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# REPRESENTATIONS OF THE MOORE-PENROSE INVERSE OF $2 \times 2$ BLOCK OPERATOR VALUED MATRICES

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Abstract. We obtain necessary and sufficient conditions for $2 \times 2$ block operator valued matrices to be Moore-Penrose (MP) invertible and give new representations of such MP inverses in terms of the individual blocks.

## 1. Introduction

The Moore-Penrose inverse (for short MP inverse) has proved helpful in systems theory, difference equations, differential equations and iterative procedures. It would be useful if these results could be extended to infinite dimensional situations. Applications could then be made to denumerable systems theory, abstract Cauchy problems, infinite systems of linear differential equations, and possibly partial differential equations and other interesting subjects (see, for example [1, 2] and [7, 8]).

Let $\mathcal{H}$ and $\mathcal{K}$ be separable, infinite dimensional and complex Hilbert spaces. Denote by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. For an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $A^*$ denote the range, the null space and the adjoint of $A$, respectively. The identity onto a closed subspace $\mathcal{M}$ is denoted by $I_{\mathcal{M}}$ or $I$ if there is no confusion. For $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, if there exists an operator $T^+ \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying the following four operator equations

$$(1) \qquad TT^+T = T, \quad T^+TT^+ = T^+, \quad TT^+ = (TT^+)^*, \quad T^+T = (T^+T)^*,$$

then $T^+$ is called the MP inverse of $T$. It is well known that $T$ has the MP inverse if and only if $\mathcal{R}(T)$ is closed and the MP inverse of $T$ is unique (see [8, 11, 16]).

In recent years, representations and characterizations of the MP inverse for matrices or operators on a Hilbert space have been considered by many authors (see [1, 2], [5], [8, 9, 10, 11, 12, 13, 14, 15, 16]). In this paper, we are mainly

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interested in MP invertibilities and representations of the MP inverse for $2\times 2$ block operator valued matrices with specified properties on a Hilbert space. Applying these results, we can obtain the MP inverses of $2\times 2$ block operator valued matrices with specified properties.

## 2. Some lemmas

In this section we shall give some lemmas.

**Lemma 1** ([3, 4])**.** *Let $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ have a closed range. Then $A$ has the form*

$$A=\begin{pmatrix} A_1 & 0\\ 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{R}(A^*)\\ \mathcal{N}(A)\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{N}(A^*)\end{pmatrix},\tag{2}$$

*where $A_1$ is invertible. In this case $A^+=A_1^{-1}\oplus 0$.*

**Lemma 2** ([6, 9])**.** *Let $A$ and $B$ be in $\mathcal{B}(\mathcal{H})$. Then*
(1) $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}((AA^*+BB^*)^{\frac{1}{2}})$.
(2) *$\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A)=\mathcal{R}(AA^*)$.*
(3) *If $A\geq 0$ is a positive operator in $\mathcal{B}(\mathcal{H})$, then $\overline{\mathcal{R}(A^{\frac{1}{2}})}=\overline{\mathcal{R}(A)}$.*

**Lemma 3.** *The $2\times 2$ block operator valued matrix $\begin{pmatrix} A & B\\ 0 & 0\end{pmatrix}$ is MP invertible if and only if $\mathcal{R}(A)+\mathcal{R}(B)$ is closed, and*

$$\begin{pmatrix} A & B\\ 0 & 0\end{pmatrix}^+=\begin{pmatrix} A^*(AA^*+BB^*)^+ & 0\\ B^*(AA^*+BB^*)^+ & 0\end{pmatrix}.\tag{3}$$

*Proof.* Put $T=\begin{pmatrix} A & B\\ 0 & 0\end{pmatrix}$. Then $\mathcal{R}(T)=\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(AA^*+BB^*)^{\frac{1}{2}}$. This implies that $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(AA^*+BB^*)$ is closed by Lemma 2. So $(AA^*+BB^*)^+$ exists if $T^+$ exists. From $T^+=T^*(TT^*)^+$ we have

$$T^+=\begin{pmatrix} A^* & 0\\ B^* & 0\end{pmatrix}\begin{pmatrix} (AA^*+BB^*)^+ & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix} A^*(AA^*+BB^*)^+ & 0\\ B^*(AA^*+BB^*)^+ & 0\end{pmatrix}. \qquad\square$$

Additionally, we include some formulae here for later use.

**Corollary 4.** (1) *The $2\times 2$ block operator valued matrix $\begin{pmatrix} A & 0\\ B & 0\end{pmatrix}$ is MP invertible if and only if $\mathcal{R}(A^*)+\mathcal{R}(B^*)$ is closed, and*

$$\begin{pmatrix} A & 0\\ B & 0\end{pmatrix}^+=\begin{pmatrix} (A^*A+B^*B)^+A^* & (A^*A+B^*B)^+B^*\\ 0 & 0\end{pmatrix}.\tag{4}$$

(2) *The $2\times 2$ block operator valued matrix $\begin{pmatrix} 0 & 0\\ B & A\end{pmatrix}$ is MP invertible if and only if $\mathcal{R}(A)+\mathcal{R}(B)$ is closed, and*

$$\begin{pmatrix} 0 & 0\\ B & A\end{pmatrix}^+=\begin{pmatrix} 0 & B^*(AA^*+BB^*)^+\\ 0 & A^*(AA^*+BB^*)^+\end{pmatrix}.\tag{5}$$

Let $A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$. The next lemma gives the MP inverse representation of the $2 \times 2$ block upper triangular operator valued matrix $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ in the case that $A$ or $B$ is invertible.

**Lemma 5.** *Let $A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K}), C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $B$ be invertible. Then the $2 \times 2$ block operator valued matrix $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is MP invertible if and only if $\mathcal{R}(A)$ is closed, and*

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{+} = \begin{pmatrix} A^{+} - A^{+}C\triangle C^{*}(I - AA^{+}) & -A^{+}C\triangle B^{*} \\ \triangle C^{*}(I - AA^{+}) & \triangle B^{*} \end{pmatrix},$$

*where* $\triangle = (B^{*}B + C^{*}(I - AA^{+})C)^{-1}$.

*Proof.* First, by Corollary 4, for an arbitrary invertible operator $M$, $\left(\begin{smallmatrix} 0 & N \\ 0 & M \end{smallmatrix}\right)$ is MP invertible and

$$\begin{aligned} \begin{pmatrix} 0 & N \\ 0 & M \end{pmatrix}^{+} &= \left(\begin{pmatrix} 0 & 0 \\ N^{*} & M^{*} \end{pmatrix}^{+}\right)^{*} \\ &= \begin{pmatrix} 0 & 0 \\ (N^{*}N + M^{*}M)^{-1}N^{*} & (N^{*}N + M^{*}M)^{-1}M^{*} \end{pmatrix}. \end{aligned}$$

Second, let $B$ be invertible. Since $\mathcal{R}(A)$ is closed, $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ has the form

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & 0 & C_1 \\ 0 & A_1 & C_2 \\ 0 & 0 & B \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A) \\ \mathcal{R}(A^{*}) \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(A^{*}) \\ \mathcal{R}(A) \\ \mathcal{K} \end{pmatrix},$$

where $A_1$ as an operator from $\mathcal{R}(A^{*})$ onto $\mathcal{R}(A)$ is invertible. Now, let $N = (0, C_1)$, $M = \left(\begin{smallmatrix} A_1 & C_2 \\ 0 & B \end{smallmatrix}\right)$ and $\triangle = (B^{*}B + C^{*}(I - AA^{+})C)^{-1} = (B^{*}B + C_1^{*}C_1)^{-1}$. It is easy to check that

$$\begin{aligned} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{+} &= \begin{pmatrix} 0 & 0 \\ (N^{*}N + M^{*}M)^{-1}N^{*} & (N^{*}N + M^{*}M)^{-1}M^{*} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ -A_1^{-1}C_2\triangle C_1^{*} & A_1^{-1} & -A_1^{-1}C_2\triangle B^{*} \\ \triangle C_1^{*} & 0 & \triangle B^{*} \end{pmatrix} \\ &= \begin{pmatrix} A^{+} - A^{+}C\triangle C^{*}(I - AA^{+}) & -A^{+}C\triangle B^{*} \\ \triangle C^{*}(I - AA^{+}) & \triangle B^{*} \end{pmatrix}. \end{aligned}$$

□

Similar to the proof of Lemma 5, we have the following result.

**Lemma 6.** *Let $A \in \mathcal{B}(\mathcal{H}), C \in \mathcal{B}(\mathcal{K},\mathcal{H}), B \in \mathcal{B}(\mathcal{K})$ and $A$ be invertible. Then the $2 \times 2$ block operator valued matrix $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is MP invertible if and only if $\mathcal{R}(B)$ is closed, and*

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{+} = \begin{pmatrix} A^{*}\triangle & -A^{*}\triangle CB^{+} \\ (I - B^{+}B)C^{*}\triangle & B^{+} - (I - B^{+}B)C^{*}\triangle CB^{+} \end{pmatrix},$$

*where* $\triangle = (AA^* + C(I - B^+B)C^*)^{-1}$.

**Lemma 7.** *Let* $A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{H},\mathcal{K}), C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ *and* $B$ *be invertible. Then the* $2 \times 2$ *block operator valued matrix* $\left(\begin{smallmatrix} A & C \\ B & 0 \end{smallmatrix}\right)$ *is MP invertible if and only if* $\mathcal{R}(C)$ *is closed, and*

$$\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}^+ = \begin{pmatrix} \triangle A^*(I - CC^+) & \triangle B^* \\ C^+ - C^+A\triangle A^*(I - CC^+) & -C^+A\triangle B^* \end{pmatrix},$$

*where* $\triangle = (B^*B + A^*(I - CC^+)A)^{-1}$.

## 3. The MP inverses of $2 \times 2$ block operator valued matrices

In this section we will give the MP inverse representations of $2 \times 2$ block operator valued matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{6}$$

where $A \in \mathcal{B}(\mathcal{H}), D \in \mathcal{B}(\mathcal{K})$, $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$.

Let us recall that operators $S, T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ are said to be $*$-orthogonality, denoted by $S \perp^* T$, whenever $ST^* = 0$ and $S^*T = 0$ (see [4]). If $S \perp^* T$, then it is easy to check that $(S+T)^+ = S^+ + T^+$. From this result we can get the following results.

**Theorem 8.** *Let* $M$ *be defined as Eqn.*(6).

(1) *If* $AC^* + BD^* = 0$, $\mathcal{R}(A) + \mathcal{R}(B)$ *and* $\mathcal{R}(C) + \mathcal{R}(D)$ *are closed, then* $M$ *is MP invertible and*

$$M^+ = \begin{pmatrix} A^*(AA^* + BB^*)^+ & C^*(DD^* + CC^*)^+ \\ B^*(AA^* + BB^*)^+ & D^*(DD^* + CC^*)^+ \end{pmatrix}.$$

(2) *If* $A^*B + C^*D = 0$, $\mathcal{R}(A^*) + \mathcal{R}(C^*)$ *and* $\mathcal{R}(B^*) + \mathcal{R}(D^*)$ *are closed, then* $M$ *is MP invertible and*

$$M^+ = \begin{pmatrix} (A^*A + C^*C)^+A^* & (A^*A + C^*C)^+C^* \\ (D^*D + B^*B)^+B^* & (D^*D + B^*B)^+D^* \end{pmatrix}.$$

*Proof.* Let

$$S = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}.$$

Since $\mathcal{R}(A) + \mathcal{R}(B)$ and $\mathcal{R}(C) + \mathcal{R}(D)$ are closed, by Lemma 3 and Corollary 4 we have

$$S^+ = \begin{pmatrix} A^*(AA^* + BB^*)^+ & 0 \\ B^*(AA^* + BB^*)^+ & 0 \end{pmatrix}, \quad T^+ = \begin{pmatrix} 0 & C^*(DD^* + CC^*)^+ \\ 0 & D^*(DD^* + CC^*)^+ \end{pmatrix}.$$

From $AC^* + BD^* = 0$ we get that $S \perp^* T$. So

$$M^+ = \begin{pmatrix} A^*(AA^* + BB^*)^+ & C^*(DD^* + CC^*)^+ \\ B^*(AA^* + BB^*)^+ & D^*(DD^* + CC^*)^+ \end{pmatrix}.$$

(2) Similar to the proof of (1), the details are omitted. $\square$

If we set $S' = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ and $T' = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ such that $S' \perp^* T'$, we can get the following results.

**Theorem 9.** *Let $M$ be defined as Eqn.(6), $\mathcal{R}(A)$, $\mathcal{R}(D)$ be closed such that $AC^* = 0$ and $D^*C = 0$.*

*(1) If $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, then $M$ is MP invertible if and only if $\mathcal{R}(C)$ and $\mathcal{R}(B_0)$ are closed and*

$$M^+ = \begin{pmatrix} A^+ & C^+ \\ B_0^+ + (D^+D + B_0^+B_0 - B_0^+B)\triangle(B - B_0)^*(I - B_0B_0^+) & (D^+D + B_0^+B_0 - B_0^+B)\triangle D^* \end{pmatrix},$$

*where $\triangle = (D^*D + (B - B_0)^*(I - B_0B_0^+)(B - B_0))^+, B_0 = (I - AA^+)B(I - D^+D)$.*

*(2) If $\mathcal{R}(D^*) \cap \mathcal{R}(B^*) = \{0\}$, then $M$ is MP invertible if and only if $\mathcal{R}(C)$ and $\mathcal{R}(B_0)$ are closed and*

$$M^+ = \begin{pmatrix} A^*\triangle(AA^+ + B_0B_0^+ - BB_0^+) & C^+ \\ B_0^+ + (I - B_0^+B_0)(B - B_0)^*\triangle(AA^+ + B_0B_0^+ - BB_0^+) & D^+ \end{pmatrix},$$

*where $\triangle = (AA^* + (B - B_0)(I - B_0^+B_0)(B - B_0)^*)^+$, $B_0 = (I - AA^+)B(I - D^+D)$.*

*(3) If $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ and $\mathcal{R}(D^*) \cap \mathcal{R}(B^*) = \{0\}$, then $M$ is MP invertible if and only if $\mathcal{R}(C)$ and $\mathcal{R}(B)$ are closed and*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^+ = \begin{pmatrix} A^+ & C^+ \\ B^+ & D^+ \end{pmatrix}.$$

*Proof.* (1) Since $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, $\mathcal{R}(A)$ and $\mathcal{R}(D)$ are closed, $S'$ has the form

(7)
$$S' = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & 0 & B_1 & B_2 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A) \\ \mathcal{R}(A^*) \\ \mathcal{R}(D^*) \\ \mathcal{N}(D) \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(A^*) \\ \mathcal{R}(A) \\ \mathcal{R}(D) \\ \mathcal{N}(D^*) \end{pmatrix}.$$

From Lemma 3 and Lemma 7 we know that $S'$ is MP invertible if and only if

$$\mathcal{R}(B_2) = \mathcal{R}((I - AA^+)B(I - D^+D))$$

is closed and $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^+ = \begin{pmatrix} 0 & 0 \\ \mathcal{T}^+ & 0 \end{pmatrix}$, where $\mathcal{T} = \begin{pmatrix} 0 & B_1 & B_2 \\ A_1 & 0 & 0 \\ 0 & D_1 & 0 \end{pmatrix}$.

If we replace $A, B$ and $C$ by $(0, B_1)$, $A_1 \oplus D_1$ and $B_2$ in Lemma 5, respectively, then we have

$$\mathcal{T}^+ = \begin{pmatrix} 0 & A_1^{-1} & 0 \\ \triangle' B_1^*(I - B_2B_2^+) & 0 & \triangle' D_1^* \\ B_2^+ - B_2^+B_1\triangle' B_1^*(I - B_2B_2^+) & 0 & -B_2^+B_1\triangle' D_1^* \end{pmatrix},$$

where $\triangle' = (D_1^* D_1 + B_1^*(I - B_2 B_2^+)B_1)^{-1}$. Hence

$$\begin{aligned}
&\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^+ \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_1^{-1} & 0 & 0 \\ \triangle' B_1^*(I - B_2 B_2^+) & 0 & \triangle' D_1^* & 0 \\ B_2^+ - B_2^+ B_1 \triangle' B_1^*(I - B_2 B_2^+) & 0 & -B_2^+ B_1 \triangle' D_1^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} A^+ & 0 \\ \begin{matrix} B_0^+ + (D^+D + B_0^+ B_0 - \\ B_0^+ B)\triangle(B - B_0)^*(I - B_0 B_0^+) \end{matrix} & (D^+D + B_0^+ B_0 - B_0^+ B)\triangle D^* \end{pmatrix},
\end{aligned}$$

where $\triangle = (D^*D + (B - B_0)^*(I - B_0 B_0^+)(B - B_0))^+, B_0 = (I - AA^+)B(I - D^+D)$.

Since $AC^* = 0, D^*C = 0$, we have $S' \perp^* T'$. So

$$\begin{aligned}
&\begin{pmatrix} A & B \\ C & D \end{pmatrix}^+ \\
&= \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^+ + \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}^+ \\
&= \begin{pmatrix} A^+ & C^+ \\ \begin{matrix} B_0^+ + (D^+D + B_0^+ B_0 - \\ B_0^+ B)\triangle(B - B_0)^*(I - B_0 B_0^+) \end{matrix} & (D^+D + B_0^+ B_0 - B_0^+ B)\triangle D^* \end{pmatrix}.
\end{aligned}$$

(2) Similar to the proof of (1), the details are omitted.

(3) Note that if $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ and $\mathcal{R}(D^*) \cap \mathcal{R}(B^*) = \{0\}$, then $B_1 = 0$ in Eqn.(7). □

If we set $S_0 = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and $T_0 = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ such that $S_0 \perp^* T_0$, we can get the following results.

**Theorem 10.** *Let $M$ be defined as Eqn.(6), $\mathcal{R}(B)$, $\mathcal{R}(C)$ be closed such that $BD^* = 0$ and $C^*D = 0$.*

(1) *If $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, then $M$ is MP invertible if and only if $\mathcal{R}(A_0)$ and $\mathcal{R}(D)$ are closed and*

$$M^+ = \begin{pmatrix} \begin{matrix} A_0^+ + (C^+C + A_0^+ A_0 - \\ A_0^+ A)\triangle_0(A - A_0)^*(I - A_0 A_0^+) \end{matrix} & (C^+C + A_0^+ A_0 - A_0^+ A)\triangle_0 C^* \\ B^+ & D^+ \end{pmatrix},$$

*where $\triangle_0 = (C^*C + (A - A_0)^*(I - A_0 A_0^+)(A - A_0))^+, A_0 = (I - BB^+)A(I - C^+C)$.*

(2) *If* $\mathcal{R}(A^*) \cap \mathcal{R}(C^*) = \{0\}$, *then* $M$ *is MP invertible if and only if* $\mathcal{R}(A_0)$ *and* $\mathcal{R}(D)$ *are closed and*

$$M^+ = \begin{pmatrix} A_0^+ + (I - A_0^+ A_0)(A - A_0)^* \triangle_0 (BB^+ + A_0 A_0^+ - AA_0^+) & C^+ \\ B^* \triangle_0 (BB^+ + A_0 A_0^+ - AA_0^+) & D^+ \end{pmatrix},$$

*where* $\triangle = (BB^* + (A - A_0)(I - A_0 A_0^+)(A - A_0)^*)^+$, $A_0 = (I - BB^+)A(I - C^+C)$.

(3) *If* $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ *and* $\mathcal{R}(A^*) \cap \mathcal{R}(C^*) = \{0\}$, *then* $M$ *is MP invertible if and only if* $\mathcal{R}(A)$ *and* $\mathcal{R}(D)$ *is closed and*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^+ = \begin{pmatrix} A^+ & C^+ \\ B^+ & D^+ \end{pmatrix}.$$

*Proof.* (1) Since $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, $\mathcal{R}(B)$ and $\mathcal{R}(C)$ are closed, $S_0$ has the form
(8)

$$S_0 = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(C) \\ \mathcal{R}(C^*) \\ \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(B^*) \\ \mathcal{R}(B) \\ \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{pmatrix}.$$

From Lemma 5 we have $S_0$ is MP inverse if and only if

$$\mathcal{R}(A_1) = \mathcal{R}((I - BB^+)A(I - C^+C))$$

is closed and $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^+ = \begin{pmatrix} \mathcal{T}_0^+ & 0 \\ 0 & 0 \end{pmatrix}$, where $\mathcal{T}_0 = \begin{pmatrix} A_1 & A_2 & 0 \\ 0 & 0 & B_1 \\ 0 & C_1 & 0 \end{pmatrix}$. By Lemma 5 we have

$$\mathcal{T}_0^+ = \begin{pmatrix} A_1^+ - A_1^+ A_2 \triangle_0' A_2^* (I - A_1 A_1^+) & 0 & -A_1^+ A_2 \triangle_0' C_1^* \\ \triangle_0' A_2^* (I - A_1 A_1^+) & 0 & \triangle_0' C_1^* \\ 0 & B_1^{-1} & 0 \end{pmatrix},$$

where $\triangle_0' = (C_1^* C_1 + A_2^* (I - A_1 A_1^+) A_2)^{-1}$. Hence

$$\begin{aligned} &\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^+ \\ &= \begin{pmatrix} A_1^+ - A_1^+ A_2 \triangle_0' A_2^* (I - A_1 A_1^+) & 0 & -A_1^+ A_2 \triangle_0' C_1^* & 0 \\ \triangle_0' A_2^* (I - A_1 A_1^+) & 0 & \triangle_0' C_1^* & 0 \\ 0 & B_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{matrix} A_0^+ + (C^+C + A_0^+ A_0 - \\ A_0^+ A) \triangle_0 (A - A_0)^* (I - A_0 A_0^+) \end{matrix} & (C^+C + A_0^+ A_0 - A_0^+ A) \triangle_0 C^* \\ B^+ & 0 \end{pmatrix}, \end{aligned}$$

where $\triangle_0 = (C^*C + (A - A_0)^*(I - A_0 A_0^+)(A - A_0))^+, A_0 = (I - BB^+)A(I - C^+C)$.

Since $BD^* = 0$ and $C^*D = 0$, we have $S_0 \perp^* T_0$. So

$$\begin{aligned}
&\begin{pmatrix} A & B \\ C & D \end{pmatrix}^+ \\
&= \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^+ + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}^+ \\
&= \begin{pmatrix} \begin{array}{c} A_0^+ + (C^+C + A_0^+A_0 - \\ A_0^+A)\triangle_0(A - A_0)^*(I - A_0A_0^+) \\ B^+ \end{array} & \begin{array}{c} (C^+C + A_0^+A_0 - A_0^+A)\triangle_0 C^* \\ D^+ \end{array} \end{pmatrix}.
\end{aligned}$$

Similarly, we can prove (2) and (3), so the details are omitted. □

Let $A$ and $D$ be MP invertible. Denoted by

$$X_1 = \mathcal{R}(A^*),\ X_2 = \mathcal{N}(A),\ X_3 = \mathcal{R}(D^*),\ X_4 = \mathcal{N}(D),$$
$$Y_1 = \mathcal{R}(A),\ Y_2 = \mathcal{N}(A^*),\ Y_3 = \mathcal{R}(D),\ Y_4 = \mathcal{N}(D^*)$$

and

$$I_0 = I \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \oplus I.$$

Then $M$ as an operator from $\sum_{i=1}^4 X_i$ into $\sum_{i=1}^4 Y_i$ has the following operator matrix form

$$M = \begin{pmatrix} A_1 & 0 & B_1 & B_3 \\ 0 & 0 & B_4 & B_2 \\ C_1 & C_3 & D_1 & 0 \\ C_4 & C_2 & 0 & 0 \end{pmatrix} = I_0^* \begin{pmatrix} A_1 & B_1 & 0 & B_3 \\ C_1 & D_1 & C_3 & 0 \\ 0 & B_4 & 0 & B_2 \\ C_4 & 0 & C_2 & 0 \end{pmatrix} I_0. \tag{9}$$

where $A_1$ and $D_1$ are invertible. Put

$$\begin{aligned} A_0 &= \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & B_3 \\ C_3 & 0 \end{pmatrix}, \\ C_0 &= \begin{pmatrix} 0 & B_4 \\ C_4 & 0 \end{pmatrix}, \quad D_0 = \begin{pmatrix} 0 & B_2 \\ C_2 & 0 \end{pmatrix}. \end{aligned} \tag{10}$$

Then

$$M = I_0^* \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} I_0,\ M^+ = I_0^* \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}^+ I_0.$$

The generalized Schur complement (see [13, 17]) plays an important role in the study of the MP invertibilities. Next we give some expressions according to the generalized Schur complement. We use some notations. Let

$$K = AA^+B(I - D^+D),\ \ H = DD^+C(I - A^+A),\ \ E = (I - AA^+)B(I - D^+D),$$
$$F = (I - DD^+)C(I - A^+A),\ \ S = DD^+(D - CA^+B)D^+D,$$

$$R = \begin{pmatrix} A^+ + A^+BS^+CA^+ & -A^+BS^+ \\ -S^+CA^+ & S^+ \end{pmatrix}. \tag{11}$$

Then we have the following general result.

**Theorem 11.** *Let $S$ as an operator from $\mathcal{R}(D^*)$ onto $\mathcal{R}(D)$ be invertible, $\mathcal{R}(A)$ and $\mathcal{R}(D)$ be closed such that*

$$(I - AA^+)BD^+D = 0,\ (I - DD^+)CA^+A = 0.$$

*Then $M$ is MP invertible if and only if $\mathcal{R}(E)$ and $\mathcal{R}(F)$ are closed and*

$$\begin{aligned} M^+ &= \begin{pmatrix} 0 & F^+ \\ E^+ & 0 \end{pmatrix} + \left[ (R^*)^+ + \begin{pmatrix} 0 & (I - F^+F)H^* \\ (I - E^+E)K^* & 0 \end{pmatrix} \right] \\ &\quad \times R^* \left[ I + R \begin{pmatrix} K(I - E^+E)K^* & 0 \\ 0 & H(I - F^+F)H^* \end{pmatrix} R^* \right]^+ \\ &\quad \times R \begin{pmatrix} I - KE^+ & 0 \\ 0 & I - HF^+ \end{pmatrix}. \end{aligned}$$

*Proof.* Note that

$$\begin{aligned} D - CA^+B &= \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} C_1 & C_3 \\ C_4 & C_2 \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_1 & B_3 \\ B_4 & B_2 \end{pmatrix} \\ &= \begin{pmatrix} D_1 - C_1A_1^{-1}B_1 & -C_1A_1^{-1}B_3 \\ -C_4A_1^{-1}B_1 & -C_4A_1^{-1}B_3 \end{pmatrix}. \end{aligned}$$

From $D - CA^+B$ as an operator from $\mathcal{R}(D^*)$ onto $\mathcal{R}(D)$ invertible, we obtain that $D_1 - C_1A_1^{-1}B_1$ is invertible. So $A_0$ is invertible and

(12)
$$A_0^{-1} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}^{-1} = \begin{pmatrix} A_1^{-1} + A_1^{-1}B_1S_1^{-1}C_1A_1^{-1} & -A_1^{-1}B_1S_1^{-1} \\ -S_1^{-1}C_1A_1^{-1} & S_1^{-1} \end{pmatrix},$$

where $S_1 = D_1 - C_1A_1^{-1}B_1$. Let $R$ be defined as Eqn.(11), a direct calculation shows that

$$R = I_0^* \begin{pmatrix} A_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} I_0.$$

Note that

$$\begin{aligned} I_0^* \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix} I_0 &= \begin{pmatrix} 0 & AA^+B(I - D^+D) \\ DD^+C(I - A^+A) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & K \\ H & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} I_0^* \begin{pmatrix} 0 & 0 \\ 0 & D_0 \end{pmatrix} I_0 &= \begin{pmatrix} 0 & (I - AA^+)B(I - D^+D) \\ (I - DD^+)C(I - A^+A) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix}, \end{aligned}$$

and

$$I_0^*\begin{pmatrix} 0 & 0 \\ 0 & D_0^+ \end{pmatrix} I_0 = \begin{pmatrix} 0 & F^+ \\ E^+ & 0 \end{pmatrix},$$

$$I_0^*\begin{pmatrix} I & 0 \\ 0 & I-D_0^+D_0 \end{pmatrix} I_0 = \begin{pmatrix} I-F^+F & 0 \\ 0 & I-E^+E \end{pmatrix}.$$

From $(I-AA^+)BD^+D=0$ and $(I-DD^+)CA^+A=0$ we obtain $B_4=0$ and $C_4=0$. Hence $C_0=0$. Put $\triangle_0=(A_0A_0^*+B_0(I-D_0^+D_0)B_0^*)^{-1}$. Then

$$\begin{aligned}
& I_0^*\begin{pmatrix} \triangle_0 & 0 \\ 0 & 0 \end{pmatrix} I_0 \\
&= I_0^*\begin{pmatrix} (A_0^*)^{-1}[I+(A_0)^{-1}B_0(I-D_0^+D_0)B_0^*(A_0^*)^{-1}]^{-1}(A_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} I_0 \\
&= I_0^*\begin{pmatrix} (A_0^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix}\left[I+\begin{pmatrix} A_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I-D_0^+D_0 \end{pmatrix}\right. \\
&\quad \left.\times\begin{pmatrix} 0 & 0 \\ B_0^* & 0 \end{pmatrix}\begin{pmatrix} (A_0^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right]^+\begin{pmatrix} A_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} I_0 \\
&= R^*\left(I+R\begin{pmatrix} 0 & K \\ H & 0 \end{pmatrix}\begin{pmatrix} I-F^+F & 0 \\ 0 & I-E^+E \end{pmatrix}\begin{pmatrix} 0 & H^* \\ K^* & 0 \end{pmatrix}R^*\right)^+ R \\
&= R^*\left[I+R\begin{pmatrix} K(I-E^+E)K^* & 0 \\ 0 & H(I-F^+F)H^* \end{pmatrix}R^*\right]^+ R.
\end{aligned}$$

By Lemma 6, we have

$$\begin{aligned}
M^+ &= I_0^*\begin{pmatrix} A_0 & B_0 \\ 0 & D_0 \end{pmatrix}^+ I_0 \\
&= I_0^*\begin{pmatrix} A_0^*\triangle_0 & -A_0^*\triangle_0B_0D_0^+ \\ (I-D_0^+D_0)B_0^*\triangle_0 & D_0^+-(I-D_0^+D_0)B_0^*\triangle_0B_0D_0^+ \end{pmatrix} I_0 \\
&= I_0^*\left\{\begin{pmatrix} 0 & 0 \\ 0 & D_0^+ \end{pmatrix}+\left[\begin{pmatrix} A_0^* & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} I & 0 \\ 0 & I-D_0^+D_0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B_0^* & 0 \end{pmatrix}\right]\right. \\
&\quad \left.\times\begin{pmatrix} \triangle_0 & 0 \\ 0 & 0 \end{pmatrix}\left[I-\begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & D_0^+ \end{pmatrix}\right]\right\} I_0 \\
&= \begin{pmatrix} 0 & F^+ \\ E^+ & 0 \end{pmatrix}+\left[(R^*)^++\begin{pmatrix} I-F^+F & 0 \\ 0 & I-E^+E \end{pmatrix}\begin{pmatrix} 0 & H^* \\ K^* & 0 \end{pmatrix}\right] \\
&\quad \times R^*\left[I+R\begin{pmatrix} K(I-E^+E)K^* & 0 \\ 0 & H(I-F^+F)H^* \end{pmatrix}R^*\right]^+ \\
&\quad \times R\left[I-\begin{pmatrix} 0 & K \\ H & 0 \end{pmatrix}\begin{pmatrix} 0 & F^+ \\ E^+ & 0 \end{pmatrix}\right]
\end{aligned}$$

$$= \begin{pmatrix} 0 & F^+ \\ E^+ & 0 \end{pmatrix} + \left[(R^*)^+ + \begin{pmatrix} 0 & (I-F^+F)H^* \\ (I-E^+E)K^* & 0 \end{pmatrix}\right]$$

$$\times R^* \left[I + R\begin{pmatrix} K(I-E^+E)K^* & 0 \\ 0 & H(I-F^+F)H^* \end{pmatrix} R^*\right]^+$$

$$\times R \begin{pmatrix} I-KE^+ & 0 \\ 0 & I-HF^+ \end{pmatrix}. \qquad \square$$

In Campbell and Meyer's book, they stated that the MP inverse of an upper block triangular matrix $\mathcal{T} = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ is still an upper block triangular if and only if $\mathcal{R}(B) \subset \mathcal{R}(A)$ and $\mathcal{R}(B^*) \subset \mathcal{R}(D^*)$. We can show this result holds in the infinite dimensional case and is a very special case of Theorem 11.

**Corollary 12.** *Let* $A \in \mathcal{B}(\mathcal{H}), D \in \mathcal{B}(\mathcal{K}), B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $\mathcal{R}(A)$ *and* $\mathcal{R}(D)$ *be closed. Then* $\mathcal{T}^+ = \left(\begin{smallmatrix} A^+ & -A^+BD^+ \\ 0 & D^+ \end{smallmatrix}\right)$ *if and only if* $\mathcal{R}(B) \subset \mathcal{R}(A)$ *and* $\mathcal{R}(B^*) \subset \mathcal{R}(D^*)$.

*Proof.* ($\Longleftarrow$) If $\mathcal{R}(B) \subset \mathcal{R}(A)$ and $\mathcal{R}(B^*) \subset \mathcal{R}(D^*)$, by Theorem 11 we have $K, H, E$ and $F$ are all equal to 0, $S = D$ and

$$R = \begin{pmatrix} A^+ & -A^+BD^+ \\ 0 & D^+ \end{pmatrix}.$$

So

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^+ = (R^*)^+R^*R = (RR^+)^*R = RR^+R = R.$$

($\Longrightarrow$) Since

$$\mathcal{T}\mathcal{T}^+ = \begin{pmatrix} AA^+ & -AA^+BD^+ + BD^+ \\ 0 & DD^+ \end{pmatrix} \quad \text{and}$$

$$\mathcal{T}^+\mathcal{T} = \begin{pmatrix} A^+A & -A^+BD^+D + A^+B \\ 0 & D^+D \end{pmatrix}$$

are selfadjoint, we have $-AA^+BD^+ + BD^+ = 0$ and $-A^+BD^+D + A^+B = 0$. From $\mathcal{T}\mathcal{T}^+\mathcal{T} = \mathcal{T}$, we have $B = AA^+B = BD^+D$. Hence $\mathcal{R}(B) \subset \mathcal{R}(A)$ and $\mathcal{R}(B^*) \subset \mathcal{R}(D^*)$. $\square$

## 4. Concluding remarks

In this paper, we derive formulae for the MP inverse of an operator matrix $M$ under some new conditions. It seems that the general representations without any conditions is difficult to find. Finally, we would like to explore further on this topic.

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