CATEGORICAL PROPERTY OF INTUITIONISTIC TOPOLOGICAL SPACES

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Abstract. We obtain some characterizations of continuous, open and closed functions in intuitionistic topological spaces. Moreover we reveal that the category of topological spaces is a bireflective full subcategory of the category of intuitionistic topological spaces.

1. Introduction

After Atanassov [1, 2, 4] introduced the concept of "intuitionistic fuzzy sets" as a generalization of fuzzy sets, it becomes a popular topic of investigation in the fuzzy set community. Many mathematical advantages of intuitionistic fuzzy sets are discussed in [3]. Çoker [7] generalized topological structures in fuzzy topological spaces to intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. Later many researchers have studied topics related to intuitionistic fuzzy topological spaces.

On the other hand, Çoker [8] introduced the concept of "intuitionistic sets" in 1996. This is a discrete form of intuitionistic fuzzy set, where all the sets are entirely the crisp sets. Still it has membership and nonmembership degrees, so this concept gives us more flexible approaches to representing vagueness in mathematical objects including engineering fields with classical set logic. In 2000, Çoker [9] also introduced the concept of intuitionistic topological spaces with intuitionistic sets, and investigated basic properties of continuous functions and compactness. He and his colleague [5, 6] also examined separation axioms in intuitionistic topological spaces.

In this paper, we obtain some characterizations of continuous, open and closed functions in intuitionistic topological spaces. Moreover we reveal that the category of topological spaces is a bireflective full subcategory of the category of intuitionistic topological spaces.

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2. Preliminaries

Here we list some definitions and properties of intuitionistic topological spaces which we shall use frequently in the following sections.

Definition 2.1 ([8]). Let X be a nonempty set. An *intuitionistic set* (IS for short) A is an object having the form

$$A = \langle X, A^1, A^2 \rangle,$$

where A^1 and A^2 are subsets of X satisfying $A^1 \cap A^2 = \emptyset$. The set A^1 is called the set of members of A, while A^2 is called the set of nonmembers of A. Every crisp set A on a nonempty set X is obviously an IS having the form $\langle X, A, A^c \rangle$.

Definition 2.2 ([8]). Let $A = \langle X, A^1, A^2 \rangle$ and $B = \langle X, B^1, B^2 \rangle$ be IS's on X. Then

(1) $A \subseteq B$ if and only if $A^1 \subseteq B^1$ and $B^2 \subseteq A^2$.
(2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
(3) $\overline{A} = \langle X, A^2, A^1 \rangle$.
(4) $\cap A_i = \langle X, \cap A_i^1, \cup A_i^2 \rangle$.
(5) $\cup A_i = \langle X, \cup A_i^1, \cap A_i^2 \rangle$.
(6) $\square A = \langle X, A^1, (A^1)^c \rangle$.
(7) $\Diamond A = \langle X, (A^2)^c, A^2 \rangle$.
(8) $\emptyset_\sim = \langle X, \emptyset, X \rangle$ and $X_\sim = \langle X, X, \emptyset \rangle$.

Let f be a function from a set X to a set Y. Let $A = \langle X, A^1, A^2 \rangle$ be an IS in X and $B = \langle Y, B^1, B^2 \rangle$ an IS in Y. Then the *preimage* $f^{-1}(B)$ is an IS in X defined by

$$f^{-1}(B) = \langle X, f^{-1}(B^1), f^{-1}(B^2) \rangle$$

and the *image* $f(A)$ is an IS in Y defined by

$$f(A) = \langle Y, f(A^1), f_-(A^2) \rangle,$$

where $f_-(A^2) = (f((A^2)^c))^c$.

Theorem 2.3 ([8]). *Let $A, A_i (i \in J)$ be IS's in X, $B, B_j (j \in K)$ IS's in Y, and $f : X \to Y$ a function. Then*

(1) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$, $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$.
(2) $A \subseteq f^{-1}(f(A))$, *and if f is injective, then $A = f^{-1}(f(A))$.*
(3) $f(f^{-1}(B)) \subseteq B$, *and if f is surjective, then $f(f^{-1}(B)) = B$.*
(4) $f^{-1}(\cup B_i) = \cup f^{-1}(B_i)$, $f^{-1}(\cap B_i) = \cap f^{-1}(B_i)$.
(5) $f(\cup A_i) = \cup f(A_i)$; $f(\cap A_i) \subseteq \cap f(A_i)$, *and if f is injective, then $f(\cap A_i) = \cap f(A_i)$.*
(6) $f^{-1}(Y_\sim) = X_\sim$, $f^{-1}(\emptyset_\sim) = \emptyset_\sim$.
(7) $f(\emptyset_\sim) = \emptyset_\sim$; $f(X_\sim) = Y_\sim$, *if f is surjective.*
(8) *If f is surjective, then $\overline{f(A)} \subseteq f(\overline{A})$. If, furthermore, f is injective, then we have $\overline{f(A)} = f(\overline{A})$.*
(9) $(f^{-1}(\overline{B})) = \overline{f^{-1}(B)}$.

Definition 2.4 ([7])**.** An intuitionistic topology (IT for short) on a nonempty set X is a family τ of IS's in X satisfying the following axioms:

(1) $\emptyset_\sim$, $X_\sim \in \mathcal{T}$,
(2) $G_1 \cap G_2 \in \mathcal{T}$ for any $G_1, G_2 \in \mathcal{T}$,
(3) $\bigcup G_i \in \mathcal{T}$ for any arbitrary family $\{G_i \mid i \in J\} \subseteq \mathcal{T}$.

In this case the pair $(X, \mathcal{T})$ is called an *intuitionistic topological space* (ITS for short) and any IS in $\mathcal{T}$ is called an *intuitionistic open* set (IOS for short) in X. The complement $\overline{A}$ of an IOS A is called an *intuitionistic closed* set (ICS for short) in X.

Example 2.5. For any topological space (X, τ), we trivially have an ITS $(X, \mathcal{T})$, where $\mathcal{T} = \{\langle X, A, A^c \rangle \mid A \in \tau\}$.

Example 2.6. Consider the set $X = \mathbb{R}$ and take the family

$$\mathcal{S} = \{\langle \mathbb{R}, (a, b), (-\infty, a] \rangle \mid a, b \in \mathbb{R}\}$$

of IS's in $\mathbb{R}$. In this case $\mathcal{S}$ generates an IT $\mathcal{T}$ on $\mathbb{R}$, which is called the "usual left intuitionistic topology" on $\mathbb{R}$. The base $\mathcal{B}$ for this IT can be written in the form $\mathcal{B} = \{X_\sim\} \cup \mathcal{S}$ while $\mathcal{T}$ consists of the following IS's:

$\emptyset_\sim, X_\sim$;

$\langle \mathbb{R}, \bigcup(a_i, b_i), (-\infty, c] \rangle$, where $a_i, b_i, c \in \mathbb{R}, \{a_i \mid i \in J\}$ is bounded from below, $c = \inf\{a_i \mid i \in J\}$;

$\langle \mathbb{R}, \bigcup(a_i, b_i), \emptyset \rangle$, where $a_i, b_i, c \in \mathbb{R}, \{a_i \mid i \in J\}$ is not bounded from below.

Definition 2.7 ([7])**.** Let $(X, \mathcal{T})$ be an ITS and $A = \langle X, A^1, A^2 \rangle$ be an IS in X. Then the *interior* and *closure* of A are defined by

$$\mathrm{cl}(A) = \bigcap\{K \mid K \text{ is an ICS in } X \text{ and } A \subseteq K\},$$

$$\mathrm{int}(A) = \bigcup\{G \mid G \text{ is an IOS in } X \text{ and } G \subseteq A\}.$$

Theorem 2.8 ([9])**.** *For any IS A in $(X, \mathcal{T})$, the following properties hold*:

$$\mathrm{cl}(\overline{A}) = \overline{\mathrm{int}(A)},$$

$$\mathrm{int}(\overline{A}) = \overline{\mathrm{cl}(A)}.$$

Theorem 2.9 ([9])**.** *Let $(X, \mathcal{T})$ be an ITS and A, B be IS's in X. Then the following properties hold*:

(1) $\mathrm{int}(A) \subseteq A$.
(2) $A \subseteq B \Rightarrow \mathrm{int}(A) \subseteq \mathrm{int}(B)$.
(3) $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.
(4) $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.
(5) $\mathrm{int}(X_\sim) = X_\sim$.
(6) $A \subseteq \mathrm{cl}(A)$.
(7) $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
(8) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
(9) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

(10) $\mathrm{cl}(\emptyset_\sim) = \emptyset_\sim$.

Definition 2.10 ([7])**.** Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two ITS's and let $f : X \to Y$ be a function. Then f is said to be *continuous* if and only if the preimage of each IS in $\mathcal{T}'$ is an IS in $\mathcal{T}$.

Definition 2.11 ([7])**.** Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two ITS's and let $f : X \to Y$ be a function. Then f is said to be *open* if and only if the image of each IS in $\mathcal{T}$ is an IS in $\mathcal{T}'$.

Theorem 2.12 ([9])**.** *$f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if the preimage of each ICS in $\mathcal{T}'$ is an ICS in $\mathcal{T}$.*

Theorem 2.13 ([9])**.** *The following are equivalents to each other:*

 (1) *$f : X \to Y$ is continuous.*
 (2) *$f^{-1}(\mathrm{int}(B)) \subseteq \mathrm{int}(f^{-1}(B))$ for each IS B in Y.*
 (3) *$\mathrm{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}(B))$ for each IS B in Y.*

For categorical terminologies used in the following sections refer to [10].

3. More about continuous, open, and closed functions

In this section, we obtain some characterizations of continuous, open and closed functions in intuitinistic topological spaces.

Theorem 3.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be ITS's. Then the following are equivalents:*

 (1) *$f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous.*
 (2) *$f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$ for each IS A in X.*

Proof. $(1) \Rightarrow (2)$ Let f be a continuous function and A any IS in X. Let $f(A) = B$. Then $A \subseteq f^{-1}(B)$. Thus, by Theorem 2.13, we have

$$f(\mathrm{cl}(A)) \subseteq f(\mathrm{cl}(f^{-1}(B))) \subseteq f(f^{-1}(\mathrm{cl}(B))) \subseteq \mathrm{cl}(B) = \mathrm{cl}(f(A)).$$

$(2) \Rightarrow (1)$ Let B be any IS in Y. By (2),

$$f(\mathrm{cl}(f^{-1}(B))) \subseteq \mathrm{cl}(ff^{-1}(B)).$$

Thus

$$\mathrm{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}(ff^{-1}(B))) \subseteq f^{-1}(\mathrm{cl}(B)).$$

Thus it follows by Theorem 2.13.   □

Theorem 3.2. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be an injection. If f is continuous, then $\mathrm{int}(f(A)) \subseteq f(\mathrm{int}(A))$ for each IS A in X*

Proof. Let A be an IS in X. Then by Theorem 2.13, we have

$$f^{-1}(\mathrm{int}(f(A))) \subseteq \mathrm{int}(f^{-1}(f(A))).$$

Since f is injective,

$$f(f^{-1}(\mathrm{int}(f(A)))) \subseteq \mathrm{int}(f(A)) \subseteq f(\mathrm{int}(f^{-1}(f(A)))) = f(\mathrm{int}(A)).$$

Thus $\mathrm{int}(f(A)) \subseteq f(\mathrm{int}(A))$.   □

Example 3.3. Let $X = \{a, b, c, d\}$ and consider the intuitionistic topology

$$\mathcal{T} = \{\emptyset_\sim, X_\sim, A_1\}, \mathcal{T}' = \{\emptyset_\sim, X_\sim, B_1, B_2\},$$

where $A_1 = \langle X, \{a, d\}, \{b, c\}\rangle, B_1 = \langle X, \{a\}, \{b, c, d\}\rangle, B_2 = \langle X, \{a, b\}, \{c, d\}\rangle$. Define $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ by $f(a) = f(d) = a, f(b) = f(c) = b$. Then $f^{-1}(B_1) = A_1, f^{-1}(B_2) = X_\sim$. Thus f is continuous and not injective. Let $A = \langle X, \{a, b, d\}, \{c\}\rangle$ be an IS in $(X, \mathcal{T})$. Then $f(\mathrm{int}(A)) = B_1$, and $\mathrm{int}(f(A)) = B_2$. Thus $\mathrm{int}(f(A)) \not\subseteq f(\mathrm{int}(A))$.

Theorem 3.4. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be ITS's. Then the following are equivalents*:

    (1) $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ *is open.*
    (2) $f(\mathrm{int}(A)) \subseteq \mathrm{int}(f(A))$ *for each IS A in X.*
    (3) $\mathrm{int}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{int}(B))$ *for each IS B in Y.*

Proof. $(1) \Rightarrow (2)$ Let f be an open function. Since $f(\mathrm{int}(A))$ is an open set contained in $f(A)$, $f(\mathrm{int}(A)) \subseteq \mathrm{int}(f(A))$ by definition of interior.

    $(2) \Rightarrow (3)$ Let B be any IS in Y. Then $f^{-1}(B)$ is an IS in X. By (2),

$$f(\mathrm{int}(f^{-1}(B)) \subseteq \mathrm{int}(ff^{-1}(B)) \subseteq \mathrm{int}(B).$$

Thus we have

$$\mathrm{int}(f^{-1}(B)) \subseteq f^{-1}f(\mathrm{int}(f^{-1}(B))) \subseteq f^{-1}(\mathrm{int}(B)).$$

    $(3) \Rightarrow (1)$ Let A be any IOS in X. Then $\mathrm{int}(A) = A$ and $f(A)$ is an IS in Y. By (3),

$$A = \mathrm{int}(A) \subseteq \mathrm{int}(f^{-1}f(A)) \subseteq f^{-1}(\mathrm{int}(f(A))).$$

Hence we have

$$f(A) \subseteq ff^{-1}(\mathrm{int}(f(A))) \subseteq \mathrm{int}(f(A)) \subseteq f(A).$$

Thus $f(A) = \mathrm{int}(f(A))$ and hence $f(A)$ is an IOS in Y. Therefore f is an open function. □

Definition 3.5. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two ITS's and let $f : X \to Y$ be a function. Then f is said to be *closed* if and only if the image of each ICS in X is an ICS in Y.

Example 3.6. Let $X = \{a, b, c\}$ and consider the intuitionistic topology

$$\mathcal{T} = \{\emptyset_\sim, X_\sim, A_1, A_2\},$$

where $A_1 = \langle X, \{a\}, \{b, c\}\rangle, A_2 = \langle X, \{a, b\}, \{c\}\rangle$. Define $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ by $f(a) = f(b) = a, f(c) = b$ and $g : (X, \mathcal{T}) \to (X, \mathcal{T})$ by $g(a) = g(b) = b, g(c) = c$. Then $f(A_1) = \langle X, \{a\}, \{b, c\}\rangle = A_1 \in \mathcal{T}, f(A_2) = \langle X, \{a\}, \{b, c\}\rangle = A_1 \in \mathcal{T}, f(X_\sim) = \langle X, \{a, b\}, \{c\}\rangle = A_2 \in \mathcal{T}$, and $f(\emptyset_\sim) = \langle X, \emptyset, X\rangle = \emptyset_\sim \in \mathcal{T}$. Thus f is an open function. However, $f(\overline{A_2}) = f(\langle X, \{c\}, \{a, b\}\rangle) = \langle X, \{b\}, \{a, c\}\rangle$ is not an ICS in Y. Thus f is not a closed function.

    Moreover $g(\overline{A_1}) = g(\langle X, \{b, c\}, \{a\}\rangle) = \langle X, \{b, c\}, \{a\}\rangle = \overline{A_1}$, $g(\overline{A_2}) = g(\langle X, \{c\}, \{a, b\}\rangle) = \langle X, \{c\}, \{a, b\}\rangle = \overline{A_2}$, $g(\overline{\emptyset_\sim}) = g(\langle X, X, \emptyset\rangle) = \langle X, \{b, c\}, \{a\}\rangle$

$= \overline{A_1}$, and $g(\overline{X_\sim}) = g(\langle X, \emptyset, X \rangle) = \langle X, \emptyset, X \rangle = \overline{X_\sim}$. Thus g is a closed function. However, $g(A_1) = \langle X, \{b\}, \{a, c\} \rangle$ is not an IOS in Y. Thus g is not an open function.

Theorem 3.7. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a bijection. Then f is open if and only if f is closed.*

Proof. Let f be an open function, and let F be an intuitionistic closed set in X. Then $F = \overline{G}$ for some IOS $G = \langle X, G^1, G^2 \rangle$ in X. Hence

$$f(F) = f(\overline{G}) = f(\langle X, G^2, G^1 \rangle) = \langle Y, f(G^2), (f(G^1)^c)^c \rangle$$
$$= \langle Y, f(G^2), f(G^1) \rangle = \overline{\langle Y, f(G^1), f(G^2) \rangle},$$

which is an ICS in Y, because f is an open function. Similarly the converse can be proved. □

Theorem 3.8. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be ITS's. Then the following statements are equivalent*:

    (1) $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ *is a closed function.*
    (2) $\mathrm{cl}(f(A)) \subseteq f(\mathrm{cl}(A))$ *for each IS A in X.*

Proof. (1) $\Rightarrow$ (2) Let A be any IS in X. Clearly, $\mathrm{cl}(A)$ is an ICS in X. Since f is a closed function, $f(\mathrm{cl}(A))$ is an ICS in Y. Thus we have

$$\mathrm{cl}(f(A)) \subseteq \mathrm{cl}(f(\mathrm{cl}(A))) = f(\mathrm{cl}(A)).$$

    (2) $\Rightarrow$ (1) Let A be any ICS in X. Then $\mathrm{cl}(A) = A$. By (2),

$$\mathrm{cl}(f(A)) \subseteq f(\mathrm{cl}(A)) = f(A) \subseteq \mathrm{cl}(f(A)).$$

Thus $f(A) = \mathrm{cl}(f(A))$ and hence $f(A)$ is an ICS in Y. Therefore f is a closed function. □

4. Categorical properties

    Now we are ready to study the relation between the category of topological spaces and the category of intuitionistic topological spaces.

    Let **Top** be the category of all topological spaces and continuous functions, and **ITop** the category of all intuitionistic topological spaces and continuous functions.

    Now we define some functors between **Top** and **ITop**.

Theorem 4.1. *Define $G_1, G_2 : $ **ITop** $\to$ **Top** by*

$$G_1(X, \mathcal{T}) = (X, G_1(\mathcal{T})) \quad and \quad G_1(f) = f,$$

where $G_1(\mathcal{T}) = \{A^1 \mid \langle X, A^1, A^2 \rangle \in \mathcal{T}\}$,

$$G_2(X, \mathcal{T}) = (X, G_2(\mathcal{T})) \quad and \quad G_2(f) = f,$$

where $G_2(\mathcal{T}) = \{(A^2)^c \mid \langle X, A^1, A^2 \rangle \in \mathcal{T}\}$. Then G_1 and G_2 are functors.

Proof. Trivial. □

Theorem 4.2. *Define $F_0 : \mathbf{Top} \to \mathbf{ITop}$ by*

$$F_0(X, \tau) = (X, F_0(\tau)) \quad and \quad F_0(f) = f,$$

where $F_0(\tau) = \{\langle X, A, A^c \rangle \mid A \in \tau\}$. Then F_0 is a functor.

Proof. Clearly $F_0(\tau)$ is an IT. Next, we show that if $f : (X, \tau) \to (Y, \tau')$ is continuous then $f : (X, F_0(\tau)) \to (Y, F_0(\tau'))$ is continuous. Let $B = \langle Y, B, B^c \rangle \in F_0(\tau')$. Then $B \in \tau'$, so $f^{-1}(B) \in \tau$. Since $f^{-1}(B^c) = (f^{-1}(B))^c$, we have $\langle X, f^{-1}(B), f^{-1}(B^c) \rangle = \langle X, f^{-1}(B), (f^{-1}(B))^c \rangle \in F_0(\tau)$. Thus F_0 is a functor. $\square$

Theorem 4.3. *Define $F_1 : \mathbf{Top} \to \mathbf{ITop}$ by*

$$F_1(X, \tau) = (X, F_1(\tau)) \quad and \quad F_1(f) = f,$$

where $F_1(\tau) = \{\langle X, A, \emptyset \rangle \mid A \in \tau\}$. Then F_1 is a functor.

Proof. Clearly $F_1(\tau)$ is an IT. Next, we show that if $f : (X, \tau) \to (Y, \tau')$ is continuous then $f : (X, F_1(\tau)) \to (Y, F_1(\tau'))$ is continuous. Let $B = \langle Y, B, \emptyset \rangle \in F_1(\tau')$. Then $B \in \tau'$, so $f^{-1}(B) \in \tau$. Thus $\langle X, f^{-1}(B), \emptyset \rangle \in F_1(\tau)$. Thus F_1 is a functor. $\square$

Theorem 4.4. *Define $F_2 : \mathbf{Top} \to \mathbf{ITop}$ by*

$$F_2(X, \tau) = (X, F_2(\tau)) \quad and \quad F_2(f) = f,$$

where $F_2(\tau) = \{\langle X, \emptyset, A^c \rangle \mid A \in \tau\}$. Then F_2 is a functor.

Theorem 4.5. *The functor $F_1 : \mathbf{Top} \to \mathbf{ITop}$ is a left adjoint of the functor $G_1 : \mathbf{ITop} \to \mathbf{Top}$.*

Proof. For any (X, τ) in $\mathbf{Top}$, $1_X : (X, \tau) \to G_1 F_1(X, \tau) = (X, \tau)$ is a fuzzy continuous function. Consider $(Y, \tau') \in \mathbf{ITop}$ and a fuzzy continuous function $f : (X, \tau) \to G_1(Y, \tau')$. In order to show that $f : F_1(X, \tau) = (X, F_1(\tau)) \to (Y, \tau')$ is a continuous function, let $B = \langle Y, B^1, B^2 \rangle \in \tau'$. Then $B^1 \in G_1(\tau')$. Since $f : (X, \tau) \to G_1(Y, \tau') = (X, G_1(\tau'))$ is continuous, $f^{-1}(B^1) \in \tau$. Thus $\langle X, f^{-1}(B), \emptyset \rangle \in F_1(\tau)$. Hence $f : F_1(X, \tau) \to (Y, \tau')$ is continuous. Therefore 1_X is a G_1-universal function for (X, τ) in $\mathbf{Top}$. $\square$

Theorem 4.6. *The functor $F_2 : \mathbf{Top} \to \mathbf{ITop}$ is a left adjoint of the functor $G_2 : \mathbf{ITop} \to \mathbf{Top}$.*

Proof. For any (X, τ) in $\mathbf{Top}$, $1_X : (X, \tau) \to G_2 F_2(X, \tau) = (X, \tau)$ is a fuzzy continuous function. Consider $(Y, \tau') \in \mathbf{ITop}$ and a fuzzy continuous function $f : (X, \tau) \to G_2(Y, \tau')$. In order to show that $f : F_2(X, \tau) = (X, F_2(\tau)) \to (Y, \tau')$ is a continuous function, let $B = \langle Y, B^1, B^2 \rangle \in \tau'$. Then $(B^2)^c \in G_2(\tau')$. Since $f : (X, \tau) \to G_2(Y, \tau') = (X, G_2(\tau'))$ is continuous, $(f^{-1}(B^2))^c = f^{-1}((B^2)^c) \in \tau$. Thus $\langle X, \emptyset, f^{-1}(B^2) \rangle \in F_2(\tau)$. Hence $f : F_2(X, \tau) \to (Y, \tau')$ is continuous. Therefore 1_X is a G_2-universal function for (X, τ) in $\mathbf{Top}$. $\square$

Let $\mathbf{ITop}_1$ the category of all ITS's whose elements are of the form $\langle X, A^1, \emptyset \rangle$, and continuous functions.

Theorem 4.7. *Two categories* $\mathbf{Top}$ *and* $\mathbf{ITop}_1$ *are isomorphic.*

Proof. Define $F_1 : \mathbf{Top} \to \mathbf{ITop}_1$ by

$$F_1(X, \tau) = (X, F_1(\tau)) \quad \text{and} \quad F_1(f) = f,$$

where $F_1(\tau) = \{\langle X, A, \emptyset \rangle \mid A \in \tau\}$. Consider the restriction $G_1 : \mathbf{ITop}_1 \to \mathbf{Top}$ of the functor G_1 in Definition 4.1. Then F_1 and G_1 are functors. Clearly $G_1 F_1(X, \tau) = G_1(X, F_1(\tau)) = (X, G_1(F_1(\tau)) = (X, \tau)$. Moreover $F_1 G_1(X, \mathcal{T}) = (X, \mathcal{T})$. Hence the result follows. $\square$

Theorem 4.8. *The category* $\mathbf{ITop}_1$ *is a bireflective full subcategory of* $\mathbf{ITop}$.

Proof. Clearly $\mathbf{ITop}_1$ is a full subcategory of $\mathbf{ITop}$. Take any $(X, \mathcal{T})$ in $\mathbf{ITop}$. Define $\mathcal{T}^* = \{\langle X, A^1, \emptyset \rangle \mid A = \langle X, A^1, A^2 \rangle \in \mathcal{T}\}$. Then $(X, \mathcal{T}^*) \in \mathbf{ITop}_1$ and $1_X : (X, \mathcal{T}) \to (X, \mathcal{T}^*)$ is a continuous function. Consider $(Y, \mathcal{U}) \in \mathbf{ITop}_1$ and a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$. We need only to check that $f : (X, \mathcal{T}^*) \to (Y, \mathcal{U})$ is a continuous function. Let $B = \langle Y, B^1, \emptyset \rangle \in \mathcal{U}$. Since $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is continuous, $f^{-1}(B) = \langle X, f^{-1}(B^1), f^{-1}(\emptyset) \rangle \in \mathcal{T}$. By definition of $\mathcal{T}^*$, $f^{-1}(B) = \langle X, f^{-1}(B^1), \emptyset \rangle \in \mathcal{T}^*$. Hence $f : (X, \mathcal{T}^*) \to (Y, \mathcal{U})$ is a continuous function. $\square$

Corollary 4.9. *The category* $\mathbf{Top}$ *is a bireflective full subcategory of* $\mathbf{ITop}$.

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), 87–96.

[2] ———, *More on intuitionistic fuzzy sets*, Fuzzy Sets and Systems **33** (1989), no. 1, 37–45.

[3] ———, *Answer to D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade's paper: Terminological difficulties in fuzzy set theory—the case of 'intuitionistic fuzzy sets'*, Fuzzy Sets and Systems **156** (2005), no. 3, 496–499.

[4] K. T. Atanassov and S. P. Stoeva, *Intuitionistic fuzzy sets*, in Proceedings of the Polish Symposium on Interval & Fuzzy Mathematics (Poznań, 1983), (Poznań), Wydawn. Politech. Poznań. (1983), 13–16.

[5] S. Bayhan and D. Çoker, *On separation axioms in intuitionistic topological spaces*, Int. J. Math. Math. Sci. **27** (2001), no. 10, 621–630.

[6] ———, *Pairwise separation axioms in intuitionistic topological spaces*, Hacet. J. Math. Stat. **34S** (2005), 101–114.

[7] D. Çoker, *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy Sets and Systems **88** (1997), 81–89.

[8] ———, *An introduction to fuzzy subspaces in intuitionistic fuzzy topological spaces*, J. Fuzzy Math. **4** (1996), no. 4, 749–764.

[9] ———, *An introduction to intuitionistic topological spaces*, Bulletin for Studies and Exchanges on Fuzziness and its Applications **81** (2000), 51–56.

[10] H. Herrlich and G. E. Strecker, *Category Theory: an introduction*. Boston, Mass.: Allyn and Bacon Inc., 1973.

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