hypothesis of Case 4.

Case 5. All inner abelian subgroups in G are isomorphic to $M_{(m,2)}$.

By Lemma 2.4, $G = H * C_G(H)$. Assume $H = \langle a, b \mid a^{p^m} = b^{p^2} = 1, a^b = a^{1+p^{m-1}}\rangle$. If there is an element x not contained in H, then $|\langle b, x\rangle| \ge p^3$, and hence $\langle b, x\rangle \trianglelefteq G$. By Proposition 2.6, $G' \le \langle b, x\rangle \cap C_G(H) = \langle b^p, x\rangle$. It means that there are integers s, t such that $a^{p^{m-1}} = b^{sp}x^{tp}$. If $p \nmid s$, $\langle a, b^s x^t\rangle$ is an inner abelian group which has a cyclic maximal subgroup $\langle a\rangle$, this is contrary to our hypothesis. Hence $G' \le \langle x\rangle$. Assume $a^{p^{m-i}} = x^{kp^{n-i}}$, where $(k, p) = 1$ and $|\langle a\rangle \cap \langle x\rangle| = p^i$, $i \ge 1$. If $m \le n$, $\langle ax^{-kp^{n-m}}, b\rangle \cong M_{(m-i,2,1)}$, a contradiction. If $m > n$, $\langle a^{p^{m-n}} x^{-k}, b\rangle$ is a nonnormal subgroup of order $p^{m+2-i} \ge p^3$, we get a contradiction again. Thus $C_G(H) \le H$, we get a group of type (xiii).

Conversely, the group of type (xiii) satisfy the hypothesis of Case 5.

($\Leftarrow$) We prove that the groups listed in Theorem 4.3 are not isomorphic to each other.

Obviously, it remains to prove the groups appear in Case 1 are not isomorphic to each other. Note that $d(G) = 6$ for a group (v) but $d(G) \le 5$ for others. Since groups (iii), (iv), (vi) and (viii) are 2-groups, and groups (i), (ii), and (vii) are not 2-groups, we discuss these two cases separately.

For groups (i), (ii) and (vii), $Z(G)$ is cyclic for groups (i) and (ii) but not cyclic for a group (vii). For a group (i), $d(G) = 4$ and $\exp(G) > p$ but for a group (ii), $d(G) = 5$ or $d(G) = 4$ and $\exp(G) = p$. Hence groups (i), (ii) and (vii) are not isomorphic to each other.

For groups (iii), (iv), (vi) and (viii), $Z(G)$ is cyclic for groups (iii) and (iv) but not cyclic for groups (vi) and (viii). If groups (vi) and (viii) have the same order, then $n = 3$, $\exp(G) = 8$ for a group (viii) but $\exp(G) = 4$ for a group (vi). If groups (iii) and (iv) have the same order, then $d(G) = 4$ for a group (iii) but $d(G) = 5$ for a group (iv). Hence groups (iii), (iv), (vi) and (viii) are not isomorphic to each other.

Finally, we prove groups (i)–(xiii) satisfy the hypothesis of Theorem 4.3.

It is easy to see that $|G'| = p$ for the groups (i)–(xiii). By Theorem 2.3, there is a subgroup of order $\geq p^2$ which is nonnormal for the groups (i)–(xiii). Assume $N \leq G$, $|N| \geq p^3$, we will prove that $N \trianglelefteq G$ for the groups (i)–(xiii). If $N \ntrianglelefteq G$, then:

(I) For groups (i)–(ix), $\mho_1(G)$ is cyclic, and $G' \leq \mho(G)$ when $\mho_1(G) \neq 1$. Since $N \ntrianglelefteq G$, $G' \nleq N$. Thus $N' = 1$ and $\mho_1(N) = 1$. Hence N is elementary abelian. It is easy to check that all elementary abelian subgroups of G not containing G' are of order $\leq p^2$, a contradiction.

(II) For groups (x) and (xiii), there is a cyclic subgroup C of G such that $G' \leq C \leq Z(G)$ and $|G : C| = p^3$. Since $N \ntrianglelefteq G$, $G' \nleq N$. We have $N \cap C = 1$. Hence $NC = G$, $N \trianglelefteq G$, a contradiction.

(III) For groups (xi) and (xii), noting that $\exp(G) = 4 = |\mho_1(G)|$, we always have $N \geq \mho_1(G) \geq G'$ since $|N| \geq p^3$. A contradiction. □

Lemma 4.4. *Assume $|G'| = p^2$. If $K \leq G' \cap Z(G)$, $|K| = p$, $G/K \cong D_8 * C_{2^n}$, $M_{(1,1,1)} * C_{p^n}$ or $D_8 * Q_8$ where $n \geq 2$, then $G' \cong C_p^2$, $n = 2$, $\exp(G) = p^2$ and $G' \leq Z(G)$.*

Proof. By Proposition 2.7, we can take $H \leq G$ such that $\overline{H} = HK/K \cong M_{(2,1)}$ when $\overline{G} = G/K \cong M_{(1,1,1)} * C_{p^n}$. On the other hand, it is easy to see we can take H such that $\overline{H} \cong Q_8$ when $\overline{G} \cong D_8 * C_{2^n}$ or $D_8 * Q_8$. If G' is cyclic, it follows from $\overline{H}' = \overline{G}'$ that $H' = G'$. Thus H is a group of order p^4, $H' \cong C_{p^2}$ and $H/\mho_1(H') \cong M_{(2,1)}$ or Q_8. By the classification of groups of order p^4, such group does not exist, hence $G' \cong C_p^2$.

If $\overline{G} \cong D_8 * C_{2^n}$ or $M_{(1,1,1)} * C_{p^n}$, we claim that $n = 2$. Taking $Z \leq G$ such that $Z/K = Z(\overline{G})$, we have $Z = K \times C$ since $Z(\overline{G})$ is cyclic and $G' \leq Z$, where $C \cong C_{2^n}$ or C_{p^n} respectively. If $n \geq 3$, then $G' \nleq C \trianglelefteq G$. Hence $C \leq Z(G)$ by Proposition 2.6. Now $|G : Z(G)| \leq |G : Z| = 4$ or p^2 respectively. It follows that $|G'| \leq p$, a contradiction.

Now we always have $\mho_1(G)K/K = \mho_1(G/K) = (G/K)' = G'/K'$. Hence $\mho_1(G) \leq G'$, $\exp(G) = p^2$.

Finally, we prove that $G' \leq Z(G)$. If $G/K \cong D_8 * C_4$ or $M_{(1,1,1)} * C_{p^2}$, then $G/K = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^{p^2} = \bar{b}^p = \bar{c}^p = \bar{1},\ [\bar{a}, \bar{b}] = [\bar{a}, \bar{c}] = \bar{1},\ [\bar{b}, \bar{c}] = \bar{a}^p \rangle$. Thus

$G = \langle a, b, c\rangle$ and $G' = \langle a^p\rangle \times K$. Since $[a, b] \in K \leq Z(G)$, $[a^p, b] = [a, b]^p = 1$. Similarly, $[a^p, c] = 1$. Hence $G' \leq Z(G)$. If $G/K \cong D_8 * Q_8$, then $G/K = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \mid \bar{a}^4 = \bar{c}^2 = \bar{d}^2 = \bar{1},\ \bar{a}^2 = \bar{b}^2 = [\bar{a}, \bar{b}] = [\bar{c}, \bar{d}],\ [\bar{a}, \bar{c}] = [\bar{a}, \bar{d}] = [\bar{b}, \bar{c}] = [\bar{b}, \bar{d}] = \bar{1}\rangle$. It follows that $[a^2, c] = [a^2, d] = [b^2, c] = [b^2, d] = 1$, and hence $G' = \langle a^2\rangle \times K = \langle b^2\rangle \times K \leq Z(G)$. □

Theorem 4.5. *Assume $|G'| = p^2$. Then G is one of the following non-isomorphic groups*:

(i) *G is a p-group of maximal class of order p^4*;

(ii) $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ [a, b] = 1,\ [a, c] = b^2,\ [b, c] = a^2b^2\rangle$;

(iii) $G = \langle a, b, c \mid a^4 = b^4 = 1,\ c^2 = b^2,\ [a, b] = 1,\ [a, c] = b^2,\ [b, c] = a^2\rangle$;

(iv) $G = \langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1,\ [a, b] = 1,\ [a, c] = b^p,\ [b, c] = a^pb^{wp}\rangle$, *where* $w = 1, 2, \ldots, \frac{p-1}{2}$ *and* $1 + \frac{w^2}{4}$ *is not a square modulo* p;

(v) $G = \langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1,\ [a, b] = 1,\ [a, c] = b^{\nu p},\ [b, c] = a^pb^{wp}\rangle$, *where ν is a non-square residue modulo p,* $w = 0, 1, \ldots, \frac{p-1}{2}$ *and* $\nu + \frac{w^2}{4}$ *is not a square modulo* p;

(vi) $G = \langle a, b, c, d \mid a^4 = b^4 = 1,\ c^2 = a^2b^2,\ d^2 = a^2,\ [a, b] = a^2,\ [c, d] = a^2b^2,\ [a, c] = [b, d] = 1,\ [b, c] = [a, d] = b^2\rangle$.

Proof. ($\Rightarrow$) By hypothesis, $|G| \geq p^4$. If $|G| = p^4$, by the classification of groups of order p^4, G is a p-group of maximal class, we get the group of type (i). Assume $|G| \geq p^5$. By Lemmas 2.2 and 2.3, there exists $K \trianglelefteq G$ such that $|G' : K| = p$ and G/K is isomorphic to one of the groups $M_{(m,1)}$, $D_8 * C_{2^n}$, $M_{(1,1,1)} * C_{p^n}$ and $D_8 * Q_8$, where $m \geq 3$ and $n \geq 2$. By Lemma 4.4, $n = 2$. We proceed in the following four cases.

Case 1. $G/K \cong M_{(m,1)} = \langle \bar{a}, \bar{b} \mid \bar{a}^{p^m} = \bar{b}^p = \bar{1},\ [\bar{a}, \bar{b}] = \bar{a}^{p^{m-1}}\rangle$.

Since $K \leq G'$, $G = \langle a, b\rangle$. Moreover, $o(a) \geq p^3$ since $m \geq 3$. Then $\langle a\rangle \trianglelefteq G$. It follows that G is metacyclic and has a cyclic subgroup $\langle a\rangle$ of index p. By [4, I, Satz 14.9], there is no such group with $|G| \geq p^5$ and $|G'| = p^2$.

Case 2. $G/K \cong D_8 * C_4 = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^4 = \bar{b}^2 = \bar{c}^2 = \bar{1},\ [\bar{b}, \bar{c}] = \bar{a}^2,\ [\bar{a}, \bar{b}] = [\bar{a}, \bar{c}] = \bar{1}\rangle$.

Since $K \leq G'$, $G = \langle a, b, c\rangle$. By Lemma 4.4, $\exp(G) = 4$, $G' = \langle a^2\rangle \times K \leq Z(G)$. Noting that $|\langle b, c\rangle| \geq 8$, we have $\langle b, c\rangle \trianglelefteq G$. Since $G = \langle b, c\rangle\langle a\rangle$, $|\langle b, c\rangle| \geq 16$. Without loss of generality we can assume $o(b) = 4$. Hence $K = \langle b^2\rangle$. Thus we have

$$G = \langle a, b, c \mid a^4 = b^4 = 1,\ c^2 = b^{2v},\ [b, c] = a^2b^{2k},\ [a, b] = b^{2s},\ [a, c] = b^{2t}\rangle,$$

where k, v, s, t are 0 or 1, and either $s \neq 1$ or $t \neq 1$.

If $v = 0$, then $\langle a, c\rangle \trianglelefteq G$, $b^2 = [a, b]$ or $[a, c]$. It follows that $b^2 \in \langle a, c\rangle$, and hence $[a, c] \neq 1$, that is, $t = 1$. By calculations, we get $|\langle ab, c\rangle| \geq 8$. Hence $\langle ab, c\rangle \trianglelefteq G$. It is easy to see $G' \leq \langle ab, c\rangle$. Since $G' \leq Z(G)$, we have $[ab, c] \neq (ab)^2$. By calculations, we get $s \neq k$. If $s = 0$, then $k = 1$, we get a group of type (ii). If $s = 1$, then $k = 0$. Let $b_1 = abc$. Then $b_1^2 = b^2$, $[a, b_1] = 1$, $[b_1, c] = a^2b^2$. So G is isomorphic to the group of type (ii).

Assume $v = 1$. If $t = 0$, then $s = 1$. If $k = 1$, letting $c' = abc$, then $c'^2 = 1$. It is reduced to the case in the above paragraph. So we may assume $k = 0$. Letting $b_1 = c$, $c_1 = b$, we get a group of type (iii). If $s = 0$, by an argument similar to that in the case $t = 0$, we still get a group of type (iii). If $s = t = 1$, letting $c_1 = abc$, then $\bar{c}_1{}^2 = 1$ and $[a, c_1] = 1$. It is reduced to the cases we discussed.

Case 3. $G/K \cong M_{(1,1,1)} * C_{p^2} = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^{p^2} = \bar{b}^p = \bar{c}^p = 1,\ [\bar{a}, \bar{b}] = [\bar{a}, \bar{c}] = \bar{1},\ [\bar{b}, \bar{c}] = \bar{a}^p \rangle$.

By Lemma 4.4, we have $\exp(G) = p^2$, $G' = \langle a^p \rangle \times K \leq Z(G)$. Since $p > 2$, G is p-abelian. By an argument similar to that in Case 2, we can assume that

$$G = \langle a, b, c \mid a^{p^2} = b^{p^2} = 1, c^p = b^{kp},\ [a, b] = b^{sp},\ [a, c] = b^{tp},\ [b, c] = a^p b^{wp} \rangle,$$

where $0 \leq k, s, t, w < p$.

Replacing c by cb^{-k}, we may assume $k = 0$. It follows from $\langle a, c \rangle \trianglelefteq G$ that $G' \leq \langle a, c \rangle$, and hence $t \neq 0$. Replacing b by $c^{-is}b$, where $it \equiv 1 \pmod p$, we have $s = 0$.

If $t \equiv i^2 \pmod p$ for some i, replacing a by a^j and c by c^j, where $ij \equiv 1 \pmod p$, we have $t = 1$. Replacing a by a^{-1} and c by c^{-1}, we have $[b, c] = a^p b^{(p-w)p}$. Thus we can assume $w \leq \frac{p-1}{2}$ without loss of generality. Now we claim that $1 + \frac{w^2}{4}$ is not a square modulo p. Assume $1 + \frac{w^2}{4} \equiv i^2 \pmod p$ for some i. Write $j = i - \frac{w}{2}$ and let $H = \langle a^j b, c^j \rangle$. Then H is a nonnormal subgroup of order p^3, this is contrary to our hypothesis. We get a group of type (iv).

If t is not a square modulo p, taking a fixed number ν which is a non-square residue modulo p, then there exists i such that $t = i^2\nu$. Replacing a by a^j and c by c^j, where $ij \equiv 1 \pmod p$, we have $t = \nu$. By an argument similar to that in the above paragraph, we get a group of type (v).

Case 4. $G/K \cong D_8 * Q_8 = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \mid \bar{a}^4 = \bar{b}^2 = \bar{1},\ [\bar{a}, \bar{b}] = \bar{a}^2,\ \bar{c}^2 = \bar{d}^2 = \bar{a}^2,\ [\bar{c}, \bar{d}] = \bar{a}^2,\ [\bar{a}, \bar{c}] = [\bar{a}, \bar{d}] = [\bar{b}, \bar{c}] = [\bar{b}, \bar{d}] = \bar{1} \rangle$.

Since $K \leq G'$, $G = \langle a, b, c, d \rangle$. By Lemma 4.4, $\exp(G) = 4$, $G' = \langle a^2 \rangle \times K \leq Z(G)$.

Suppose that $H/K = \overline{H} \leq \overline{G} = G/K$. We have the following observation.

If $\overline{H} = \langle \bar{x}, \bar{y} \mid \bar{x}^4 = \bar{y}^2 = 1, \bar{x}^{\bar{y}} = \bar{x}^{-1} \rangle \cong D_8$, then $o(y) = 4$, and hence $H \cong M_{(2,2)}$.

In fact, by Lemma 2.4, $\overline{G} = \overline{H} * C_{\overline{G}}(\overline{H})$, we can assume $x = a$, $y = b$. Noting that $C_{\langle c,d \rangle}(b) \neq 1$, we may assume $[b, c] = 1$ without loss of generality. By hypothesis, $\langle b, c \rangle \trianglelefteq G$ and $\langle a, b \rangle \trianglelefteq G$. If $b^2 = 1$, then $[\langle b, c \rangle, \langle a, d \rangle] \leq \langle b, c \rangle \cap G' = \langle c^2 \rangle$ and $[\langle a, b \rangle, \langle c, d \rangle] \leq K \cap \langle a, b \rangle = 1$. Hence $G' = \langle c^2 \rangle$, this is contrary to $|G'| = p^2$. Thus $o(b) = 4$, $K = \langle b^2 \rangle$. Moreover, $\langle \bar{a}, \overline{ab} \rangle \cong D_8$, and hence $(ab)^2 = a^2b^2[a, b] \neq 1$. We have $[a, b] = a^2$, $H \cong M_{(2,2)}$.

Noting that $C_{\langle c,d\rangle}(a) \neq 1$, we may assume $[a,c]=1$ without loss of generality. Let $\overline{H_1} = \langle \overline{ca}, \bar{d}\rangle \cong D_8$. By the observation above, we have $(ca)^2 = c^2a^2[a,c] = c^2a^2 \neq 1$, and hence $c^2 = a^2b^2$. Moreover, $d^2 = [ca,d] = [c,d][a,d]$, it follows that $[c,d] = d^2[a,d]$. Let $\overline{H_2} = \langle \overline{da}, \bar{c}\rangle \cong D_8$. In the same way, we have $(da)^2 = d^2a^2[a,d] \neq 1$, $d^2[a,d] = a^2b^2$. Hence $[c,d] = c^2 = a^2b^2$.

If $d^2 = c^2$, then $[\langle b\rangle, \langle c,d\rangle] \leq \langle c,d\rangle \cap K = 1$, $(bc)^2 = b^2c^2 = a^2$. $[d,c] = [d,bc] \in \langle a,bc\rangle \cap G' = \langle a^2\rangle$, contradicting the fact that $[c,d] = a^2b^2$. Hence $d^2 = a^2$. It follows that $[a,d] = d^2[c,d] = b^2$.

If $[b,c]=1$, then $[a,d] \in \langle a,bc\rangle \cap G' = \langle a^2\rangle$, contradicting the fact $[a,d] = b^2$. Hence $[b,c] = b^2$.

Finally, if $[b,d] = b^2$, letting $d_1 = cd$, then $d_1^2 = d^2$, $[a,d_1] = [a,d]$, $[c,d_1] = [c,d]$ and $[b,d_1] = 1$. Thus we can assume $[b,d] = 1$ and get a group of type (vi).

($\Leftarrow$) We prove that those groups listed in Theorem 4.5 are not isomorphic to each other. It suffices to show that the groups of type (ii), (iii), (iv) and (v) are not isomorphic to each other.

For groups (ii) and (iii), their order are 2^5. To prove that these two groups are not isomorphic, we observe the following facts: (1) both of groups (ii) and (iii) have a maximal subgroup H which isomorphic to $C_4 \times C_4$, and (2) there is an involution in $G \setminus H$ for a group (ii) but not for a group (iii).

For the groups (iv) and (v), their order are $p^5 (p>2)$.

Firstly, we prove that these groups of the same type but with different values of parameters are not isomorphic to each other.

It is easy to see $Z(G) = G'$ and hence $\langle a,b\rangle$ is the unique abelian maximal subgroup.

For groups of type (iv), suppose the group with parameter w_1 is isomorphic to the group with parameter w_2. Since $\langle a,b\rangle$ char G, we may assume $a_2 = a_1^{i_1}b_1^{j_1}$, $b_2 = a_1^{i_2}b_1^{j_2}$ and $c_2 = a_1^{i_3}b_1^{j_3}c_1^k$ satisfying $a_i^{p^2} = b_i^{p^2} = c_i^p = 1$, $[a_i,b_i]=1$, $[a_i,c_i] = b_i^p$ and $[b_i,c_i] = a_i^pb_i^{w_ip}$, where $i=1,2$, $\left|\begin{smallmatrix} i_1 & j_1 \\ i_2 & j_2 \end{smallmatrix}\right| \not\equiv 0 \pmod p$ and $p \nmid k$. Since $(a_1^{i_2}b_1^{j_2})^p = b_2^p = [a_2,c_2] = [a_1^{i_1}b_1^{j_1}, a_1^{i_3}b_1^{j_3}c_1^k]$, we have (1) $j_1k \equiv i_2 \pmod p$, (2) $i_1k + w_1j_1k \equiv j_2 \pmod p$; since $(a_1^{i_1}b_1^{j_1})^p(a_1^{i_2}b_1^{j_2})^{w_2p} = a_2^pb_2^{w_2p} = [b_2,c_2] = [a_1^{i_2}b_1^{j_2}, a_1^{i_3}b_1^{j_3}c_1^k]$, we have (3) $j_2k \equiv i_1 + w_2i_2 \pmod p$, (4) $i_2k + w_1j_2k \equiv j_1 + w_2j_2 \pmod p$.

By (1) and (4), we have $j_1(k^2-1) + j_2(kw_1 - w_2) \equiv 0 \pmod p$; by (1), (2) and (3), we have $i_1(k^2-1) + i_2(kw_1-w_2) \equiv 0 \pmod p$. Since $\left|\begin{smallmatrix} i_1 & j_1 \\ i_2 & j_2 \end{smallmatrix}\right| \not\equiv 0 \pmod p$, we have $k^2 - 1 \equiv 0 \pmod p$, $kw_1 - w_2 \equiv 0 \pmod p$. Since $1 \leq w_i \leq \frac{p-1}{2}$, $w_1 = w_2$.

Similarly, groups of type (v) with different values of parameters are not isomorphic to each other.

Secondly, we prove that a group (iv) is not isomorphic to a group (v). Suppose the group (iv) with parameter w_1 is isomorphic to the group (v) with parameter w_2. We may assume $a_2 = a_1^{i_1}b_1^{j_1}$, $b_2 = a_1^{i_2}b_1^{j_2}$ and $c_2 = a_1^{i_3}b_1^{j_3}c_1^k$

satisfying $a_i^{p^2} = b_i^{p^2} = c_i^p = 1$, $[a_i, b_i] = 1$, $[a_1, c_1] = b_1^p$, $[a_2, c_2] = b_2^{\nu p}$ and $[b_i, c_i] = a_i^p b_i^{w_i p}$, where $i = 1, 2$, $|\begin{smallmatrix} i_1 & j_1 \\ i_2 & j_2 \end{smallmatrix}| \not\equiv 0 \pmod p$ and $p \nmid k$. By a similar calculation, we have $j_1(k^2\nu^{-1} - 1) + j_2(kw_1 - w_2) \equiv 0 \pmod p$ and $i_1(k^2\nu^{-1} - 1) + i_2(kw_1 - w_2) \equiv 0 \pmod p$. Hence $\nu \equiv k^2$, a contradiction.

Finally, we prove that the groups (i)–(vi) satisfy the hypothesis of Theorem 4.5.

For groups (i)–(vi), it is easy to see that $|G'| = p^2$.

For the group (i), it is easy to see that it satisfies the hypothesis of Theorem 4.5.

For groups (ii), (iii), (iv) and (v), $|G| = p^5$.

We prove that for any subgroup K of G' of order p, $\overline{G} = G/K$ is a Dedekind group or a group listed in Theorem 2.3. It is easy to check that $\overline{G} \cong D_8 * C_4$ for group (ii), $\overline{G} \cong D_8 * C_4$ or $Q_8 \times C_2$ for group (iii). For groups (iv) and (v), we already know that $G/\langle b^p \rangle \cong M_{(1,1,1)} * C_{p^2}$, so we assume $K = \langle a^p b^{xp} \rangle$. Let $\overline{M} = \langle \bar{a}\bar{b}^x, \bar{c} \rangle$. Note $(\bar{a}\bar{b}^x)^p = \bar{c}^p = \bar{1}$, $[\bar{a}\bar{b}^x, \bar{c}] = \bar{b}^{p(\nu^i - x^2 + xw)}$, where $i = 0$ for group (iv) and $i = 1$ for group (v). Since $\nu^i - x^2 + xw = \nu^i + \frac{w}{4} - (x - \frac{w}{2})^2 \not\equiv 0 \pmod p$, $M \cong M_{(1,1,1)}$. Noting $|\overline{G}'| = p$ and $\exp(\overline{G}) = p^2$, by Lemma 2.4, $\overline{G} \cong M_{(1,1,1)} * C_{p^2}$.

Now it suffices to prove all subgroups of order p^3 are normal. Assume $H \leq G$, $|H| = p^3$ and $H \ntrianglelefteq G$. Then $G' \nleq H$. Note $G' = Z(G)$ is of order p^2. If $H \cap G' = 1$, then $G = G' \times H$, this contradicts $H \ntrianglelefteq G$. Hence $|H \cap G'| = p$. Thus $G/H \cap G'$ is a Dedekind group or a group listed in Theorem 2.3. So $H/H \cap G' \trianglelefteq G/H \cap G'$, and hence $H \trianglelefteq G$, a contradiction.

For the group (vi), by calculations, we have the following facts: $\Omega_1(G) = Z(G) = G' = \langle a^2 \rangle \times \langle b^2 \rangle$. Let H be a nonnormal subgroup of G. Then $|H \cap G'| \leq 2$. Since $H \cap \mho_1(G) = \mho_1(H)$, H has a unique element of order 2. It follows that H is either a cyclic group or a quaternion group. If $|H| \geq 8$, then H is a quaternion group. Taking $K \leq G'$ and $K \neq H'$, and letting $\overline{G} = G/K$. Since $\overline{H} = HK/K \cong Q_8$ and $|\overline{G}'| = 2$, $\overline{G} = \overline{H} * C_{\overline{G}}(\overline{H})$ by Lemma 2.4. Let $C/K = C_{\overline{G}}(\overline{H})$. Then $G = HC$, $H \cap C = H'$. Hence G/H' has an elementary abelian subgroup of order 8. But one can easily check that $G/K \cong D_8 * Q_8$ for any $1 < K < G'$ and $D_8 * Q_8$ has no elementary abelian subgroup of order 8, a contradiction. $\square$

Remark 4.6. The group of type (vi) in Theorem 4.5 is the same as the group of type (1) in [8, Proposition 10]. In fact, for (vi), let $a_1 = b, b_1 = a, c_1 = c$ and $d_1 = bd$. Then (vi) is reduced to (1) by an easy calculation. By [8, Remark] this means the group (vi) is an $\mathcal{M}_2$-group and MI-group. (An $\mathcal{M}_2$-group is a p-group all of whose subgroups of index p^2 are metacyclic, but there is at least one subgroup of index p is not; An MI-group is a group all of whose maximal subgroups are isomorphic.)

Summarizing, we have the following:

Theorem 4.7. *Let G be a finite p-group. Then all subgroups of G of order $\geq p^3$ are normal if and only if G is either a Dedekind group, or a group listed in Theorems 2.3, 4.3, and 4.5.*

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