J. Korean Math. Soc. **46** (2009), No. 6, pp. 1119–1137
DOI 10.4134/JKMS.2009.46.6.1119

ON HYPERBOLIC 3-MANIFOLDS WITH SYMMETRIC HEEGAARD SPLITTINGS

Soo Hwan Kim and Yangkok Kim

Abstract. We construct a family of hyperbolic 3-manifolds by pairwise identifications of faces in the boundary of certain polyhedral 3-balls and prove that all these manifolds are cyclic branched coverings of the 3-sphere over certain family of links with two components. These extend some results from [5] and [10] concerning with the branched coverings of the whitehead link.

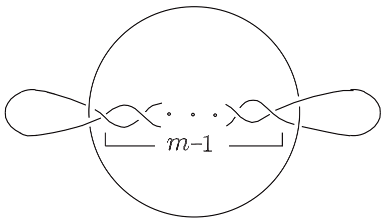
1. Introduction

There are two well known results about the realization of closed 3-manifolds. One is that any closed orientable 3-manifold can be obtained by Dehn surgeries on the components of an oriented link in the 3-sphere. The other one says that any closed 3-manifold can be represented as a branched covering of some link in the 3-sphere. So if we consider a link in the 3-sphere, we can construct many classes of closed orientable 3-manifolds by considering its branched coverings or Dehn surgeries along it. The description of closed 3-manifolds as polyhedral 3-balls, whose finitely many boundary faces are glued together in pairs, is a further standard way to construct 3-manifolds (see [3], [4], [10], [11], and [12]). If the polyhedral 3-ball admits a geometric structure and the face identification is performed by means of geometric isometries, then the same geometric structure is inherited by the quotient manifold (see [10], [12], and [15]). Many authors have studied the connections between the face identification procedure and the representation of closed 3-manifolds as branched coverings of the 3-sphere. In [10] Helling, Kim and Mennicke considered a family of polyhedral 3-balls $\mathcal{P}_n$ depending on a positive integer n, and for any coprime positive integers n and k, they defined a pairwise gluing of faces in the boundary of $\mathcal{P}_n$ yielding a closed orientable 3-manifold $\mathcal{M}_{n,k}$. In the sequel, they proved that $\mathcal{M}_{n,k}$ is an n-fold strongly cyclic covering of the 3-sphere branched over the Whitehead link and classified, up to isometry, those coverings. More general

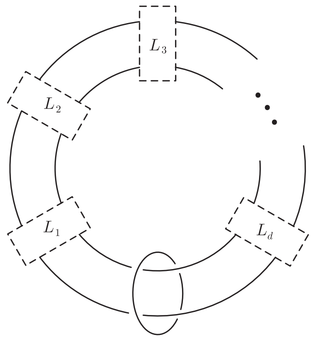
Received August 16, 2004; Revised March 22, 2008.
2000 *Mathematics Subject Classification.* Primary 57M12, 57M25; Secondary 57M50, 57N10.
Key words and phrases. crystallization, cyclic branched covering, symmetric Heegaard splitting.

cases (that is the branched coverings of the Whitehead link which are not necessarily strongly cyclic) were handled by A. Cavicchioli and L. Paoluzzi [5], when $\gcd(n,k) = d > 1$.

In this paper we consider the following classes of links $\mathcal{W}_m$ and $\mathcal{L}_{(m,d)}$ for positive integers m and d as shown in Figure 1, where the index $m-1$ in Figure 1a denotes the number of half twists and each L_i in box denotes the $(\frac{1}{m})$-rational tangle. We note that $\mathcal{W}_m$ is a link of two components which is the rational $(\frac{4m}{2m-1})$-tangle. Moreover $\mathcal{W}_m$ and $\mathcal{L}_{(m,d)}$ extend the Whitehead link $\mathcal{W}_2$ and $\mathcal{L}_{(2,1)}$. We construct an infinite family of 3-manifolds $\mathcal{M}(2m+1,n,k)$ by the identification of oppositely oriented boundary faces of a polyhedral 3-cell $\mathcal{P}(2m+1,n)$ for positive integers m,n,k. Then we shall deal with the combinatorial representation of $\mathcal{M}(2m+1,n,k)$ by a special class of edge-colored graphs, called crystallizations (see for example [2], [8], [6], and [14]).



a. The link $\mathcal{W}_m$



b. $\mathcal{L}_{(m,d)}$

FIGURE 1

By applying *LCG* moves on crystallizations ([13] and [16]), we prove that $\mathcal{M}(2m+1,2d,d)$ is a 2-fold strongly cyclic covering of the 3-sphere branched over $\mathcal{L}_{(m,d)}$. In the sense of Birman and Hilden [1], we obtain the result that the symmetric extension of a Heegaard splitting for $\mathcal{M}(2m+1,2d,d)$ represents $\mathcal{M}(2m+1,n,k)$ as (n/d)-fold strongly cyclic coverings of the 3-sphere branched over $\mathcal{L}_{(m,d)}$, where $\gcd(n,k) = d$. Moreover $\mathcal{M}(2m+1,n,k)$ is an n-fold cyclic covering of the 3-sphere branched over $\mathcal{W}_m$, where the branched indices of its components are n and n/d, respectively. We note that $\mathcal{L}_{(m,d)}$ is hyperbolic for $m > 1$ and so does $\mathcal{M}(2m+1,n,k)$ for $m > 1$. The results extend the corresponding ones of Helling-Kim-Mennicke [10] and Cavicchioli-Paoluzzi [5], where $m = 2$ and $\gcd(n,k) = 1$, and $m = 2$, $\gcd(n,k) = d > 2$, respectively.

2. Construction of the manifold $\mathcal{M}(2m+1,n,k)$

In this section we construct an infinite family of 3-manifold $\mathcal{M}(2m+1,n,k)$ by considering a combinatorial polyhedron together with an identification of

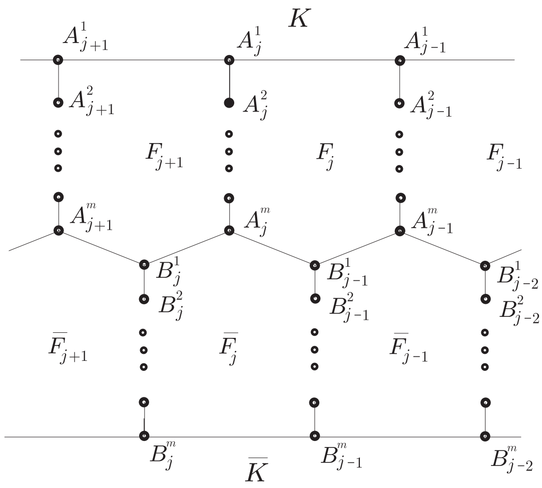


FIGURE 2. $\mathcal{P}(2m+1,n)$

pairs of faces on its boundary. For positive integers m,n and k, let $\mathcal{P}(2m+1,n)$ be a polyhedron whose boundary, which can be regarded as the 2-sphere, consists of two n-gons in the northern and southern hemispheres, and $2n$ $(2m+1)$-gons in the equatorial zone as shown in Figure 2. Then $\mathcal{P}(2m+1,n)$ has $2n+2$ faces, $2nm+2n$ edges and $2nm$ vertices.

We define the boundary cycles of two n-gons and $2n$ $(2m+1)$-gons as follow:

$$\begin{aligned}
K &: A_1^1 A_2^1 \cdots A_n^1\\
\overline{K} &: B_1^m B_2^m \cdots B_n^m\\
F_j &: A_j^m A_j^{m-1} \cdots A_j^1 A_{j-1}^1 A_{j-1}^2 \cdots A_{j-1}^m B_{j-1}^1\\
\overline{F}_j &: A_j^m B_j^1 B_j^2 \cdots B_j^m B_{j-1}^m B_{j-1}^{m-1} \cdots B_{j-1}^2 B_{j-1}^1
\end{aligned}$$

for $j=1,\ldots,n$ (see Figure 2).

We now define the face identification of $2n$ $(2m+1)$-gons as follows: for each $j=1,\ldots,n$,

$$T_j : \begin{array}{l} F_j \to \overline{F}_{j+k} \\ \left\{ \begin{array}{l} A_j^m \to A_{j+k}^m, \\ A_j^l \to B_{j+k}^{m-l} \text{ for } 1 \le l \le m-1, \\ A_{j-1}^1 \to B_{j+k}^m, \\ A_{j-1}^l \to B_{j+k-1}^{m-l+2} \text{ for } 2 \le l \le m, \\ B_{j-1}^1 \to B_{j+k-1}^1, \end{array} \right. \end{array}$$

where the indices are taken mod n.

Consider the oriented edges:

$$x_j = (A_j^1, A_{j+1}^1) \text{ and } u_j = (B_{j-1}^1, A_j^m).$$

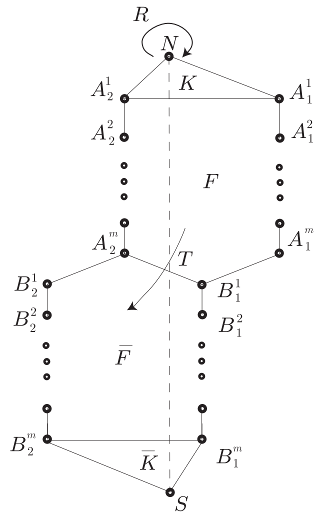


FIGURE 3. The quotient space $\mathcal{P}(2m+1,n)/\sim$

Then the identifications T_j naturally induce the face identification of two n-gons. Moreover each oriented edge u_j has $\frac{n}{d}$ equivalent edges for $j = 1, 2, \ldots, n$. Now we calculate the Euler characteristic of the cellular complex $\mathcal{K}(2m+1, n, k)$ induced by the face identifications of the polyhedron $\mathcal{P}(2m+1, n)$. We note that there is a rotational symmetry $\sim$ by

$$A_j^i \to A_{j-1}^i \quad \text{and} \quad B_j^i \to B_{j-1}^i \text{ for all } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

of $\mathcal{P}(2m+1, n)$. Thus it suffices to consider our case in the quotient space $\mathcal{P}(2m+1,n)/\sim$ with $\gcd(n,k) = 1$ described in Figure 3, where N and S are the centers of two n-gons in $\mathcal{P}(2m+1, n)$.

Indeed, it is easy to see that all edges of two $(2m+1)$-gons except (B_1^1, A_2^m) are equivalent under the repeated applications of composition actions of Q, R and T;

$$Q : \begin{cases} A_2^1 \to B_2^m \\ A_1^1 \to B_1^m \\ N \to S \end{cases} \quad \text{and } R : \begin{cases} N \to N \\ S \to S \\ A_2^i \to A_1^i \\ B_2^i \to B_1^i \end{cases}$$

for all $i = 1, 2, \ldots, m$, and

$$T : \begin{cases} A_1^1 \to B_2^m \\ A_1^l \to B_1^{m-l+2} & \text{for } 2 \le l \le m \\ B_1^1 \to B_1^1 \\ A_2^m \to A_2^m \\ A_2^l \to B_2^{m-l} & \text{for } 1 \le l \le m-1. \end{cases}$$

This means that the complex $\mathcal{K}(2m+1,n,k)$ has d vertices, $n+d$ edges, $n+1$ two-cells and 1 three-cell. So it follows that the resulting complex $\mathcal{M}(2m+1,n,k)$ is a closed oriented 3-manifold by the theorem of Seifert and Threlfall: *a complex formed by identifying the faces of a polyhedron will be a closed manifold if and only if its Euler characteristic equals zero.*

3. Crystallizations and LCG moves

We introduce crystallizations and *LCG* moves (for detail, see [13] and [16]). For a given multigraph Γ, $V(\Gamma)$ and $E(\Gamma)$ denote the sets of vertices and edges of Γ (both finite) respectively. An edge-coloration on a graph $\Gamma = (V(\Gamma), E(\Gamma))$ is a map $\gamma : E(\Gamma) \to \Delta = \{0,1,2,3\}$ such that $\gamma(e) \neq \gamma(f)$ for any two adjacent edges e, f. A pair (Γ, γ) is called a 4-colored graph if Γ is regular of valency 4. For a subset Δ' of Δ, we set $\Gamma_{\Delta'} = (V(\Gamma), \gamma^{-1}(\Delta'))$. Each connected component of $\Gamma_{\Delta'}$ is called a Δ'-residue of degree k, where k is the order of the component. The number of $\{ij\}$-residues of (Γ, γ) is denoted by $g_{ij}(\Gamma, \gamma)$. A 4-colored graph (Γ, γ) is said to be contracted if $\Gamma_{\Delta\setminus\{i\}}$ is connected for each $i \in \Delta$. If (Γ, γ) is a 4-colored graph, then the associated pseudocomplex $K(\Gamma)$ is defined as follows:

(1) take a 3-simplex $\sigma^3(v)$ for each $v \in V(\Gamma)$ and label its vertices by different elements of Δ;
(2) if $v, w \in V(\Gamma)$ are joined by an i-colored edge, then identify the 2-faces of $\sigma^3(v)$ and $\sigma^3(w)$ opposite to the vertices labelled by i, so that equally labelled vertices are identified together.

In this case (Γ, γ) is said to represent $K(\Gamma)$ and every space homeomorphic to it. A contracted 4-colored graph representing a closed connected 3-manifold M is said to be a *crystallization* of M.

For the drawing of a crystallization we first fix two arbitrary colors, for example, 0 and 1. Then we draw all $\{01\}$-residues as circles, and draw all the third colored edges, say 2-colored edges, by connecting $\{01\}$-residues. Finally we express each 3-colored edge by denoting the initial and terminal vertices by a, a^* or simply a, a if there is no confusion. This expression is called a crystallization based on $(01;2)$. In (Γ, γ) an *$\{ij\}$-residue $\{a,b,c,d\}$ of degree* 4 is called *standard* if four vertices a, b, c and d in Γ have the following two properties:

(1) a, b, c and d are vertices of a $\{ij\}$-residue of degree 4;
(2) a, b, c and d are vertices of mutually distinct $\Delta\setminus\{ij\}$-residues.

In particular, when $\{a,b,c,d\} = \{a,a^*,b,b^*\}$, we simply write a standard $\{ij\}$-residue $\{a,b\}$ or a standard 2-residue $\{a,b\}$ for a standard $\{ij\}$-residue $\{a,a^*,b,b^*\}$ of degree 4. We note that every closed connected 3-manifold admits a crystallization, and that a manifold can have non-isomorphic crystallizations. However there is a set of moves, called moves I, II, and A, which connects any two crystallizations of a manifold (see [7]). We now introduce a *LCG* move which is equivalent to moves I, II, and A.

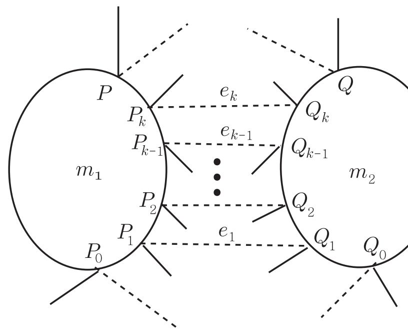


FIGURE 4. An extended 1-dipole

Let m_1 and m_2 be two distinct $\{01\}$-residues in Γ with $\Delta = \{0, 1, 2, 3\}$. We say that m_1 and m_2 are joined by consecutive 3-colored edges $e_1, e_2, \ldots, e_k$ if there exists a partial ordering of vertices $P_0, P_1, \ldots, P_{k-1}, P_k, P$ of m_1 and $Q_0, Q_1, \ldots, Q_{k-1}, Q_k, Q$ of m_2 such that

(1) P_i is joined to Q_i by a 3-colored edge e_i for $1 \leq i \leq k$;
(2) $\overline{P_0P_1}$ and $\overline{Q_0Q_1}$ belong to a $\{03\}$ or $\{13\}$-residue of degree greater than 4;
(3) $\overline{P_kP}$ and $\overline{Q_kQ}$ belong to a $\{03\}$ or $\{13\}$-residue of degree greater than 4.

Let m_1 and m_2 be two distinct $\{01\}$-residues in Γ which are joined by consecutive 3-colored edges $e_1, e_2, \ldots, e_k$. Then a partial subgraph of Γ formed by vertices $P_1, \ldots, P_k$ of m_1 and $Q_1, \ldots, Q_k$ of m_2, joined by consecutive 3-colored edges is said to be *an extended* 1*-dipole* if m_1 and m_2 belong to different $\{012\}$-residues (see Figure 4). In this case, we simply say that Γ has an extended 1-dipole generated by $(P_1, \ldots, P_k; Q_1, \ldots, Q_k)$. We recall that a subgraph of Γ formed by two vertices X, Y joined by 2 edges with colors i, j will be called a dipole $\{X, Y\}$ of type 2 if and only if X and Y belong to distinct components of $\Gamma_{\Delta\setminus\{i,j\}}$.

Let Γ have an extended 1-dipole consisting of two $\{01\}$-residues m_1, m_2 and consecutive 3-edges $e_1, e_2, \ldots, e_k$. Then we construct a 4-colored graph Γ' from Γ as follows:

(1) Remove all 0, 1 and 3-colored edges ending at P_i or Q_i, and all P_i or Q_i for $1 \leq i \leq k$ from Γ.
(2) Connect 2-colored edges ending at Q_i and P_i for $1 \leq i \leq k$.
(3) Connect 0 or 1-colored edges ending at Q and P, and Q_0 and P_0

In this case we say that Γ' is obtained from Γ by *eliminating an extended* 1*-dipole* or Γ is obtained from Γ' by *adding an extended* 1*-dipole.*

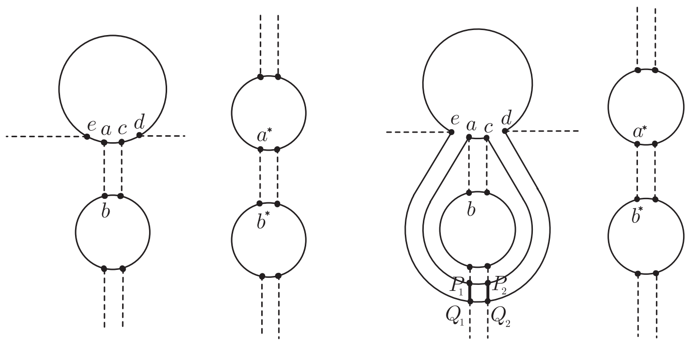
We say that Γ' is obtained from Γ by a *linear cut-and-glue move* (or a *LCG-move*) if there exists a non-contracted 4-colored graph $\overline{\Gamma}$ such that $\overline{\Gamma}$ is obtained from Γ by adding an extended 1-dipole and Γ' is obtained from $\overline{\Gamma}$ by eliminating an extended 1-dipole. Two crystallizations are said to be *LCG-equivalent* if they can be joined by a finite sequence of *LCG* moves.

Theorem 1 ([7] and [13])**.** *Let M and M' be closed 3-manifolds, and (Γ, γ) and (Γ', γ') be two crystallizations of M and M' respectively. Then the following statements are equivalent*:

(1) *M is homeomorphic with M',*
(2) *(Γ, γ) and (Γ', γ') are (I, II)-equivalent,*
(3) *(Γ, γ) and (Γ', γ') are A-equivalent,*
(4) *(Γ, γ) and (Γ', γ') are LCG-equivalent.*

Theorem 2. *Let M be a closed 3-manifold and (Γ, γ) be a crystallization of M. If (Γ, γ) has a standard $\{23\}$-residue, then there is a crystallization (Γ', γ') of M such that $g_{01}(\Gamma', \gamma') = g_{01}(\Gamma, \gamma) - 1$.*

Proof. Let (Γ, γ) have a standard $\{23\}$-residue $\{a, b\}$. Then we assume, without loss of generality, that (Γ, γ) contains a part shown in Figure 5a, where the circles denote $\{01\}$-residues and the dashed lines denote 2-colored edges.



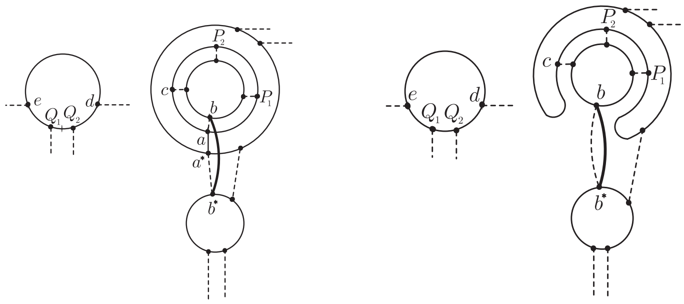
a. (Γ, γ) b. $(\overline{\Gamma}, \overline{\gamma})$

Figure 5

We now change this part and keep the other part unchanged to get a 4-colored graph $(\overline{\Gamma}, \overline{\gamma})$. First we draw two parallel lines which

(1) have starting points a, e and ending points c, d, respectively and
(2) circumscribe only one $\{01\}$-residue which contains b.

Then we remove lines $\overline{ae}$, $\overline{bd}$ and replace lines $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$ by thick lines which denote 3-colored edges as shown in Figure 5b. We have a 4–colored graph $(\overline{\Gamma}, \overline{\gamma})$ which contains an extended 1-dipole generated by $(P_1, P_2; Q_1, Q_2)$ as shown in Figure 5b or Figure 6a. From the construction, it is clear that (Γ, γ) is obtained from $(\overline{\Gamma}, \overline{\gamma})$ by eliminating an extended 1-dipole generated by $(P_1, P_2; Q_1, Q_2)$.



a. $(\overline{\Gamma}, \overline{\gamma})$ b.

FIGURE 6

We note that $(\overline{\Gamma}, \overline{\gamma})$ has a standard $\{23\}$-residue $\{a, b\}$. Moreover $(a; a)$ generates an extended 1-dipole. By eliminating it, we have a 4-colored graph with a dipole $\{b, b^*\}$ of type 2 (see Figure 6b). Indeed a crystallization in Figure 6b is obtained from (Γ, γ) by a *LCG*-move. Finally, by cancelling the dipole $\{b, b^*\}$ of type 2, we have a crystallization (Γ', γ') of M such that $g_{01}(\Gamma', \gamma') = g_{01}(\Gamma, \gamma) - 1$. □

Let (Γ, γ) be a crystallization of a closed 3-manifold M with a standard 2-residue $\{a, b\}$. Then a crystallization (Γ', γ') of M constructed by using the method in the proof of Theorem 2 is said to be *generated from* (Γ, γ) *by LCG-moves.*

4. The manifold $\mathcal{M}(2m+1, n, k)$ as a branched covering

There are two presentations corresponding to a spine of $\mathcal{M}(2m+1, n, k)$. One is the description of polyhedral 3-balls, whose finitely many boundary faces are glued together in pairs, and the other one is a (n/d)-symmetric Heegaard splitting (or (n/d)-symmetric crystallization). The latter is easily obtained from a combinatorial complex triangulating $\mathcal{M}(2m+1, n, k)$ by pairwise identification of boundary 2-cells. By [1], $\mathcal{M}(2m+1, n, k)$ is an (n/d)-fold cyclic covering of the 3-sphere branched over a link of bridge number $\leq \frac{p-1+g}{p-1}$, where g is the genus of $\mathcal{M}(2m+1, n, k)$ and $p = \frac{n}{d}$. By the rotational symmetry, it suffices to prove the result in case $n = 2d$. We now consider the polyhedral schemata $\mathcal{P}(2m+1, 2d)$, which defines the closed orientable 3-manifold $\mathcal{M}(2m+1, 2d, d)$ as a quotient of a triangulated 3-ball B^3 by pairwise identification of its boundary 2-cells (see Figure 2). Triangulate $\mathcal{P}(2m+1, 2d)$ into a simplicial complex $K(2m+1, 2d)$ by using stellar subdivisions (for example see a triangulation of $K(7, 4)$ in Figure 7). We note that outside of the exterior circle we have a vertex and the corresponding stellar subdivision. The configuration is a simplicial tessellation of the 2-sphere ∂B^3 consisting of $2(2m+1)d + 2$ vertices, $6(2m+1)d$ edges, and $4(2m+1)d$ triangles. Let w be a point in the interior

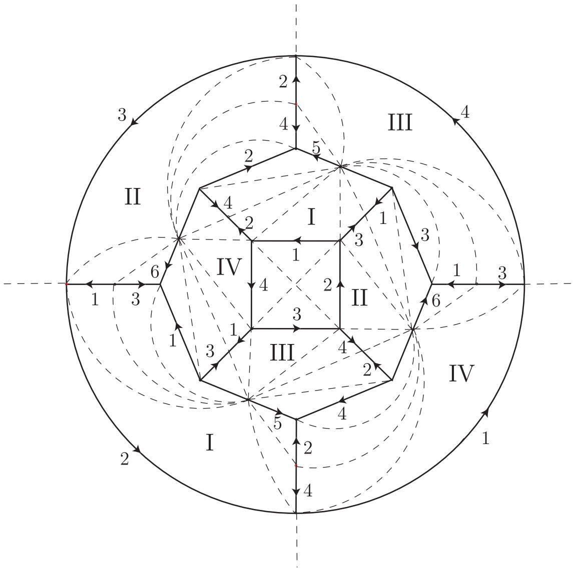


FIGURE 7. $K(7,4)$

of B^3. Now $K(2m+1,2d)$ is just the simplicial join from w on the above tessellation. Identify the two copies of each triangle in $\partial K(2m+1,2d)$ so that the corresponding oriented edges carrying the same label are glued together. The identification produces a pseudocomplex $\widetilde{K}(2m+1,2d,d)$ which triangulates $\mathcal{M}(2m+1,2d,d)$ consisting of $2d+2$ vertices, $2d+2+4(2m+1)d$ edges, $8(2m+1)d$ triangles, and $4(2m+1)d$ tetrahedra. Indeed $\widetilde{K}(2m+1,2d,d)$ is a colored complex. We now construct the 4-colored graph $\widetilde{\Gamma}(2m+1,2d,d)$ associated to $\widetilde{K}(2m+1,2d,d)$ as follows. The vertices of $\widetilde{\Gamma}(2m+1,2d,d)$ are the elements of

$$\widetilde{V}(2m+1,2d,d) = (\{0,2d+1\} \times \mathbb{Z}_{2d}) \cup (\{1,2,\ldots,2d\} \times \mathbb{Z}_{4m}).$$

For coloring edges of $\widetilde{\Gamma}(2m+1,2d,d)$ we consider the following four fixed-point-free involutions on $\widetilde{V}(2m+1,2d,d)$:

$$\begin{aligned}
v_0(i,j) &= (i, j+(-1)^j),\\
v_1(i,j) &= (i, j-(-1)^j),\\
v_2(i,j) &= \begin{cases} (2d+1, i-1) & \text{if } j=0,\\ (0,1-i) & \text{if } j=2m,\\ (i+(-1)^{i+1}\mu(j), j) & \text{otherwise,} \end{cases}\\
v_3(i,j) &= \begin{cases} (i+d, 2m+1-j) & \text{if } i\in\{1,2,\ldots,d\},\ i \text{ odd},\\ (i+d, 2m-1-j) & \text{if } i\in\{1,2,\ldots,d\},\ i \text{ even},\\ (2d+1, 1-j) & \text{if } i=0, \end{cases}
\end{aligned}$$

where $\mu : \mathbb{Z}_{4m}\backslash\{0, 2m\} \to \{+1, -1\}$ is the function defined by

$$\mu(j) = \begin{cases} +1 & \text{if } 1 \leq j \leq 2m-1 \\ -1 & \text{if } 2m+1 \leq j \leq 4m-1 \end{cases}$$

and the arithmetic is either mod $2d$ or mod $4m$. We now define the 4-colored graph $\widetilde{\Gamma}(2m+1, 2d, d)$ as follows: for each $i \in \Delta$ two vertices x and y in $\widetilde{V}(2m+1, 2d, d)$ are joined by an i-colored edge if and only if $y = v_i(x)$. The geometrical shape of $\widetilde{\Gamma}(2m+1, 2d, d)$ consists of $2d$ circles C_i which are $\{01\}$-residues of length $4m$ cyclically set on the plane following the natural order of the set $\{1, 2, \ldots, 2d\}$, and of two circles C_0, C_{2d+1} of length $2d$ which are $\{01\}$-residues of length $2d$. For each $i \in \{0, \ldots, 2d\}$, the vertex set of C_i consists of pairs (i, j) for any $j \in \mathbb{Z}_{4m}$. The vertex sets of C_0 and C_{2d+1} consist of pairs (i, j) for any $j \in \mathbb{Z}_{2d}$. We cyclically order the vertices (i, j) of each C_i following the natural order of j in $\mathbb{Z}_{4m}$(or $\mathbb{Z}_{2d}$) so that all these orderings induce the clockwise (resp. anti-clockwise) orientation of the plane when i is odd (resp. even). There are exactly $2m-1$ 2-colored edges between C_i and C_{i+1} (resp. C_i and C_{i-1}) for any $i = 1, \ldots, 2d$ (here $0 \equiv 2d$). There is exactly one 2-colored edge between C_i and C_{2d+1} (resp. C_i and C_0) for any $i = 1, \ldots, 2d$. Furthermore there are exactly $4m$ 3-colored edges between C_i and C_{i+d} for any $i = 1, \ldots, d$, and $2d$ 3-colored edges between C_0 and C_{2d+1}. For the simple notation of vertices, we order vertices linearly by the lexicographic ordering:

$$(i, j) < (k, l) \text{ if } i < k, \text{ or } i = k \text{ and } j < l.$$

First we do numbering for the vertices of C_1 to C_d by the lexicographic ordering. That is, we use numbers 1 to $4md$ for the numbering of the vertices of C_1 to C_d, for example 1 for $(1, 0)$, 2 for $(1, 1)$ and so on. We then order the vertices of C_0 by using numbers $4md+1$ to $4md+2d$. For the rest of vertices in $C_{d+1}, \ldots, C_{2d+1}$ we do as follows; if the vertex is connected to a vertex numbered k by a 3-colored edge, then we number the vertex by k. (For example see Figure 8 for the crystallization of $\widetilde{\Gamma}(2m+1, 4, 2)$ with numbered vertices.) Summarizing the above argument we have:

Lemma 1. *The closed orientable* 3*-manifold* $\mathcal{M}(2m+1, 2d, d)$ *is represented by a* 4*-colored graph* $\widetilde{\Gamma}(2m+1, 2d, d)$.

We also note that in the above method of stellar subdivision which triangulates $\mathcal{P}(2m+1, 2d)$ into a simplicial complex $K(2m+1, 2d)$ is independent of d. Thus we can prove the following theorems by using the special numbers d without loss of generality. Indeed we use the same method as one in [5].

Theorem 3. *The closed orientable* 3*-manifold* $\mathcal{M}(2m+1, 2d, d)$ *is a* 2*-fold strongly cyclic covering of the* 3*-sphere branched over* $\mathcal{L}_{(m,d)}$*, where d is even. Furthermore,* $\mathcal{M}(2m+1, 2d, d)$ *is a* $2d$*-fold cyclic branched covering of* $\mathcal{W}_m$ *in the* 3*-sphere, where the branched indices of its components are* $2d$ *and* 2*, respectively.*

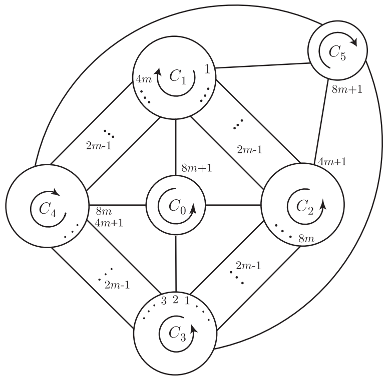


FIGURE 8. The 4-colored graph $\widetilde{\Gamma}(2m+1,4,2)$

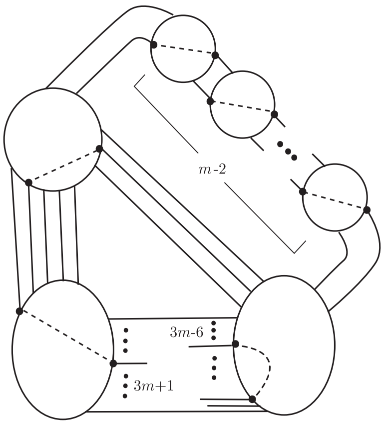
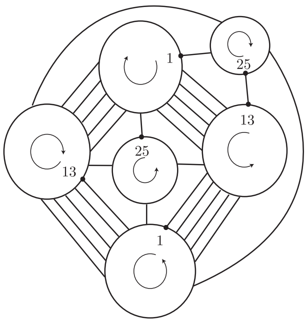


FIGURE 9. A 2-symmetric crystallization $\Gamma(2m+1,4,2)$ inducing an link

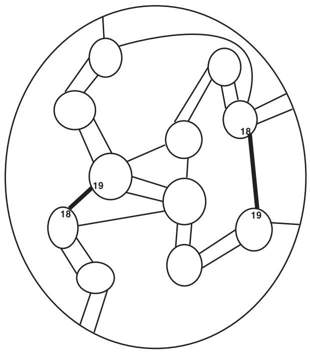
Proof. We note that $\mathcal{M}(2m+1,2d,d)$ can be represented by a 4-colored graph $\widetilde{\Gamma}(2m+1,2d,d)$ by Lemma 1. Thus it suffices to prove that a 2-symmetric crystallization $\Gamma(2m+1,2d,d)$ is obtained from $\widetilde{\Gamma}(2m+1,2d,d)$ by a finite sequence of LCG-moves. For simplicity we now restrict our case to $d=2$. One can immediately extend the construction for the general cases by a simple iteration. We let a subgraph $\widetilde{\Gamma}(2m+1,4,2)$ be regularly embedded in the plane. To see standard 2-residues specifically, we consider a crystallization based on $(02;1)$. Then it consists of $(4m-4)$ $\{02\}$-colored cycles of length 4 and four $\{02\}$-colored cycles of length 6. Now the crystallization $\Gamma(2m+1,4,2)$ contains $(4m+2)$ standard $\{02\}$-residues. By applying a LCG move for a standard $\{02\}$-residue, we have a 4-colored graph $\Gamma'(2m+1,4,2)$ such that $g_{02}(\Gamma'(2m+1,4,2)) = g_{02}(\Gamma(2m+1,4,2)) - 1$. Note that $g_{02}(\Gamma(2m+1,4,2)) = 4m$ and $g_{02}((\Gamma'(2m+1,4,2))) = 4m-1$. Repeating this procedure for $(3m-1)$ standard $\{02\}$-residues, we obtain a crystallization $\overline{\Gamma}(2m+1,4,2)$ with $g_{02}(\overline{\Gamma}(2m+1,4,2)) = 4m-(3m-1) = m+1$, which yields a 2-symmetric crystallization as shown in Figure 9, where dashed lines denote the axis of a 2-symmetric Heegaard splitting induced by a 2-symmetric crystallization. Thus $\mathcal{M}(2m+1,4,2)$ is the 2-fold covering of the 3-sphere branched over a link. Using Reidemeister moves, it is immediate to verify that this link is equivalent to a 3-bridge link $\mathcal{L}_{(m,2)}$.

For the second statement, we simply note that $\mathcal{L}_{(m,d)}$ has a component which can be regarded as the axis of symmetry of order d. This symmetry produces a d-fold cyclic branched covering of $\mathcal{W}_m$ in the 3-sphere, where the branched indices of its components are $2d$ and 2, respectively. □

We now explicitly describe the procedures described in Theorem 3 for $\mathcal{M}(7, 4, 2)$ which is the case $m = 3$ and $d = 2$. On the basis of the triangulation in Figure 7 we have a crystallization $\widetilde{\Gamma}(7,4,2)$ based on $(01;2)$ as shown in Figure 10a. We now change the crystallization based on $(01;2)$ to one based on $(12;0)$ to see standard $\{03\}$-residues (see Figure 10b). We note that there



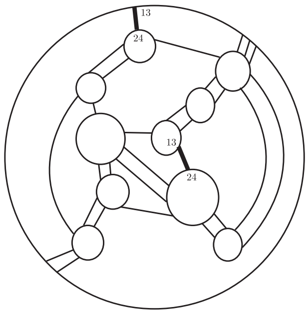
a. Based on $(01;2)$



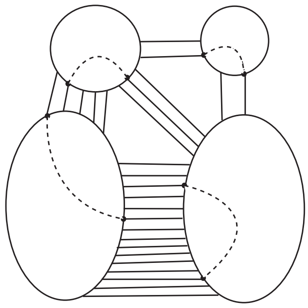
b. Based on $(12;0)$

FIGURE 10

is a standard $\{03\}$-residue $\{18,19\}$. Applying a LCG move for $\{18,19\}$, we have a 4-colored graph as shown in Figure 11a. We can do the same job for a standard $\{03\}$-residue $\{13,24\}$ in a 4-colored graph. One application of a LCG move reduces the number of $\{03\}$-residues by one. We can continue this procedure 8 times to get a crystallization shown in Figure 11b which is a 2-symmetric crystallization of $\mathcal{M}(7,4,2)$. Thus $\mathcal{M}(7,4,2)$ is the 2-fold covering of the 3-sphere branched over a 4-bridge link. Using Reidemeister moves, it is immediate to verify that this link is equivalent to $\mathcal{L}_{(3,2)}$.



a. Based on $(01;2)$

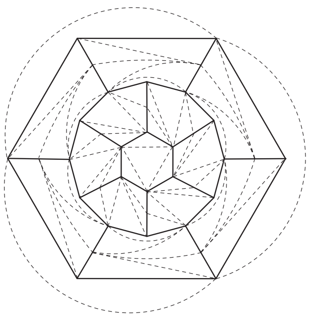


b. 2-symmetric crystallization

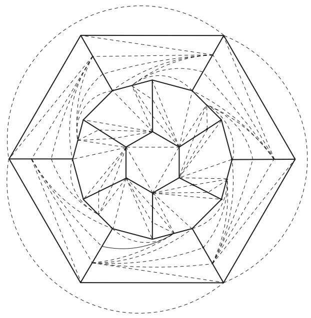
FIGURE 11

Theorem 4. *The closed orientable* 3*-manifold* $\mathcal{M}(2m+1, 2d, d)$ *is a* 2*-fold strongly cyclic covering of the* 3*-sphere branched over* $\mathcal{L}_{(m,d)}$*, where* d *is odd. Furthermore,* $\mathcal{M}(2m+1, 2d, d)$ *is* $2d$*-fold cyclic branched covering of* $\mathcal{W}_m$ *in the* 3*-sphere, where the branched indices of its components are* $2d$ *and* 2*, respectively.*

Proof. We consider the case $d = 3$ that can be immediately extended to the construction for the general cases by a simple iteration as in Theorem 3. That is, we claim that $\mathcal{P}(2m+1, 6)$ can be represented by a 2-symmetric crystallization $\Gamma(2m+1, 6, 3)$. Let $\mathcal{P}(2m+1, 6)$ be the polyhedral schemata which defines the closed orientable 3-manifold $\mathcal{M}(2m+1, 6, 3)$ as a quotient of a triangulated 3-ball B^3 by pairwise identification of its boundary 2-cells. Triangulate $\mathcal{P}(2m+1, 6)$ into a simplicial complex $\widetilde{K}(2m+1, 6, 3)$. There are two distinct ways to triangulate $\mathcal{P}(2m+1, 6)$ depending on the parity of m. For example, we see two triangulations for $\mathcal{P}(7, 6)$ and $\mathcal{P}(9, 6)$, where $m = 3$ and 4, respectively in Figure 12. We can handle the general cases by a natural extension.



a. $\widetilde{K}(7, 6, 3)$



b. $\widetilde{K}(9, 6, 3)$

FIGURE 12

By the same way in the proof of Theorem 3, we can construct a crystallization $\widetilde{\Gamma}(2m+1,6,3)$ associated to $\widetilde{K}(2m+1,6,3)$ as follows.

Case i) $m = 2t+1$ and $t \geq 0$.

The vertices of $\widetilde{\Gamma}(2m+1,6,3)$ are the elements of

$$\widetilde{V}(2m+1,6,3) = \{(i,j(i)) \mid i = 1,2,\ldots,18\} \cup \{O,O'\},$$

where

$$\begin{cases} 1 \leq j(i) \leq m+3 & \text{if } i = 1,2,3, \\ 1 \leq j(i) \leq 2m+2 & \text{if } i = 4,5,6, \\ 1 \leq j(i) \leq m+3 & \text{if } i = 7,8,9, \\ 1 \leq j(i) \leq 2 & \text{if } i = 10,11,12, \\ 1 \leq j(i) \leq 2m-4 & \text{if } i = 13,14,15, \\ 1 \leq j(i) \leq 2m-8 & \text{if } i = 16,17,18. \end{cases}$$

For colored edges, we consider a permutation η which is of order 3:

$$\eta = (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)$$

on $X = \{0,1,2,\ldots,18\}$ and a subset V_1 of $\widetilde{V}(2m+1,6,3)$;

$$V_1 = \{(i,j(i)) | i = 1,4,7,10,13,16\ \} \cup \{O,O'\}$$

with the following edge-colorations;

$v_0(O') = O$,
$v_0(1,1) = (11,2)$, $v_0(1,2) = (12,1)$, $v_0(1,3) = (9,m+2)$,
$v_0(1,4) = (9,m+3)$, $v_0(1,j) = (18,m+1-j)$, $5 \leq j \leq m$,
$v_0(1,m+1) = (6,4)$, $v_0(1,m+2) = (6,5)$,
$v_0(4,1) = (1,2)$, $v_0(4,2) = (5,m+4)$, $v_0(4,3) = (5,m+5)$,
$v_0(4,4) = (2,m+1)$, $v_0(4,5) = (2,m+2)$,
$v_0(4,j) = (16,2m+2)$, $6 \leq j \leq m+3$, $v_0(4,m+4) = (6,2)$,
$v_0(4,m+5) = (6,3)$, $v_0(4,j) = (18,3m-2-j)$, $m+6 \leq j \leq 2m+1$,
$v_0(4,2m+2) = (9,1)$,
$v_0(7,1) = (5,2m+2)$, $v_0(7,2) = (5,1)$, $v_0(7,j) = (14,m-1-j)$, $3 \leq j \leq m$,
$v_0(7,m+1) = (1,m+3)$, $v_0(7,m+2) = (2,3)$, $v_0(7,m+3) = (2,4)$,
$v_0(10,2) = (3,1)$, $v_0(10,1) = (2,2)$,

$$v_0(14,j) = \begin{cases} (9,m+1-j), & \text{if } 1 \leq j \leq m-2, \\ (5,2m+2-j), & \text{if } m-1 \leq j \leq 2m-4, \end{cases}$$

$$v_0(16,j) = \begin{cases} (2,m+1-j), & \text{if } 1 \leq j \leq m-4, \\ (5,3m-2-j), & \text{if } m-3 \leq j \leq 2m-8, \end{cases}$$

$$\begin{aligned} v_1(1,j) &= (1,j-(-1)^j), \text{ where } j \in \mathbb{Z}_{m+3}, \\ v_1(4,j) &= (4,j+(-1)^j), \text{ where } j \in \mathbb{Z}_{2m+2}, \\ v_1(13,j) &= (13,j+1),\ 1 < j < 2m-2,\ j \text{ even}, \\ v_1(13,1) &= (3,m+3), \text{ and } v_1(13,2m-4) = (3,m+2), \\ v_1(16,2j) &= (16,2j-1),\ 1 < 2j \leq 2m-8,\ j \text{ even},\ j \neq m-3, \\ v_1(10,1) &= (7,m+2), \text{ and } v_1(10,2) = O', \end{aligned}$$

$$
\begin{aligned}
v_2(1,1) &= O, \\
v_2(4,1) &= (13, m-2) \text{ and } v_0(4,2) = (13, m-1), \\
v_2(4,3) &= (16, 2m-8), \text{ and } v_0(4,4) = (17,1), \\
v_2(4,j) &= (8, m+4-j),\ 5 \le j \le m+2, \\
v_2(4,j) &= (1, 2m+4-j),\ m+3 \le j \le 2m+2, \\
v_2(7,j) &= (7, j-(-1)^j),\ \text{ where } j \in \mathbb{Z}_{m+3}, \\
v_2(10,1) &= (8,m),\ \text{ and } v_2(10,2) = (8, m+1), \\
v_2(13,j) &= (13, j-1),\ 1 < 2j < 2m-4,\ j \text{ even},\ j \neq m-1, \\
v_2(16,j) &= (16, j+1),\ 1 < j < 2m-8,\ j \text{ even},
\end{aligned}
$$

$$
\begin{aligned}
v_3(1,j) &= (1, j+(-1)^j),\ \text{ where } j \in \mathbb{Z}_{m+3}, \\
v_3(4,j) &= (4, j-(-1)^j),\ \text{ where } j \in \mathbb{Z}_{2m+2}, \\
v_3(7,j) &= (7, j-(-1)^j),\ \text{ where } j \in \mathbb{Z}_{m+3}, \\
v_3(10,1) &= (10,2), \\
v_3(13,j) &= (13, 2m-3-i), \\
v_3(16,j) &= (16, 2m-7-i).
\end{aligned}
$$

We define an action of η on V by

$$
\begin{aligned}
\eta(v,w) &= (\eta(v), w) \text{ for } (v,w) \in V \backslash \{O, O'\}, \text{ and} \\
&\quad\ \eta \text{ fixes } O \text{ and } O'.
\end{aligned}
$$

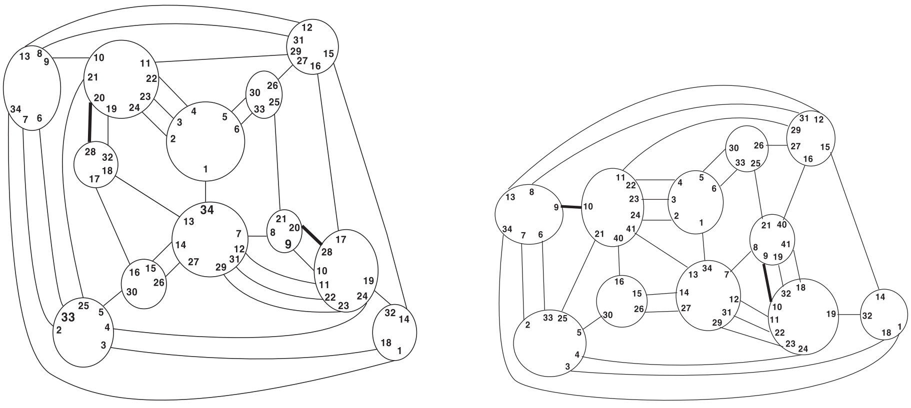
Furthermore for edge-colorations,

$$\text{if } y = v_i(x) \text{ for } x,\ y \text{ in } V, \text{ then we define } \eta(y) = v_{\eta(i)}(\eta(x)).$$

Since $V_1 \cup \eta(V_1) \cup \eta^2(V_1) = V$, a function η and an edge-coloration of V_1 produce a crystallization $\widetilde{\Gamma}(2m+1, 6, 3)$ associated to $\widetilde{K}(2m+1, 6, 3)$.

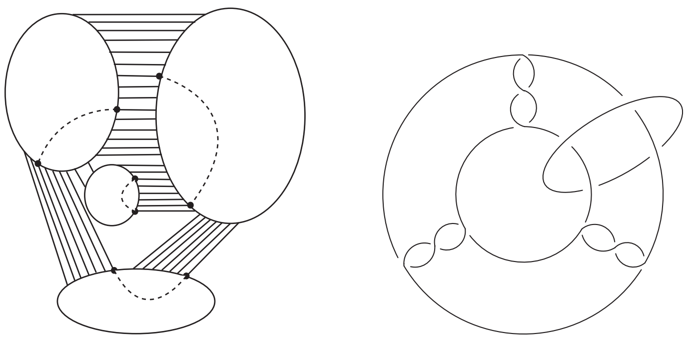
For example, we consider the case $\widetilde{\Gamma}(7, 6, 3)$. One can easily extend to the general case $\widetilde{\Gamma}(2m+1, 6, 3)$ by a simple extension. We have an crystallization based on $(01; 2)$ associated to $\widetilde{K}(7, 6, 3)$ in Figure 13a. There exist 10 standard $\{01\}$-residues. By the same method in Theorem 3 and example following, we reduce the number of $\{01\}$-residues.

Note that the result does not depend on the choice of a standard 2-residue. First we remove a standard 2-residue $\{20, 28\}$ as shown in Figure 13a. We apply a LCG move for $\{20, 28\}$ to get Figure 13b which has a standard 2-residue $\{9, 10\}$. We apply a LCG move for $\{9, 10\}$. We can continue the same procedure until we remove all standard 2-residues. Note that each time we can reduce the genus by one. Finally we obtain the 2-symmetric crystallization $\Gamma(7, 6, 3)$ as shown in Figure 14a, where dashed lines denote the axis of a 2-symmetric Heegaard splitting induced by a 2-symmetric crystallization.



a. b.

FIGURE 13



a. $\Gamma(7,6,3)$ b. The link $\mathcal{L}_{(3,3)}$

FIGURE 14

By applying Reidemeister moves on a 4-bridge link induced by a 2-symmetric crystallization $\Gamma(7,6,3)$, we can easily verify that the link is equivalent to the link $\mathcal{L}_{(3,3)}$ as shown in Figure 14b.

Case ii) $m = 2t$

We can apply the same argument for case i) with the following edge-colorations of V_1;

$v_0(O) = O'$,

$v_0(1,1) = (11,2)$, $v_0(1,2) = (12,1)$, $v_0(1,j) = (18, m+2-j)$, $3 \le j \le m+1$,

$v_0(1, m+2) = (6,2)$,

$v_0(1, m+3) = (6,3)$,

$v_0(1, m+4) = (7, m+2)$,

$v_0(4,1) = (9,2)$, $v_0(4,2) = (2, m+2)$,

$$v_0(4,j) = \begin{cases} (15, 2m+2-j), & \text{if } 3 \le j \le m+2, \\ (18, 3m-3-j), & \text{if } m+3 \le j \le 2m-1, \end{cases}$$

$v_0(4,2m) = (9,1)$,

$v_0(7,1) = (5,2m)$, $v_0(7,2) = (5,1)$, $v_0(7,j) = (18, m+2-j)$, $3 \le j \le m+1$,

$v_0(7,m+2) = (1,m+4)$, $v_0(7,m+3) = (2,3)$, $v_0(7,m+4) = (2,4)$,

$v_0(10,2) = (3,1)$, $v_0(10,1) = (2,2)$,

$$v_0(13,j) = \begin{cases} (9, m+2-j), & \text{if } 1 \le j \le m-1, \\ (5, 2m-2-j), & \text{if } m \le j \le 2m-2, \end{cases}$$

$$v_0(16,j) = \begin{cases} (2, m+2-j), & \text{if } 1 \le j \le m-3, \\ (5, 3m-3-j), & \text{if } m-2 \le j \le 2m-6, \end{cases}$$

$$\begin{aligned} v_1(1,j) &= (1, j+(-1)^j), \text{ where } j \in \mathbb{Z}_{m+3}, \\ v_1(4,j) &= (4, j-(-1)^j), \text{ where } j \in \mathbb{Z}_{2m+2}, \\ v_1(7,j) &= (7, j+(-1)^j), \\ v_1(13,j) &= (13, 2m-3-j), 1 \le j \le 2m-4, \\ v_1(16,j) &= (16, 2m-5-j), 1 \le j \le 2m-6, \\ v_1(10,1) &= (10,2), \end{aligned}$$

$$\begin{aligned} v_2(1,1) &= O, \\ v_2(1,j) &= (4, 2m+2-j),\ 2 \le j \le m+1, \\ v_2(1,m+2) &= (8,2), \\ v_2(1,m+3) &= (14, 2m-2),\ v_2(1,m+4) = (14,1), \\ v_2(4,1) &= (13, m-1),\ v_2(4,2) = (16,1), \\ v_2(4,j) &= (8, m+3-j),\ 3 \le j \le m, \\ v_2(7,j) &= (7, j-(-1)^j), \text{ where } j \in \mathbb{Z}_{m+4}, \\ v_2(10,1) &= (8, m+1),\ v_2(10,2) = (8, m+2), \\ v_2(13,j) &= (13, j-1),\ 1 < j < 2m-2, j \text{ even},\ j \ne m, \\ v_2(16,j) &= (16, j+1),\ 1 < j < 2m-6 \text{ and } j \text{ even}, \\ v_2(16, 2m-6) &= (13, m), \end{aligned}$$

and

$$\begin{aligned} v_3(1,j) &= (1, j-(-1)^j), \text{ where } j \in \mathbb{Z}_{m+3}, \\ v_3(4,j) &= (4, j+(-1)^j), \text{ where } j \in \mathbb{Z}_{2m+2}, \\ v_3(7,m+3) &= (10,1),\ v_3(7,m+4) = (16, m-3), \\ v_3(7,1) &= (16, m-2), \\ v_3(10,2) &= O', \\ v_3(13,1) &= (3, m+4),\ v_3(13, 2m-4) = (3, m+3), \\ v_3(13,j) &= (13, j+1),\ j \text{ even}, 1 < j < 2m-4, \\ v_3(16,j) &= (16, j-1),\ j \text{ even}, 1 \le j \le 2m-6,\ j \ne m-2. \end{aligned}$$

The same argument for Theorem 3 can be applied for the second statement. □

We denote by $\mathcal{O}_{n/d}(\mathcal{L}_{(m,d)})$ an orbifold whose underlying space is the 3-sphere and whose singular set is $\mathcal{L}_{(m,d)}$ with branched index n/d. Similarly by $\mathcal{O}_{n,n/d}(\mathcal{W}_m)$ we denote an orbifold whose underlying space is the 3-sphere and whose singular set is $\mathcal{W}_m$ with the branched indices of its components are n and n/d, respectively. Then we obtain the following result.

Theorem 5. *The closed connected orientable* 3*-manifold* $\mathcal{M}(2m+1, n, k)$ *is an* (n/d)*-fold strongly cyclic covering of the* 3*-sphere branched over* $\mathcal{L}_{(m,d)}$*, where* $d = (n, k)$*. Furthermore,* $\mathcal{M}(2m+1, n, k)$ *is an* n*-fold cyclic branched covering of* $\mathcal{W}_m$ *in the* 3*-sphere, where the branched indices of its components are* n *and* n/d*, respectively.*

Proof. We note that $\mathcal{M}(2m+1, 2d, d)$ is a 2-fold strongly cyclic branched covering of the 3-sphere over $\mathcal{L}_{(m,d)}$ by Theorems 3 and 4. Hence $\mathcal{M}(2m+1, 2d, d)$ admits a 2-symmetric Heegaard splitting. As a sense of [1] $\mathcal{M}(2m+1, n, k)$ admits an (n/d)-symmetric Heegaard splitting by the rotational symmetry, where $d = (n, k)$. By [1], $\mathcal{M}(2m+1, n, k)$ is an (n/d)-fold strongly cyclic branched covering of the 3-sphere over $\mathcal{L}_{(m,d)}$. We note that $\mathcal{L}_{(m,d)}$ has an unknotted component which is the axis of d-symmetry of the chain (see Figure 1). Hence $\mathcal{O}_{n,n/d}(\mathcal{W}_m)$ is the 3-sphere branched over $\mathcal{W}_m$ whose branched indices of its components are n and n/d, respectively. That is, we have the following commutative diagram of branched coverings:

$$\begin{array}{ccc} \mathcal{M}(2m+1, n, k) & \xrightarrow{\ n/d\ } & \mathcal{O}_{n/d}(\mathcal{L}_{(m,d)}) \\ || & & d\downarrow \\ \mathcal{M}(2m+1, n, k) & \xrightarrow{\ n\ } & \mathcal{O}_{n,n/d}(\mathcal{W}_m) \end{array}$$

where the labels of the maps indicate the degree of the covering. □

Acknowledgements. I would like to thank the referee for an improvement of the article and other helpful comments.

References

[1] J. S. Birman and H. M. Hilden, *Heegaard splittings of branched coverings of* $\mathbb{S}^3$, Trans. Amer. Math. Soc. **213** (1975), 315–352.

[2] J. Bracho and L. Montejano, *The combinatorics of colored triangulations of manifolds*, Geom. Dedicata **22** (1987), no. 3, 303–328.

[3] A. Cavicchioli, F. Hegenbarth, and A. C. Kim, *A geometric study of Sieradski groups*, Algebra Colloq. **5** (1998), no. 2, 203–217.

[4] ———, *On cyclic branched coverings of torus knots*, J. Geom. **64** (1999), no. 1-2, 55–66.

[5] A. Cavicchioli and L. Paoluzzi, *On certain classes of hyperbolic 3-manifolds*, Manuscripta Math. **101** (2000), no. 4, 457–494.

[6] M. Ferri, *Crystallisations of* 2*-fold branched coverings of* $\mathbb{S}^3$, Proc. Amer. Math. Soc. **73** (1979), no. 2, 271–276.

[7] M. Ferri and C. Gagliardi, *Crystallisation moves*, Pacific J. Math. **100** (1982), no. 1, 85–103.
[8] M. Ferri, C. Gagliardi, and L. Grasselli, *A graph-theoretical representation of PL-manifolds—a survey on crystallizations*, Aequationes Math. **31** (1986), no. 2-3, 121–141.
[9] C. Gagliardi, *A combinatorial characterization of* 3*-manifold crystallizations*, Boll. Un. Mat. Ital. A (5) **16** (1979), no. 3, 441–449.
[10] H. Helling, A. C. Kim, and J. L. Mennicke, *Some honey-combs in hyperbolic* 3*-space*, Comm. Algebra **23** (1995), no. 14, 5169–5206.
[11] ———, *A geometric study of Fibonacci groups*, J. Lie Theory **8** (1998), no. 1, 1–23.
[12] H. M. Hilden, M. T. Lozano, and J. M. Montesinos-Amilibia, *The arithmeticity of the figure eight knot orbifolds*, Topology '90 (Columbus, OH, 1990), 169–183, Ohio State Univ. Math. Res. Inst. Publ., 1, de Gruyter, Berlin, 1992.
[13] Y. H. Im and S. H. Kim, *Heegaard splittings of the Brieskorn homology spheres that are equivalent after one stabilization*, Note Mat. **20** (2001), no. 1, 53–63.
[14] S. L. Lins and A. Mandel, *Graph-encoded* 3*-manifolds*, Discrete Math. **57** (1985), no. 3, 261–284.
[15] L. Paoluzzi and B. Zimmermann, *On a class of hyperbolic* 3*-manifolds and groups with one defining relation*, Geom. Dedicata **60** (1996), no. 2, 113–123.
[16] H. J. Song, Y. H. Im, and K. H. Ko, *LCG moves in crystallizations*, Geom. Dedicata **76** (1999), no. 3, 229–251.
[17] W. P. Thurston, *The Geometry and Topology of 3-Manifolds*, Lect. Notes, Princeton, N.J., Princeton Univ. Press, 1980.

Soo Hwan Kim
Department of Mathematics
Dongeui University
Pusan 614-714, Korea
E-mail address: `sootopology@hanmail.net`

Yangkok Kim
Department of Mathematics
Dongeui University
Pusan 614-714, Korea
E-mail address: `ykkim@dongeui.ac.kr`