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VARIATION OF PARAMETERS METHOD FOR SOLVING SIXTH-ORDER BOUNDARY VALUE PROBLEMS

Syed Tauseef Mohyud-Din, Muhammad Aslam Noor, and Asif Waheed

Abstract. In this paper, we develop a reliable algorithm which is called the variation of parameters method for solving sixth-order boundary value problems. The proposed technique is quite efficient and is practically well suited for use in these problems. The suggested iterative scheme finds the solution without any perturbation, discritization, linearization or restrictive assumptions. Moreover, the method is free from the identification of Lagrange multipliers. The fact that the proposed technique solves nonlinear problems without using the Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method. Several examples are given to verify the reliability and efficiency of the proposed method. Comparisons are made to reconfirm the efficiency and accuracy of the suggested technique.

1. Introduction

In this paper, we consider the general sixth-order boundary value problems of the type

$$y^{(vi)}(x) = f(x, y), \qquad a \le x \le b \tag{1}$$

with boundary conditions

$$\begin{aligned} y(a) &= A_1, \qquad y''(a) = A_2, \qquad y^{(iv)}(a) = A_3, \\ y(b) &= A_4, \qquad y''(b) = A_5, \qquad y^{(iv)}(b) = A_6. \end{aligned}$$

The sixth-order boundary value problems are known to arise in astrophysics; the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modeled by sixth-order boundary value problems [1-11, 14-17, 23-25, 28-32]. Glatzmaier also notice that dynamo action in some stars may be modeled by such equations see, [10]. Moreover, when an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in (see [28-32]), when this instability is

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of ordinary convection than the governing ordinary differential equation is of sixth-order, see [1-11, 14-17, 23-25, 28-32] and the references therein. Inspired and motivated by the ongoing research in this area, we implemented variation of parameters method to solve sixth-order boundary value problems. It is worth mentioning that the developed method is free from perturbation, discretization, restrictive assumptions and calculation of Adomian's polynomials. The results are very encouraging and reveal the complete reliability of the new algorithm. Several examples are given to verify the efficiency and accuracy of the proposed algorithm.

2. Variation of parameters method

Consider the general sixth-order boundary value problem

$$y^{(vi)}(x) = f(x,y), \qquad a \le x \le b \tag{2}$$

with boundary conditions

$$\begin{aligned} y(a) &= A_1, \qquad y''(a) = A_2, \qquad y^{(iv)}(a) = A_3, \\ y(b) &= A_4, \qquad y''(b) = A_5, \qquad y^{(iv)}(b) = A_6. \end{aligned}$$

The variation of parameters method [12, 13, 18-22, 26, 27] provides the general solution of equation (1) as follows

$$y = \sum_{i=1}^{n} A_i y_i(x) + \sum_{i=1}^{n} y_i(x) \int_0^x f(s,y)\, g(s)\, ds. \tag{3}$$

First term in equation (3), i.e., $\left(\sum_{i=1}^{n} A_i y_i(x)\right)$ is complementary solution of equation (1), second term in equation (3), i.e., $\left(\sum_{i=1}^{n} y_i(x) \int_0^x f(s,y)\, g(s)\, ds\right)$ is a particular solution. Particular solution is obtained by using variation of parameters method. In VPM we replace constants $(A_i's)$ in complementary solution by parameters and then we use this expression in equation (1). After making some assumptions we obtain system of equations. We solve this system of equations to find the value of unknown parameters in form of integrals and hence we have the particular solution as mentioned in the 2nd term of equation (3). In particular solution an independent variable inside integral sign is replaced by some dummy variable making function of and is written as. If we move the variable inside the integral sign which was outside in equation (3), we obtain function of and is represented by in the following equation (4). The imposition of initial and boundary conditions in equation (3) yields following iterative scheme [18-22, 26, 27],

$$y_{n+1} = h(x) + \int_a^b f(s, y_n)\, g(x,s) + \int_a^x f(s, y_n)\, g(x,s)\,. \tag{4}$$

$h(x)$ is function of x, consist of the terms that arise out side the integral after using initial or boundary conditions in equation (3). Further we use $h(x)$ as initial guess, i.e., $h(x) = y_0(x)$.

3. Numerical applications

In this section, we apply the method of variation of parameters developed in Section 2 for solving the sixth-order boundary value problems. Numerical results are very encouraging.

Example 3.1. Consider the following nonlinear boundary value problem of sixth-order

$$y^{(vi)}(x) = e^{-x}y^2(x), \qquad 0 < x < 1, \tag{5}$$

with boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = 1, \qquad y(1) = y''(1) = y^{(iv)}(1) = e. \tag{6}$$

The exact solution for this problem is

$$y(x) = e^x.$$

The method of variation of parameters gives the solution of nonlinear boundary value problem $(5, 6)$ as

$$\begin{aligned} y_{n+1}(x) &= A_1 + A_2 x + A_3\frac{x^2}{2!} + A_4\frac{x^3}{3!} + A_5\frac{x^4}{2!} + A_6\frac{x^5}{5!} \\ &\quad + \int_0^x e^{-s}y_n^2(s)\left(-\frac{s^5}{120} + \frac{xs^4}{24} - \frac{x^2s^3}{12} + \frac{x^3s^2}{12} - \frac{x^4s}{24} + \frac{x^5}{120}\right)ds. \end{aligned}$$

Imposing the boundary conditions will yield

$$\begin{aligned} A_1 &= 1, \\ A_2 &= \frac{1}{120}\left(\frac{307e}{3} - \frac{472}{3}\right) - \frac{1}{120}\int_0^1 e^{-s}y_n^2(s)\left(-s^2 + 5s^4 - \frac{20}{3}s^3 + \frac{8}{3}s\right)ds, \\ A_3 &= 1, \\ A_4 &= \frac{1}{6}(5e-8) - \frac{1}{6}\int_0^1 e^{-s}y_n^2(s)\left(-s^3 + 3s^2 - 2s\right)ds, \\ A_5 &= 1, \\ A_6 &= e - 1 + \int_0^1 e^{-s}y_n^2(s)(s-1)\,ds. \end{aligned}$$

Consequently, following approximants are obtained:

$$\begin{aligned} y_0(x) &= 1 + 1.006979226xx + \frac{1}{2}x^2 + .1553169206x^3 + \frac{1}{24}x^4 + .01431901523x^5, \\ y_1(x) &= 1 + .9999853819x + .500000000x^2 + .1666907042x^3 \\ &\quad + .0416666667x^4 + .008321486957x^5 + .00138888884x^6 \\ &\quad + .0002011822358x^7 + .00002480400282x^8 + 2.379\times 10^{-6}x^9 \\ &\quad + 2.746\times 10^{-7}x^{10} + 6.149\times 10^{-8}x^{11} - 1.884\times 10^{-9}x^{12} \\ &\quad -3.271\times 10^{-10}x^{13} - 2.053\times 10^{-11}x^{14} + 3.421\times 10^{-11}x^{15} + o\left(x^{16}\right), \end{aligned}$$

$$y_2(x) = 1 + 1.000000030x + .500000000x^2 + .1666666166x^3 + .0416666667x^4 + .00833358007x^5 + .00138888884x^6 + .0001984069000x^7 + .00002480400282x^8 + 2.756\times10^{-6}x^9 + 2.755\times10^{-7}x^{10} + 2.498\times10^{-8}x^{11} + 2.087\times10^{-9}x^{12} + 1.650\times10^{-10}x^{13} + 1.147\times10^{-11}x^{14} + 5.561\times10^{-13}x^{15} + o\left(x^{16}\right),$$

$$y_3(x) = 1 + x + .500000000x^2 + .1666666668x^3 + .0416666667x^4 + .008333333269x^5 + .00138888884x^6 + .0001984127119x^7 + .00002480158720x^8 + 2.756\times10^{-6}x^9 + 2.755\times10^{-7}x^{10} + 2.505\times10^{-8}x^{11} + 2.087\times10^{-9}x^{12} + 1.650\times10^{-10}x^{13} + 1.147\times10^{-11}x^{14} + 7.651\times10^{-13}x^{15} + o\left(x^{16}\right),$$

$$y_4(x) = 1 + x + .500000000x^2 + .1666666668x^3 + .0416666667x^4 + .008333333269x^5 + .00138888884x^6 + .0001984127119x^7 + .00002480158720x^8 + 2.756\times10^{-6}x^9 + 2.755\times10^{-7}x^{10} + 2.505\times10^{-8}x^{11} + 2.087\times10^{-9}x^{12} + 1.650\times10^{-10}x^{13} + 1.147\times10^{-11}x^{14} + 7.651\times10^{-13}x^{15} + o\left(x^{16}\right).$$

The series solution is given by

$$y(x) = 1 + x + .500000000x^2 + .1666666668x^3 + .0416666667x^4 + .008333333269x^5 + .00138888884x^6 + .0001984127119x^7 + .00002480158720x^8 + 2.756\times10^{-6}x^9 + 2.755\times10^{-7}x^{10} + 2.505\times10^{-8}x^{11} + 2.087\times10^{-9}x^{12} + 1.650\times10^{-10}x^{13} + 1.147\times10^{-11}x^{14} + 7.651\times10^{-13}x^{15} + o\left(x^{16}\right).$$

Table 3.1(Error estimates)

		*Errors			
x	Exact solution	VIM	HPM	ADM	VPM
0.0	1.000000000	0.000000	0.000000	0.000000	0.000000
0.1	1.105170918	$-1.233E-04$	$-1.233E-04$	$-1.233E-04$	$-3.400E-14$
0.2	1.221402758	$-2.354E-04$	$-2.354E-04$	$-2.354E-04$	$-2.670E-13$
0.3	1.349858808	$-3.257E-04$	$-3.257E-04$	$-3.257E-04$	$-8.890E-13$
0.4	1.491824698	$-3.855E-04$	$-3.855E-04$	$-3.855E-04$	$-2.054E-12$
0.5	1.648721271	$-4.086E-04$	$-4.086E-04$	$-4.086E-04$	$-3.862E-12$
0.6	1.822118800	$-3.919E-04$	$-3.919E-04$	$-3.919E-04$	$-6.311E-12$
0.7	2.013752707	$-3.361E-04$	$-3.361E-04$	$-3.361E-04$	$-9.369E-12$
0.8	2.225540928	$-2.459E-04$	$-2.459E-04$	$-2.459E-04$	$-1.275E-11$
0.9	2.459603111	$-1.299E-04$	$-1.299E-04$	$-1.299E-04$	$-1.609E-11$
1.0	2.718281828	$2.000E-09$	$2.000E-09$	$2.000E-09$	$-1.878E-11$

*Error = Exact solution – Series solution

Table 3.1 exhibits the errors obtained by using the variational iteration method (VIM), the homotopy perturbation method (HPM), Adomian's decomposition method (ADM) and the variation of parameters method (VPM). The table clearly indicates the improvements made by employing the proposed VPM. It is obvious that evaluation of more components of $y(x)$ will reasonably improve the accuracy of series solution.

Example 3.2. Consider the following nonlinear boundary value problem of sixth-order

$$y^{(vi)}(x) = e^{x} y^{2}(x), \qquad 0 < x < 1, \tag{7}$$

with boundary conditions

$$y(0) = y(0) = 1, y''(0) = 1, \qquad y(1) = y''(1) = e^{-1}, y'(1) = -e^{-1}. \tag{8}$$

The exact solution for this problem is

$$y(x) = e^{-x}.$$

The variation of parameters method gives the solution of nonlinear boundary value problem $(7, 8)$ as

$$\begin{aligned} y_{n+1}(x) = {} & A_1 + A_2 x + A_3 \frac{x^2}{2!} + A_4 \frac{x^3}{3!} + A_5 \frac{x^4}{2!} + A_6 \frac{x^5}{5!} \\ & + \int_0^x e^s y_n^2(s) \left(-\frac{s^5}{120} + \frac{xs^4}{24} - \frac{x^2 s^3}{12} + \frac{x^3 s^2}{12} - \frac{x^4 s}{24} + \frac{x^5}{120} \right) ds. \end{aligned}$$

Imposing the boundary conditions will yield

$$\begin{aligned} A_1 &= 1, \\ A_2 &= -1, \\ A_3 &= 1, \\ A_4 &= -33 + \frac{87}{e} - \int_0^1 e^s y_n^2(s) \left(-\frac{1}{2} s^5 + \frac{3}{2} s^4 - \frac{3}{2} s^3 + \frac{1}{2} s^2 \right) ds, \\ A_5 &= 204 - \frac{552}{e} + \int_0^1 e^s y_n^2(s) \left(-3s^5 + 8s^4 - 6s^3 + s \right) ds, \\ A_6 &= -420 + \frac{1140}{e} + \int_0^1 e^s y_n^2(s) \left(6s^5 - 15s^4 + 10s^3 - 1 \right) ds. \end{aligned}$$

Consequently, following approximants are obtained:

$$\begin{aligned} y_0(x) = {} & 1 - x + \frac{1}{2} x^2 - .1657481033 x^3 + .03877285417 x^4 \\ & - .005145308333 x^5, \end{aligned}$$

$$\begin{aligned}
y_1(x) &= 1 - x + .500000000x^2 - .1666666928x^3 + .04166673768x^4 \\
&+ .00833384272x^5 + .00138888884x^6 - .0001984127000x^7 \\
&+ .00002480158720x^8 - 2.725 \times 10^{-6}x^9 + 2.372 \times 10^{-7}x^{10} \\
&- 5.884 \times 10^{-9}x^{11} - 2.086 \times 10^{-9}x^{12} + 1.569 \times 10^{-10}x^{13} \\
&- 7.152 \times 10^{-12}x^{14} - 1.813 \times 10^{-12}x^{15} + o\left(x^{16}\right), \\
y_2(x) &= 1 - x + .500000000x^2 - .1666666670x^3 + .04166673768x^4 \\
&- .00833332794x^5 + .00138888884x^6 - .0001984127000x^7 \\
&+ .00002480158720x^8 - 2.755 \times 10^{-6}x^9 + 2.755 \times 10^{-7}x^{10} \\
&- 2.505 \times 10^{-8}x^{11} - 2.086 \times 10^{-9}x^{12} - 1.605 \times 10^{-10}x^{13} \\
&+ 1.147 \times 10^{-11}x^{14} - 7.478 \times 10^{-13}x^{15} + o\left(x^{16}\right).
\end{aligned}$$

The series solution is given by

$$\begin{aligned}
y(x) &= 1 - x + .500000000x^2 - .1666666670x^3 + .04166673768x^4 \\
&- .00833332794x^5 + .00138888884x^6 - .0001984127000x^7 \\
&+ .00002480158720x^8 - 2.755 \times 10^{-6}x^9 + 2.755 \times 10^{-7}x^{10} \\
&- 2.505 \times 10^{-8}x^{11} - 2.086 \times 10^{-9}x^{12} - 1.605 \times 10^{-10}x^{13} \\
&+ 1.147 \times 10^{-11}x^{14} - 7.478 \times 10^{-13}x^{15} + o\left(x^{16}\right).
\end{aligned}$$

Table 3.2(Error estimates)

		*Errors			
x	Exact solution	VIM	HPM	ADM	VPM
0.0	1.000000000	0.000000	0.000000	0.000000	0.000000
0.1	0.9048374180	$-2.347E-07$	$-2.347E-07$	$-2.347E-07$	$-2.170E-13$
0.2	0.8187307531	$-1.389E-06$	$-1.389E-06$	$-1.389E-06$	$-7.130E-13$
0.3	0.7408182207	$-3.307E-06$	$-3.307E-06$	$-3.307E-06$	$-1.325E-12$
0.4	0.6703200460	$-5.203E-06$	$-5.203E-06$	$-5.203E-06$	$-1.267E-11$
0.5	0.6065306597	$-6.198E-06$	$-6.198E-06$	$-6.198E-06$	$-4.468E-11$
0.6	0.5488116361	$-5.780E-06$	$-5.780E-06$	$-5.780E-06$	$-1.139E-10$
0.7	0.4965853038	$-4.082E-06$	$-4.082E-06$	$-4.082E-06$	$-2.429E-10$
0.8	0.4493289641	$-1.903E-06$	$-1.903E-06$	$-1.903E-06$	$-4.604E-10$
0.9	0.4065696597	$-3.570E-06$	$-3.570E-06$	$-3.570E-06$	$-8.025E-13$
1.0	0.3678794412	$-5.000E-06$	$-5.000E-06$	$-5.000E-06$	$-1.313E-13$

*Error = Exact solution – Series solution

Table 3.2 exhibits the errors obtained by using the variational iteration method (VIM), the homotopy perturbation method (HPM), the Adomian's decomposition method (ADM) and the variation of parameters method (VPM). The table clearly indicates the improvements made by employing the proposed VPM. It is obvious that evaluation of more components of $y(x)$ will reasonably improve the accuracy of series solution.

Example 3.3. Consider the following linear boundary value problem of sixth-order

$$y^{(vi)}(x) = -6e^{x} + y(x), \qquad 0 < x < 1, \tag{9}$$

with boundary conditions

$$\begin{aligned} &y(0) = 1,\ y''(0) = -1,\ y^{(iv)}(0) = -3, \\ &y(1) = 0,\ y''(1) = -2e,\ y(1)^{(iv)} = -4e. \end{aligned} \tag{10}$$

The exact solution of the problem is

$$y(x) = (1-x)e^{x}.$$

The variation of parameters method gives the solution of nonlinear boundary value problem $(9, 10)$ as

$$\begin{aligned} y_{n+1}(x) = {} & A_1 + A_2 x + A_3 \frac{x^2}{2!} + A_4 \frac{x^3}{3!} + A_5 \frac{x^4}{2!} + A_6 \frac{x^5}{5!} \\ & + \int_0^x \left(-6e^{-s} + y_n(s)\right)\left(-\frac{s^5}{120} + \frac{xs^4}{24} - \frac{x^2 s^3}{12} + \frac{x^3 s^2}{12} - \frac{x^4 s}{24} + \frac{x^5}{120}\right) ds. \end{aligned}$$

Imposing the boundary conditions will yield

$$\begin{aligned} A_1 &= 1, \\ A_2 &= \frac{23e}{90} - \frac{11}{15} - \int_0^1 \left(-6e^{-s} + y_n(s)\right)\left(-\frac{1}{20}s^5 + \frac{1}{24}s^4 - \frac{1}{18}s^3 + \frac{1}{45}s^2\right) ds, \\ A_3 &= 1, \\ A_4 &= -\frac{4e}{3} + 2 - \int_0^1 \left(-6e^{-s} + y_n(s)\right)\left(-\frac{1}{6}s^3 + \frac{1}{2}s^2 - \frac{1}{3}s\right) ds, \\ A_5 &= 1, \\ A_6 &= 3 - 4e + \int_0^1 \left(-6e^{-s} + y_n(s)\right)(s-1)\, ds. \end{aligned}$$

Consequently, following approximants are obtained:

$$\begin{aligned} y_0(x) = {} & 1 - .0386613106x - \frac{1}{2}x^2 - .2707292952x^3 - \frac{1}{8}x^4 - .06560939425x^5, \\ y_1(x) = {} & 1 + .00004051915000x - .500000000x^2 - .3333999574x^3 - .125000000x^4 \\ & - .03330050718x^5 - .006944444445x^6 - .00198147092x^7 \\ & - 2.480 \times 10^{-5}x^8 - 2.101 \times 10^{-5}x^9 - 2.480 \times 10^{-6}x^{10} \\ & - 3.475 \times 10^{-7}x^{11} - 1.252 \times 10^{-8}x^{12} - 9.635 \times 10^{-10}x^{13} \\ & - 6.882 \times 10^{-11}x^{14} - 4.588 \times 10^{-12}x^{15} + o\left(x^{16}\right), \end{aligned}$$

$$\begin{aligned}
y_2(x) = {} & 1 - 4.227\times10^{-7}x - .500000000x^2 - .333332640x^3 - .125000000x^4 \\
& - .03333336753x^5 - .006944444445x^6 - .001190468165x^7 \\
& - 1.736\times10^{-4}x^8 - 2.204\times10^{-5}x^9 - 2.480\times10^{-6}x^{10} \\
& - 2.504\times10^{-7}x^{11} - 2.296\times10^{-8}x^{12} - 1.933\times10^{-9}x^{13} \\
& - 1.491\times10^{-10}x^{14} - 1.042\times10^{-11}x^{15} + o\left(x^{16}\right), \\
y_3(x) = {} & 1 - 7.000\times10^{-11}x - .500000000x^2 - .333332334x^3 - .125000000x^4 \\
& - .03333333327x^5 - .006944444445x^6 - .001190476205x^7 \\
& - 1.736\times10^{-4}x^8 - 2.204\times10^{-5}x^9 - 2.480\times10^{-6}x^{10} \\
& - 2.505\times10^{-7}x^{11} - 2.296\times10^{-8}x^{12} - 1.927\times10^{-9}x^{13} \\
& - 1.491\times10^{-10}x^{14} - 1.070\times10^{-11}x^{15} + o\left(x^{16}\right).
\end{aligned}$$

The series solution is given by

$$\begin{aligned}
y(x) = {} & 1 - 7.000\times10^{-11}x - .500000000x^2 - .333332334x^3 - .125000000x^4 \\
& - .03333333327x^5 - .006944444445x^6 - .001190476205x^7 \\
& - 1.736\times10^{-4}x^8 - 2.204\times10^{-5}x^9 - 2.480\times10^{-6}x^{10} \\
& - 2.505\times10^{-7}x^{11} - 2.296\times10^{-8}x^{12} - 1.927\times10^{-9}x^{13} \\
& - 1.491\times10^{-10}x^{14} - 1.070\times10^{-11}x^{15} + o\left(x^{16}\right).
\end{aligned}$$

Table 3.3(Error estimates)

		*Errors			
x	Exact solution	VIM	HPM	ADM	VPM
0.0	1.000000000	0.000000	0.000000	0.000000	0.000000
0.1	0.99465383	−0.00040933	−0.00040933	−0.00040933	$-7.066E-12$
0.2	0.97712221	−0.00077820	−0.00077820	−0.00077820	$-1.451E-11$
0.3	0.94490117	−0.00107048	−0.00107048	−0.00107048	$-2.265E-11$
0.4	0.89509482	−0.00125787	−0.00125787	−0.00125787	$-3.164E-11$
0.5	0.82436064	−0.00132238	−0.00132238	−0.00132238	$-4.147E-11$
0.6	0.72884752	−0.00125787	−0.00125787	−0.00125787	$-5.189E-10$
0.7	0.60412581	−0.00107048	−0.00107048	−0.00107048	$-6.241E-10$
0.8	0.44510819	−0.00077820	−0.00077820	−0.00077820	$-7.229E-11$
0.9	0.24596031	−0.00040933	−0.00040933	−0.00040933	$-8.053E-13$
1.0	0.000000	0.000000	0.000000	0.000000	$-8.562E-11$

*Error = Exact solution – Series solution

Table 3.3 exhibits the errors obtained by using the variational iteration method (VIM), the homotopy perturbation method (HPM), Adomian's decomposition method (ADM) and the variation of parameters method (VPM). The table clearly indicates the improvements made by employing the proposed VPM. It is obvious that evaluation of more components of $y(x)$ will reasonably improve the accuracy of series solution.

4. Conclusion

In this paper, we have used the method of variation of parameters for finding the solution of boundary value problems for sixth-order. The method is applied in a direct way without using linearization, transformation, discretization, perturbation or restrictive assumptions. It may be concluded that the proposed technique is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result, the size reduction amounts to the improvement of performance of approach. The fact that the developed algorithm solves nonlinear problems without using the Adomian's polynomials and the identification of Lagrange multipliers are the clear advantages of this technique over the variational iteration method and the decomposition method.

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Syed Tauseef Mohyud-Din
Department of Mathematics
HITEC University
Taxila Cantt, 44000, Pakistan
E-mail address: syedtauseefs@hotmail.com

Muhammad Aslam Noor
Department of Mathematics
COMSATS Institute of Information Technology
Islamabad, 44000, Pakistan
E-mail address: noormaslam@hotmail.com

Asif Waheed
Department of Mathematics
COMSATS Institute of Information Technology
Islamabad, 44000, Pakistan
E-mail address: waheedasif@hotmail.com