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STRONG CONVERGENCE OF COMPOSITE ITERATIVE METHODS FOR NONEXPANSIVE MAPPINGS

Jong Soo Jung

Abstract. Let E be a reflexive Banach space with a weakly sequentially continuous duality mapping, C be a nonempty closed convex subset of E, $f : C \to C$ a contractive mapping (or a weakly contractive mapping), and $T : C \to C$ a nonexpansive mapping with the fixed point set $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated by a new composite iterative scheme: $y_n = \lambda_n f(x_n) + (1-\lambda_n)Tx_n$, $x_{n+1} = (1-\beta_n)y_n + \beta_n Ty_n$, $(n \geq 0)$. It is proved that $\{x_n\}$ converges strongly to a point in $F(T)$, which is a solution of certain variational inequality provided the sequence $\{\lambda_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\{\beta_n\} \subset [0, a)$ for some $0 < a < 1$ and the sequence $\{x_n\}$ is asymptotically regular.

1. Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $x, y \in C$. We use Σ_C to denote the collection of all contractions on C. That is, $\Sigma_C = \{f : C \to C \mid f \text{ is a contraction with constant } k\}$. Let $T : C \to C$ be a nonexpansive mapping (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$) and $F(T)$ denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$.

We consider the iterative scheme: for a nonexpansive mapping T, $f \in \Sigma_C$ and $\lambda_n \in (0, 1)$,

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)Tx_n, \quad n \geq 0. \tag{1.1}$$

As a special case of (1.1), the following iterative scheme

$$z_{n+1} = \lambda_n u + (1 - \lambda_n)Tz_n, \quad n \geq 0, \tag{1.2}$$

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where $u,\ z_0 \in C$ are arbitrary (but fixed), has been investigated by many authors: see, for example, Cho et al. [3], Halpern [6], Jung [7], Lions [12], Reich [17, 18], Shioji and Takahashi [19], Wittmann [20] and Xu [21]. The authors above showed that the sequence $\{z_n\}$ generated by (1.2) converges strongly to a point in the fixed point set $F(T)$ under appropriate conditions on $\{\lambda_n\}$ in either Hilbert spaces or certain Banach spaces.

The viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [14]. In 2004, Xu [22] extended Theorem 2.2 of Moudafi [14] for the iterative scheme (1.1) to a Banach space setting using the followings conditions on $\{\lambda_n\}$:

(H1) $\lim_{n\to\infty} \lambda_n = 0$; $\sum_{n=0}^{\infty} \lambda_n = \infty$ or equivalently, $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$;
(H2) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Recently, Kim and Xu [11] provided a simpler modification of Mann iterative scheme (1.3) in a uniformly smooth Banach space as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \end{cases} \tag{1.3}$$

where $u \in C$ is an arbitrary (but fixed) element, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1). They proved that $\{x_n\}$ generated by (1.3) converges to a fixed point of T under the control conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0,\ \ \lim_{n\to\infty} \beta_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \ \sum_{n=0}^{\infty} \beta_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

In this paper, motivated by above-mentioned results, as the viscosity approximation method, we consider a new composite iterative scheme for a nonexpansive mapping T:

$$\begin{cases} x_0 = x \in C, \\ y_n = \lambda_n f(x_n) + (1-\lambda_n)Tx_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n Ty_n, \end{cases} \tag{IS}$$

where $\{\lambda_n\}$, $\{\beta_n\} \subset (0,1)$. First, we prove the strong convergence of the sequence $\{x_n\}$ generated by (IS) under the suitable conditions on the control parameters $\{\lambda_n\}$ and $\{\beta_n\}$ and the asymptotic regularity on $\{x_n\}$ in a reflexive Banach space with a weakly sequentially continuous duality mapping. Moreover, we show that the strong limit is a solution of certain variational inequality. Next we study the viscosity approximation with the weakly contractive mapping to a fixed point of a nonexpansive mapping in the same Banach space. The main results improve and complement the corresponding results of [3, 6, 12, 14, 17, 18, 19, 20, 21, 22]. In particular, if $\beta_n = 0$ for all $n \geq 0$, then

(IS) reduces to (1.1). We point out that the iterative scheme (IS) is a new method for finding a fixed point of T.

2. Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x,\ y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$.

By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\} \text{ for all } x \in E$$

is called the *duality mapping* with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$ denoted by J, is referred to as the *normalized duality mapping.* It is known (cf. [4]) that a Banach space E is smooth if and only if the normalized duality mapping J is single-valued.

We say that a Banach space E has a weakly sequential continuous duality mapping if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J_\varphi(x_n) \overset{*}{\rightharpoonup} J_\varphi(x)$. For example, every l^p space $(1 < p < \infty)$ has a weakly sequentially continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$.

Let D be a subset of C. Then a mapping $Q : C \to D$ is said to be a retraction from C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \ge 0$ with $Qx+t(x-Qx) \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. In a smooth Banach space E, it is well-known [5, p. 48] that Q is a sunny nonexpansive retraction from C onto D if and only if the following condition holds:

$$\langle x - Qx, J(z - Qx)\rangle \le 0, \quad x \in C, \quad z \in D. \tag{2.1}$$

We need the following lemmas for the proof of our main results. (Lemma 2.1 was also given in Jung and Morales [9] and Lemma 2.2 is essentially Lemma 2 of Liu [13] (also see [21])).

Lemma 2.1. *Let E be a real Banach space and J be the duality mapping. Then, for any given $x,\ y \in E$, one has*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $j(x+y) \in J(x+y)$.

Lemma 2.2. *Let* $\{s_n\}$ *be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *satisfy the following conditions*:

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} \alpha_n\gamma_n < \infty$,
(iii) $\delta_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \delta_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Let μ be a continuous linear functional on l^∞ and $(a_0, a_1, \ldots) \in l^\infty$. We write $u_n(a_n)$ instead of $\mu((a_0, a_1, \ldots))$. μ is said to be *Banach limit* if μ satisfies $\|\mu\| = \mu(1) = 1$ and $u_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \ldots) \in l^\infty$. If μ is a Banach limit, the following are well-known:

(i) for all $n \geq 1, a_n \leq c_n$ implies $\mu(a_n) \leq \mu(c_n)$,
(ii) $\mu(a_{n+1}) = \mu(a_n)$,
(iii) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \ldots) \in l^\infty$.

The following lemma was given in [19].

Lemma 2.3. *Let* $a \in \mathbb{R}$ *be a real number and let a sequence* $\{a_n\} \in l^\infty$ *satisfy* $\mu_n(a_n) \leq a$ *for all Banach limit* μ. *If* $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, *then* $\limsup_{n\to\infty} a_n \leq a$.

Recall a mapping $A : C \to C$ is said to be *weakly contractive* if

$$\|Ax - Ay\| \leq \|x - y\| - \psi(\|x - y\|) \quad \text{for all } x,\ y \in K,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and strictly increasing function such that ψ is positive on $(0, \infty)$ and $\psi(0) = 0$. Obviously, the class of the weakly contractive mappings contains the class of the contractive mappings as a special case $(\psi(t) = (1-k)t)$. Rhodes [16] obtained the following result for weakly contractive mapping.

Lemma 2.4 ([16, Theorem 2])**.** *Let* (X, d) *be a complete metric space, and* A *a weakly contractive mapping on* X. *Then* A *has a unique fixed point* p *in* X. *Moreover, for* $x \in X$, $\{A^n x\}$ *converges strongly to* p.

The following lemma was given in [1, 2].

Lemma 2.5. *Let* $\{s_n\}$ *and* $\{\gamma_n\}$ *be two sequences of nonnegative real numbers and* $\{\lambda_n\}$ *a sequence of positive numbers satisfying the conditions*

(i) $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1 - \lambda_n) = 0$,
(ii) $\lim_{n\to\infty} \frac{\gamma_n}{\lambda_n} = 0$.

Let the recursive inequality

$$s_{n+1} \leq s_n - \lambda_n\psi(s_n) + \gamma_n, \quad n = 0, 1, 2, \ldots,$$

be given where $\psi(t)$ *is a continuous and strict increasing function on* $[0, +\infty)$ *with* $\psi(0) = 0$. *Then* $\lim_{n\to\infty} s_n = 0$.

Finally, the sequence $\{x_n\}$ in E is said to be *asymptotically regular* if

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \quad \text{that is,} \quad x_{n+1} - x_n \to 0.$$

3. Main results

First, using the asymptotic regularity, we study a strong convergence theorem for a composite iterative method for the nonexpansive mapping with the contractive mapping.

For abbreviation, we set the duality mapping $J := J_\varphi$. In all our proofs we assume, without loss of generality, that J is normalized.

Let $T : C \to C$ be a nonexpansive mapping. Then, for any $t \in (0,1)$ and $f \in \Sigma_C$, $tf + (1-t)T : C \to C$ defines a contraction. Thus, by the Banach contraction principle, there exists a unique fixed point x_t^f satisfying

$$\text{(R)} \qquad x_t^f = tf(x_t^f) + (1-t)Tx_t^f.$$

For simplicity we will write x_t for x_t^f provided no confusion occurs.

The following result was given by Jung [8] (see also O'Hara et al. [15] and Xu [23] for the case that $f(x) = u$ a constant). We refer Jung and Sahu [10] for the case of non-LIpschizian mappings.

Theorem J ([8]). *Let E be a reflexive Banach space with a weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of E and T nonexpansive mappings from C into itself with $F(T) \neq \emptyset$. Then $\{x_t\}$ defined by* (R) *converges strongly to a point in $F(T)$. If we define $Q : \Sigma_C \to F(T)$ by*

$$Q(f) := \lim_{t\to 0^+} x_t, \quad f \in \Sigma_C,$$

then $Q(f)$ solves a variational inequality

$$\text{(3.1)} \qquad \langle (I-f)(Q(f)), J(Q(f)-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F(T).$$

Remark 3.1. In Theorem J, if $f(x) = u \in C$ is a constant, then (3.1) become

$$\langle Qu - u, J(Qu - p)\rangle \le 0, \quad u \in C, \quad p \in F(T).$$

Hence by (2.1), Q reduces to the sunny nonexpansive retraction from C to $F(T)$. Namely $F(T)$ is a sunny nonexpansive retraction of C.

Using Theorem J, we have the following result.

Proposition 3.1. *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of E and T nonexpansive mappings from C into itself with $F(T) \neq \emptyset$. Let $f \in \Sigma_C$ and and μ a Banach limit. Let $\{y_n\}$ be a bounded sequence in C. If $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$, then*

$$\mu_n \langle (I-f)(Q(f)), J(Q(f) - y_n)\rangle \le 0,$$

where $Q : \Sigma_C \to F$ is defined by $Q(f) = \lim_{t\to 0^+} x_t$ and x_t is defined by (R).

Proof. Note that the definition of the weak continuity of duality mapping J implies that E is smooth. By Theorem J, there exists $\lim_{t\to 0^+} x_t = Q(f)$, where x_t is defined by (R).

First, we show that $\|x_t - z\| \le \frac{1}{1-k}\|f(z) - z\|$ for $t \in (0,1)$ and $z \in F(T)$ and so $\{x_t\}$, $\{Tx_t\}$ and $\{f(x_t)\}$ are bounded. To this end, let $z \in F(T)$ and $t \in (0,1)$. Then

$$x_t - z = t(f(x_t) - z) + (1-t)(Tx_t - Tz)$$

and so

$$\begin{aligned}\|x_t - z\| &\le t\|f(x_t) - z\| + (1-t)\|Tx_t - Tz\| \\ &\le t\|f(x_t) - z\| + (1-t)\|x_t - z\|.\end{aligned}$$

This gives that

$$\begin{aligned}\|x_t - z\| \le \|f(x_t) - z\| &\le \|f(x_t) - f(z)\| + \|f(z) - z\| \\ &\le k\|x_t - z\| + \|f(z) - z\|,\end{aligned}$$

and so

$$\|x_t - z\| \le \frac{1}{1-k}\|f(z) - z\|. \tag{3.2}$$

Hence $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{Tx_t\}$.

Now we can write

$$x_t - y_n = (1-t)(Tx_t - y_n) + t(f(x_t) - y_n).$$

Applying Lemma 2.1, we have

$$\|x_t - y_n\|^2 \le (1-t)^2\|Tx_t - y_n\|^2 + 2t\langle f(x_t) - y_n, J(x_t - y_n)\rangle. \tag{3.3}$$

Using $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$, we derive

$$\|Tx_t - y_n\| \le \|x_t - y_n\| + e_n,$$

where $e_n = \|y_n - Ty_n\| \to 0$ as $n \to \infty$, and

$$\langle f(x_t) - y_n, J(x_t - y_n)\rangle = \langle f(x_t) - x_t, J(x_t - y_n)\rangle + \|x_t - y_n\|^2.$$

Thus it follows from (3.3) that

$$\begin{aligned}\|x_t - y_n\|^2 \le\ & (1-t)^2(\|x_t - y_n\| + e_n)^2 \\ & + 2t(\langle f(x_t) - x_t, J(x_t - y_n)\rangle + \|x_t - y_n\|^2).\end{aligned} \tag{3.4}$$

Applying the Banach limit μ to (3.4), we have

$$\begin{aligned}\mu_n(\|x_t - y_n\|^2) \le\ & (1-t)^2\mu_n((\|x_t - y_n\| + e_n)^2) \\ & + 2t\mu_n(\langle f(x_t) - x_t, J(x_t - y_n)\rangle + \|x_t - y_n\|^2)\end{aligned} \tag{3.5}$$

and it follows from (3.5) that

$$\mu_n(\langle x_t - f(x_t), J(x_t - y_n)\rangle) \le t\mu_n(\|x_t - y_n\|^2). \tag{3.6}$$

From (3.2) and boundedness of $\{y_n\}$, it follows that

$$t\|x_t - y_n\|^2 \le t\left(\frac{1}{1-k}\|f(z) - z\| + \|z - y_n\|\right)^2 \to 0 \quad (\text{as } t \to 0).$$

Thus we conclude from Theorem J and (3.6) that

$$\begin{aligned}\mu_n(\langle (I-f)(Q(f)), J(Q(f) - y_n)\rangle) &\le \limsup_{t\to 0} \mu_n(\langle x_t - f(x_t), J(x_t - y_n)\rangle) \\ &\le 0.\end{aligned}$$

□

Using Proposition 3.1 and the asymptotic regularity on the sequence $\{x_n\}$, we obtain the first main result.

Theorem 3.1. *Let E be a reflexive Banach space with a weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of E and T nonexpansive mappings from C into itself with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ which satisfies the conditions*:

(C1) $\lim_{n\to\infty} \lambda_n = 0$; $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(C2) $\beta_n \in [0, a)$ *for some* $0 < a < 1$ *for all* $n \ge 0$.

Let $f \in \Sigma_C$ and $x_0 \in C$ chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\text{(IS)} \qquad \begin{cases} x_0 = x \in C, \\ y_n = \lambda_n f(x_n) + (1-\lambda_n)Tx_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n Ty_n, \quad n \ge 0. \end{cases}$$

If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $Q(f) \in F(T)$, where $Q(f)$ is the unique solution of the variational inequality

$$\langle (I-f)(Q(f)), J(Q(f) - p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F(T).$$

Proof. We notice that by Theorem J, there exists a solution $Q(f)$ of a variational inequality

$$\langle (I-f)(Q(f)), J(Q(f) - p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F(T).$$

Namely, $Q(f) = \lim_{t\to 0^+} x_t$, where x_t is defined by (R). We will show that $x_n \to Q(f)$.

We proceed with the following steps:

Step 1. $\|x_n - z\| \le \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\}$ for all $n \ge 0$ and all $z \in F(z)$ and so $\{x_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{Tx_n\}$ and $\{Ty_n\}$ are bounded.

Indeed, let $z \in F(T)$. Then we have

$$\begin{aligned}\|y_n - z\| &= \|\lambda_n(f(x_n) - z) + (1-\lambda_n)(Tx_n - z)\| \\ &\le \lambda_n\|f(x_n) - z\| + (1-\lambda_n)\|x_n - z\| \\ &\le \lambda_n(\|f(x_n) - f(z)\| + \|f(z) - z\|) + (1-\lambda_n)\|x_n - z\| \\ &\le \lambda_n k\|x_n - z\| + \lambda_n\|f(z) - z\| + (1-\lambda_n)\|x_n - z\| \\ &= (1-(1-k)\lambda_n))\|x_n - z\| + \lambda_n\|f(z) - z\|\end{aligned}$$

$$\leq \max\left\{\|x_n - z\|, \frac{1}{1-k}\|f(z) - z\|\right\}$$

and

$$\begin{aligned}\|x_{n+1} - z\| &= \|(1-\beta_n)(y_n - z) + \beta_n(Ty_n - z)\| \\ &\leq (1-\beta_n)\|y_n - z\| + \beta_n\|y_n - z\| \\ &= \|y_n - z\| \leq \max\left\{\|x_n - z\|, \frac{1}{1-k}\|f(z) - z\|\right\}.\end{aligned}$$

Using an induction, we obtain

$$\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\right\}$$

for all $n \geq 0$. Hence $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{Tx_n\}$, $\{Ty_n\}$ and $\{f(x_n)\}$. Moreover, it follows from condition (C1) that

$$\|y_n - Tx_n\| = \lambda_n\|f(x_n) - Tx_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.7}$$

Step 2. $\lim_{n \to\infty} \|x_{n+1} - y_n\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Indeed, by the condition (C2)

$$\begin{aligned}\|x_{n+1} - y_n\| &= \beta_n\|Ty_n - y_n\| \\ &\leq \beta_n(\|Ty_n - Tx_n| + \|Tx_n - y_n\|) \\ &\leq a(\|y_n - x_n\| + \|Tx_n - y_n\|) \\ &\leq a(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|Tx_n - y_n\|)\end{aligned}$$

which implies that

$$\|x_{n+1} - y_n\| \leq \frac{a}{1-a}(\|x_{n+1} - x_n\| + \|Tx_n - y_n\|).$$

So, by asymptotic regularity of $\{x_n\}$ and (3.7), we have $\|x_{n+1} - y_n\| \to 0$, and also

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.8}$$

Step 3. $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$ and $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$. By (3.7) and Step 2, we have

$$\begin{aligned}\|y_n - Ty_n\| &\leq \|y_n - Tx_n\| + \|Tx_n - Ty_n\| \\ &\leq \|y_n - Tx_n\| + \|x_n - y_n\| \to 0 \quad (\text{as } n \to \infty).\end{aligned}$$

Also asymptotic regularity of $\{x_n\}$ and (3.8) implies that

$$\begin{aligned}&\|y_{n+1} - y_n\| \\ &\leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - y_n\| \to 0 \quad (\text{as } n \to \infty).\end{aligned}$$

Step 4. $\limsup_{n\to\infty}\langle(I - f)(Q(f)), J(Q(f) - y_n)\rangle \leq 0$. To prove this, put

$$a_n := \langle(I - f)(Q(f)), J(Q(f) - y_n)\rangle, \quad n \geq 0.$$

Then, by $y_n - Ty_n \to 0$ in Step 3, Proposition 3.1 implies that $\mu_n(a_n) \le 0$ for any Banach limit μ. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $y_{n_j} \rightharpoonup p$ for some $p \in E$. From $y_{n+1} - y_n \to 0$ in Step 3, it follows that $y_{n_j+1} \rightharpoonup p$. From the weak sequentially continuity of duality mapping J, we have

$$w - \lim_{j\to\infty} J(Q(f) - y_{n_j+1}) = w - \lim_{j\to\infty}(J(Q(f) - y_{n_j}) = J(Q(f) - p),$$

and so

$$\begin{aligned}&\limsup_{n\to\infty}(a_{n+1} - a_n)\\ &= \lim_{j\to\infty}\langle (I-f)(Q(f)), J(Q(f) - y_{n_j+1}) - J(Q(f) - y_{n_j})\rangle = 0.\end{aligned}$$

Then Lemma 2.3 implies that $\limsup_{n\to\infty} a_n \le 0$, that is

$$\limsup_{n\to\infty}\langle (I-f)(Q(f)), J(Q(f) - y_n)\rangle \le 0.$$

Step 5. $\lim_{n\to\infty}\|x_n - Q(f)\| = 0$. By using (IS), we have

$$\begin{aligned}\|x_{n+1} - Q(f)\| &\le \|y_n - Q(f)\|\\ &= \|\lambda_n(f(x_n) - Q(f)) + (1-\lambda_n)(Tx_n - Q(f))\|.\end{aligned}$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}&\|x_{n+1} - Q(f)\|^2 \le \|y_n - Q(f)\|^2\\ &\le (1-\lambda_n)^2\|Tx_n - Q(f)\|^2 + 2\lambda_n\langle f(x_n) - Q(f), J(y_n - Q(f))\rangle\\ &\le (1-\lambda_n)^2\|x_n - Q(f)\|^2 + 2\lambda_n\langle f(x_n) - f(Q(f)), J(y_n - Q(f))\rangle\\ &\quad + 2\lambda_n\langle f(Q(f)) - Q(f), J(y_n - Q(f))\rangle\\ &\le (1-\lambda_n)^2\|x_n - Q(f)\|^2 + 2k\lambda_n\|x_n - Q(f)\|\|y_n - Q(f)\|\\ &\quad + 2\lambda_n\langle f(Q(f)) - Q(f), J(y_n - Q(f))\rangle\\ &\le (1-\lambda_n)^2\|x_n - Q(f)\|^2 + 2k\lambda_n\|x_n - Q(f)\|^2\\ &\quad + 2k\lambda_n\|x_n - Q(f)\|\|y_n - x_n\|\\ &\quad + 2\lambda_n\langle f(Q(f)) - Q(f), J(y_n - Q(f))\rangle.\end{aligned}$$

It then follows that

(3.9)

$$\begin{aligned}\|x_{n+1} - Q(f)\|^2 &\le (1 - 2(1-k)\lambda_n + \lambda_n^2)\|x_n - Q(f)\|^2\\ &\quad + 2k\lambda_n\|x_n - Q(f)\|\|y_n - x_n\|\\ &\quad + 2\lambda_n\langle f(Q(f)) - Q(f), J(y_n - Q(f))\rangle\\ &\le (1 - 2(1-k)\lambda_n)\|x_n - Q(f)\|^2 + \lambda_n^2 M^2 + 2\lambda_n kM\|y_n - x_n\|\\ &\quad + 2\lambda_n\langle (I-f)(Q(f)), J(Q(f) - y_n)\rangle,\end{aligned}$$

where $M = \sup_{n\geq 0}\|x_n - Q(f)\|$. Put

$$\alpha_n = 2(1-k)\lambda_n,$$
$$\gamma_n = \frac{\lambda_n}{2(1-k)}M^2 + \frac{kM}{1-k}\|y_n - x_n\| + \frac{1}{1-k}\langle (I-f)(Q(f)), J(Q(f)-y_n)\rangle.$$

From the condition (C1), Step 2 and Step 4, it follows that $\alpha_n \to 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, and $\limsup_{n\to\infty}\gamma_n \leq 0$. Since (3.9) reduces to

$$\|x_{n+1} - Q(f)\|^2 \leq (1-\alpha_n)\|x_n - Q(f)\|^2 + \alpha_n\gamma_n,$$

from Lemma 2.1 with $\delta_n = 0$, we conclude that $\lim_{n\to\infty}\|x_n - Q(f)\| = 0$. This completes the proof. □

Remark 3.2. If $\{\lambda_n\}$ and $\{\beta_n\}$ in Theorem 3.1 satisfy conditions

(C1) $\lim_{n\to\infty}\lambda_n = 0$; $\sum_{n=0}^{\infty}\lambda_n = \infty$;
(C2) $\beta_n \in [0, a)$ for some $0 < a < 1$ for all $n \geq 0$;
(C3) $\sum_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$; $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$; or
(C4) $\lim_{n\to\infty}\frac{\lambda_n}{\lambda_{n+1}} = 1$; $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$; or
(C5) $|\lambda_{n+1} - \lambda_n| \leq o(\lambda_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty}\sigma_n < \infty$ (the perturbed control condition); $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

then the sequence $\{x_n\}$ generated by (IS) is asymptotically regular. Now we only give the proof in case when $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the conditions (C1), (C2) and (C5). Indeed, From (IS), we have for every $n \geq 1$,

$$\begin{cases} y_n = \lambda_n f(x_n) + (1-\lambda_n)Tx_n \\ y_{n-1} = \lambda_{n-1} f(x_{n-1}) + (1-\lambda_{n-1})Tx_{n-1}, \end{cases}$$

and so, for every $n \geq 1$, we have

$$\begin{aligned} &\|y_n - y_{n-1}\| \\ &= \|(1-\lambda_n)(Tx_n - Tx_{n-1}) + \lambda_n(f(x_n) - f(x_{n-1}) \\ &\qquad + (\lambda_n - \lambda_{n-1})(f(x_{n-1}) - Tx_{n-1})\| \\ &\leq (1-\lambda_n)\|x_n - x_{n-1}\| + L|\lambda_n - \lambda_{n-1}| + k\lambda_n\|x_n - x_{n-1}\| \\ &= (1-(1-k)\lambda_n)\|x_n - x_{n-1}\| + L|\lambda_n - \lambda_{n-1}|, \end{aligned} \tag{3.10}$$

where $L = \sup\{\|f(x_n) - Tx_n\| : n \geq 0\}$.

On the other hand, by (IS), we also have for every $n \geq 1$,

$$\begin{cases} x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n \\ x_n = (1-\beta_{n-1})y_{n-1} + \beta_{n-1}Ty_{n-1}. \end{cases}$$

Simple calculations show that

$$\begin{aligned} x_{n+1} - x_n = &(1-\beta_n)(y_n - y_{n-1}) + \beta_n(Ty_n - Ty_{n-1}) \\ &+ (\beta_n - \beta_{n-1})(Ty_{n-1} - y_{n-1}). \end{aligned}$$

Then it follows that

$$\begin{aligned}\|x_{n+1} - x_n\| \le (1-\beta_n)\|y_n - y_{n-1}\| + \beta_n\|y_n - y_{n-1}\| \\ + |\beta_n - \beta_{n-1}|\|Ty_{n-1} - y_{n-1}\|.\end{aligned} \tag{3.11}$$

Substituting (3.10) into (3.11) and using the condition (C5), we derive

$$\begin{aligned}\|x_{n+1} - x_n\| &\le (1-(1-k)\lambda_n)\|x_n - x_{n-1}\| \\ &\quad + L|\lambda_n - \lambda_{n-1}| + M|\beta_n - \beta_{n-1}| \\ &\le (1-(1-k)\lambda_n)\|x_n - x_{n-1}\| \\ &\quad + L(o(\lambda_n) + \sigma_{n-1}) + M|\beta_n - \beta_{n-1}|,\end{aligned} \tag{3.12}$$

where $M = \sup\{\|Ty_n - y_n\| : n \ge 0\}$. By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\alpha_n = (1-k)\lambda_n$, $\alpha_n\gamma_n = L\ o(\lambda_n)$ and $\delta_n = L\sigma_{n-1} + M|\beta_n - \beta_{n-1}|$ in (3.12), we have

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\gamma_n + \delta_n.$$

Hence, by the conditions (C1), (C5) and Lemma 2.2, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Moreover, from (3.10) and the condition (C5), it follows that $\lim_{n\to\infty}\|y_n - y_{n-1}\| = 0$.

From this fact, we have the following:

Corollary 3.1. *Let E, C and T be the same as in Theorem 3.1. Let $\{\lambda_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ which satisfies the conditions* (C1), (C2) *and* (C5) (*or the conditions* (C1), (C2) *and* (C3), *or the conditions* (C1), (C2) *and* (C4)), *$f \in \Sigma_C$ and $x_0 \in C$ chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = \lambda_n f(x_n) + (1-\lambda_n)Tx_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n Ty_n, \quad n \ge 0. \end{cases}$$

Then $\{x_n\}$ converges strongly to $Q(f) \in F(T)$, where $Q(f)$ is the unique solution of the variational inequality

$$\langle (I-f)(Q(f)), J(Q(f)-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F(T).$$

Remark 3.3. (1) Theorem 3.1 and Corollary 3.1 improve and complement the corresponding results in Moudafi [14] and Xu [22].

(2) Even $\beta_n = 0$ in (IS), Corollary 3.1 generalizes the corresponding results in Halpern [6], Lions [12], Reich [17, 18], Shioji and Takahashi [19], Wittmann [20] and Xu [21] to the viscosity methods along with the perturb control condition (C5).

Next, we consider the viscosity approximation method with a weakly contractive mapping for the nonexpansive mapping.

Theorem 3.2. *Let E be a reflexive Banach space with a weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of E and T nonexpansive mappings from C into itself with $F(T) \ne \emptyset$. Let $\{\lambda_n\}$*

and $\{\beta_n\}$ *be sequences in* $(0,1)$ *which satisfies the conditions* (C1), (C2) *and* (C5) (*or the conditions* (C1), (C2) *and* (C3), *or the conditions* (C1), (C2) *and* (C4)). *Let* $A : C \to C$ *be a weakly contractive mapping and* $x_0 \in C$ *chosen arbitrarily. Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_0 = x \in C \\ y_n = \lambda_n A x_n + (1-\lambda_n) T x_n \\ x_{n+1} = (1-\beta_n) y_n + \beta_n T y_n, \quad n \geq 0. \end{cases}$$

Then $\{x_n\}$ *converges strongly to* $Q(Ax^*) = x^* \in F(T)$, *where* Q *is a sunny nonexpansive retraction from* C *onto* $F(T)$.

Proof. It follows from Remark 3.1 that $F(T)$ is the sunny nonexpansive retract of C. Denote by Q the sunny nonexpansive retraction of C onto F. Then $Q \circ A$ is a weakly contractive mapping of C into itself. Indeed,

$$\|Q(Ax) - Q(Ay)\| \leq \|Ax - Ay\| \leq \|x - y\| - \psi(\|x - y\|) \text{ for all } x,\ y \in C.$$

Lemma 2.4 assures that there exists a unique element $x^* \in C$ such that $x^* = Q(Ax^*)$. Such a $x^* \in C$ is an element of $F(T)$.

Now we define a iterative scheme as follows:

$$\begin{cases} z_n = \lambda_n A x^* + (1-\lambda_n) T w_n \\ w_{n+1} = (1-\beta_n) z_n + \beta_n T z_n, \quad n \geq 0. \end{cases} \tag{3.13}$$

Let $\{w_n\}$ be the sequence generated by (3.13). Then Corollary 3.1 with $f = Ax^*$ a constant assures that $\{w_n\}$ converges strongly to $Q(Ax^*) = x^*$ as $n \to \infty$. For any n, we have

$$\begin{aligned} &\|x_{n+1} - w_{n+1}\| \\ &\leq (1-\beta_n)\|y_n - z_n\| + \beta_n \|Ty_n - Tz_n\| \\ &\leq \|y_n - z_n\| \\ &\leq \lambda_n \|Ax_n - Ax^*\| + (1-\lambda_n)\|Tx_n - Tw_n\| \\ &\leq \lambda_n(\|Ax_n - Aw_n\| + \|Aw_n - Ax^*\|) + (1-\lambda_n)\|x_n - w_n\| \\ &\leq \|x_n - y_n\| - \lambda_n \psi(\|x_n - w_n\|) + \lambda_n(\|w_n - x^*\| - \psi(\|w_n - x^*\|)) \\ &\leq \|x_n - w_n\| - \lambda_n \psi(\|x_n - w_n\|) + \lambda_n \|w_n - x^*\|. \end{aligned}$$

Thus, we obtain for $s_n = \|x_n - w_n\|$ the following recursive inequality:

$$s_{n+1} \leq s_n - \lambda_n \psi(s_n) + \lambda_n \|w_n - x^*\|.$$

Since $\|w_n - x^*\| \to 0$, it follows from Lemma 2.5 that $\lim_{n\to\infty} \|x_n - w_n\| = 0$. Hence

$$\lim_{n\to\infty} \|x_n - x^*\| \leq \lim_{n\to\infty} (\|x_n - w_n\| + \|w_n - x^*\|) = 0.$$

This completes the proof. □

Remark 3.4. Theorem 3.2 (and Corollary 3.3) develops and complements the corresponding results in Cho et al. [3], Halpern [6], Lions [12], Moudafi [14], Reich [17, 18], Shioji and Takahashi [19], Wittmann [20] and Xu [21, 22].

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Department of Mathematics
Dong-A University
Busan 604-714, Korea
E-mail address: jungjs@mail.donga.ac.kr