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STRONG CONVERGENCE OF MODIFIED HYBRID ALGORITHM FOR QUASI-ϕ-ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we propose a modified hybrid algorithm and prove strong convergence theorems for a family of quasi-ϕ-asymptotically nonexpansive mappings. Our results extend and improve the results by Nakajo, Takahashi, Kim, Xu, Su and some others.

1. Introduction

Let E be a Banach space and C a nonempty subset of E. Recall that a mapping $T : C \to C$ is called uniformly Lipschitzian if there exists some $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $n \geq 1$ and $x, y \in C$.

A mapping $T : C \to C$ is called asymptotically nonexpansive [5] if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \tag{1.1}$$

for all $x, y \in C$ and $n \geq 1$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] in 1972. They proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence problems of the iterative algorithms for such a class of mappings (see, e.g., [5, 10, 13]).

It is well known that, in an infinite dimensional Hilbert space, the normal Mann's iterative algorithm has only weak convergence, in general, even for

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nonexpansive mapping. Consequently, in order to obtain strong convergence, one has to modify the normal Mann's iteration algorithm, the so called hybrid projection iteration method is such a modification.

The hybrid projection iteration algorithm was introduced by Haugazeau [6] in 1968. For 40 years, the algorithm has received rapid developments.

In 2003, Nakajo and Takahashi [9] proposed the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.2}$$

where C is a closed convex subset of H, and P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}x_0$.

In 2006, Kim and Xu [8] proposed the following modification of the Mann iteration method for asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where C is a bounded closed convex subset and

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\text{diam}C)^2 \to 0 \text{ as } n \to \infty.$$

They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(T)}x_0$.

In 2006, Carlos and Xu [3] proposed the following modification of the Ishikawa iteration method for nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad +(1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where C is a closed convex subset of H. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}x_0$.

In 2007, Su and Qin [11] proposed the following hybrid iteration method with generalized projection for relatively asymptotically nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n) \\ \qquad +(1-\alpha_n)(\kappa_n^2 \|z_n\|^2 - \|x_n\|^2 \\ \qquad +(\kappa_n^2 - 1)M - 2\langle z, \kappa_n^2 Jz_n - jx_n\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.5}$$

They proved the following convergence theorem.

Theorem SQ. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty bounded closed convex subset of E. Let $T : C \to C$ be a relatively asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$ and $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. Suppose that $\{x_n\}$ is given by (1.5), where J is the duality mapping on E and M is an appropriate constant such that $M > \|v\|^2$ for each $v \in C$. If T is uniformly continuous, then $\{x_n\}$ converges to some $q = \Pi_{F(T)}x_0$.*

The purpose of this article is to introduce a modified hybrid projection iteration algorithm and prove strong convergence theorems for a family of uniformly Lipschitzian and quasi-ϕ-asymptotically nonexpansive mappings. In order to get the strong convergence theorems for such a family of mappings, the classical hybrid projection iteration algorithm is modified and then is used to approximate the common fixed points of such a family of mappings. We remark that the classical hybrid projection iteration algorithm can be used to construct fixed points of asymptotically nonexpansive mappings but it can not be used to construct the fixed points of quasi-ϕ-asymptotically nonexpansive mappings. However, the modified hybrid projection iterative algorithm can be used to construct some common fixed points of such a family of mappings.

2. Preliminaries

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is uniformly convex, then J is uniformly norm to norm continuous on bounded subsets of E. It is also well known that if C is nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [1] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a real smooth Banach space. Let us consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \text{ for } x, y \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2,\ x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x), \tag{2.2}$$

the existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [1, 2, 4, 7, 12]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2 \text{ for all } x, y \in E \tag{2.3}$$

Remark 2.1. If E is a reflexive strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$. That is, $x = y$. One can consult [4, 12] for the details.

Let C be a closed convex subset of E, and T a mapping from C into itself. T is called ϕ-asymptotically nonexpansive, if there exists some real sequence $\{k_n\}$ with $k_n \ge 1$ and $k_n \to 1$ such that $\phi(T^n x, T^n y) \le k_n \phi(x, y)$ for all $n \ge 1$ and $x, y \in C$. T is called quasi-ϕ-asymptotically nonexpansive, if there exists some real sequence $\{k_n\}$ with $k_n \ge 1$ and $k_n \to 1$ and $F(T) \ne \emptyset$ such that $\phi(p, T^n x) \le k_n \phi(p, x)$ for all $n \ge 1,\ x \in C$ and $p \in F(T)$.

We remark that ϕ-asymptotically nonexpansive mappings with nonempty fixed point set $F(T)$ is quasi-ϕ-asymptotically nonexpansive mappings, but the converse may be not true.

We present some examples which are quasi-ϕ-asymptotically nonexpansive.

Example 1. Let E be a real line. We define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \frac{x}{2} \sin \frac{1}{x} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is a quasi-asymptotically nonexpansive mapping with the constant sequence $\{1\}$ but not asymptotically nonexpansive.

Example 2. Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ is a maximal monotone mapping such that $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1}J$ is a quasi-ϕ-asymptotically nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 3. Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a quasi-ϕ-asymptotically nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

The following lemmas are crucial for the proofs of the main results in this paper.

Lemma 2.1 ([7]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.2 ([1]). *Let C be a nonempty closed convex subset of a smooth Banach space E, $x_0 \in C$ and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \text{ for } y \in C.$$

Lemma 2.3 ([1]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \text{ for all } y \in C.$$

Lemma 2.4. *Let E be a uniformly convex and uniformly smooth Banach space, C be a closed convex subset of E, and T be a closed and quasi-ϕ-asymptotically nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Proof. We first show that $F(T)$ is closed. To see this, let $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ as $n \to \infty$, we shall prove that $p \in F(T)$. Using the definition of T, we have that $\phi(p_n, T^n p) \leq k_n \phi(p_n, p)$, which implies that $\phi(p_n, T^n p) \to 0$ as $n \to \infty$. It follows from Lemma 2.1 that $\|p_n - T^n p\| \to 0$ as $n \to \infty$, and hence $T^n p \to p$ as $n \to \infty$, which implies that $TT^n p = T^{n+1} p \to p$ as $n \to \infty$. The closedness of T implies that $Tp = p$. We next show that $F(T)$ is convex. Let $p, q \in F(T)$. We prove $w \in F(T)$, where $w = tp + (1-t)q$ for $t \in (0, 1)$. Indeed, by using (2.1) we have

$$\begin{aligned}
\phi(w, T^n w) &= \|w\|^2 - 2\langle w, JT^n w \rangle + \|T^n w\|^2 \\
&= \|w\|^2 - 2t\langle p, JT^n w \rangle - 2(1-t)\langle q, JT^n w \rangle + \|T^n w\|^2 \\
&= \|w\|^2 + t\phi(p, T^n w) + (1-t)\phi(q, T^n w) - t\|p\|^2 - (1-t)\|q\|^2 \\
&\leq \|w\|^2 + k_n t\phi(p, w) + k_n(1-t)\phi(q, w) - t\|p\|^2 - (1-t)\|q\|^2 \\
&= (k_n - 1)(t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2),
\end{aligned}$$

which implies that $\phi(w, T^n w) \to 0$ as $n \to \infty$. By Lemma 2.1, we have $T^n w \to w$ as $n \to \infty$, and hence $TT^n w = T^{n+1} w \to w$ as $n \to \infty$. Since T is closed, we have that $w = Tw$. This completes the proof. □

3. Main results

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E, and $\{T_i\}_{i\in I} : C \to C$ be a family of uniformly L_i-Lipschitzian and quasi-ϕ-asymptotically nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$, where I is an index set. Let $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in* [0,1] *such that* $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\lim_{n\to\infty} \beta_n = 1$. *Define a sequence $\{x_n\}$ in C by the following algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ z_{n,i} = J^{-1}(\beta_n J x_n + (1-\beta_n) J {T_i}^n x_n), \\ y_{n,i} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J {T_i}^n z_{n,i}), \\ C_{n,i} = \{v \in C : \phi(v, y_{n,i}) \leq \phi(v, x_n) + \xi_{n,i}\}, \\ C_n = \bigcap_{i\in I} C_{n,i}, \\ Q_0 = C, \\ Q_n = \{v \in Q_{n-1} : \langle x_n - v, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{3.1}$$

where $\xi_{n,i} = (1-\alpha_n)(k_{n,i}-1)(k_{n,i}(1-\beta_n)+1)M$, $M \geq \phi(p, x_n)$ *for all* $p \in F$, $x_n \in C$. *Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

Proof. We first show that C_n, Q_n are closed and convex for all $n \geq 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for all $n \geq 0$. Observe that the set

$$C_{n,i} = \{v \in C : \phi(v, y_{n,i}) \leq \phi(v, x_n) + \xi_{n,i}\}$$

is identical to the set

$$D_{n,i} = \{v \in C : 2\langle v, Jx_n - Jy_{n,i} \rangle \leq \|x_n\|^2 - \|y_{n,i}\|^2 + \xi_{n,i}\},$$

and $D_{n,i}$ is closed and convex, so is $C_{n,i}$ for all $n \geq 0$ and $i \in I$. Consequently, $C_n = \bigcap_{i\in I} C_{n,i} = \bigcap_{i\in I} D_{n,i}$ is closed and convex for all $n \geq 0$. □

Next, we show that $F \subset C_n$ for all $n \geq 0$. Noting that $\|\cdot\|^2$ is convex and using (2.1) for all $p \in F$ and $i \in I$, we have

$$
\begin{aligned}
\phi(p, y_{n,i}) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_i{}^n z_{n,i}) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)JT_i{}^n z_{n,i}\rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)JT_i{}^n z_{n,i}\|^2 \\
&\le \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, JT_i{}^n z_{n,i}\rangle \\
&\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|T_i{}^n z_{n,i}\|^2 \\
&= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, T_i{}^n z_{n,i}) \\
&\le \alpha_n\phi(p, x_n) + k_{n,i}(1-\alpha_n)\phi(p, z_{n,i}).
\end{aligned} \tag{3.2}
$$

In addition

$$
\begin{aligned}
\phi(p, z_{n,i}) &= \phi(p, J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_i{}^n x_n) \\
&= \|p\|^2 - 2\langle p, \beta_n Jx_n + (1-\beta_n)JT_i{}^n x_n\rangle \\
&\quad + \|\beta_n Jx_n + (1-\beta_n)JT_i{}^n x_n\|^2 \\
&= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, T_i{}^n x_n) \\
&\le \beta_n\phi(p, x_n) + k_{n,i}(1-\beta_n)\phi(p, x_n) \\
&= [(k_{n,i}-1)(1-\beta_n)+1]\phi(p, x_n).
\end{aligned} \tag{3.3}
$$

From (3.2), (3.3), we have

$$
\begin{aligned}
\phi(p, y_{n,i}) &\le \alpha_n\phi(p, x_n) + k_{n,i}(1-\alpha_n)\phi(p, z_{n,i}) \\
&\le \alpha_n\phi(p, x_n) + k_{n,i}(1-\alpha_n)[(k_{n,i}-1)(1-\beta_n)+1]\phi(p, x_n) \\
&= \phi(p, x_n) + (1-\alpha_n)(k_{n,i}-1)(k_{n,i}(1-\beta_n)+1)\phi(p, x_n) \\
&\le \phi(p, x_n) + (1-\alpha_n)(k_{n,i}-1)(k_{n,i}(1-\beta_n)+1)M \\
&= \phi(p, x_n) + \xi_{n,i},
\end{aligned}
$$

which infers that $p \in C_{n,i}$ for all $n \ge 0$ and $i \in I$. Therefore, $p \in C_n = \bigcap_{i\in I} C_{n,i}$. This proves that $F \subset C_n$ for all $n \ge 0$.

Next, we show that $F \subset Q_n$ for all $n \ge 0$. We prove this by induction. For $n = 0$, we have $F \subset Q_0 = C$. Assume that $F \subset Q_{n-1}$ for some $n \ge 1$, we plan to show that $F \subset Q_n$ for the same $n \ge 1$. Since $x_n = \Pi_{C_{n-1}\cap Q_{n-1}} x_0$, by Lemma 2.2 we have

$$\langle x_n - v, Jx_0 - Jx_n\rangle \ge 0, \quad \forall v \in C_{n-1}\cap Q_{n-1}.$$

Since $F \subset C_{n-1}\cap Q_{n-1}$ by the induction assumptions, the last inequality holds, for all $p \in F$. This together with the definition of Q_n implies that $F \subset Q_n$.

Since $x_{n+1} = \Pi_{C_n\cap Q_n} x_0 \in Q_n$ and $x_n = \Pi_{Q_n} x_0$, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0) \tag{3.4}$$

for all $n \ge 0$. Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. Further, by Lemma 2.3 we have

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \le \phi(p, x_0) - \phi(p, x_n) \le \phi(p, x_0)$$

for all $p \in F \subset Q_n$ and for all $n \geq 0$. Therefore $\{\phi(x_n, x_0)\}$ and $\{x_n\}$ are bounded. This together with (3.4) ensures that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of Q_n, we see that $Q_m \subset Q_n$ and $x_m = \Pi_{Q_m} x_0 \in Q_n$ for all $m \geq n$. By Lemma 2.3, we have, for any positive integer $m \geq n$, that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{Q_n} x_0) \leq \phi(x_m, x_0) - \phi(x_n, x_0). \tag{3.5}$$

Taking the limit in (3.5) yields

$$\lim_{n\to\infty} \phi(x_m, x_n) = 0.$$

By Lemma 2.1, we have that $x_m - x_n \to 0$ as $n, m \to \infty$, hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is a closed subset of E, we can assume that $x_n \to p \in C$ as $n \to \infty$. Finally, we show that $p = \Pi_F x_0$. We first show that $p \in F$. Taking $m = n + 1$ in (3.5) yields that $x_{n+1} - x_n \to 0$ as $n \to \infty$. Since $x_{n+1} = \Pi_{C_n \cap Q_n} \in C_n$, from the definition of C_n for all $i \in I$, we have

$$\phi(x_{n+1}, y_{n,i}) \leq \phi(x_{n+1}, x_n) + \xi_{n,i} \to 0, \quad n \to \infty,$$

and hence $x_{n+1} - y_{n,i} \to 0$ as $n \to \infty$ by Lemma 2.1. It follows that

$$\|x_n - y_{n,i}\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_{n,i}\| \to 0, \quad n \to \infty.$$

Since J is uniformly norm to norm continuous on any bounded sets of E, we conclude that

$$\lim_{n\to\infty} \|Jx_n - Jy_{n,i}\| = 0$$

for all $i \in I$. By the definition of $y_{n,i}$ and the assumption on $\{\alpha_n\}$, we deduce that $Jx_n - JT_i{}^n z_{n,i} \to 0$ as $n \to \infty$.

Since J^{-1} is also uniformly norm to norm continuous on any bounded sets of E^*, we conclude that

$$\lim_{n\to\infty} \|x_n - T_i{}^n z_{n,i}\| = 0.$$

Observe that

$$\begin{aligned}
\phi(x_n, z_{n,i}) &= \phi(x_n, J^{-1}(\beta_n Jx_n + (1 - \beta_n)JT_i{}^n x_n) \\
&= \|x_n\|^2 - 2\langle x_n, \beta_n Jx_n + (1 - \beta_n)JT_i{}^n x_n\rangle \\
&\quad + \|\beta_n Jx_n + (1 - \beta_n)JT_i{}^n x_n\|^2 \\
&= \beta_n \phi(x_n, x_n) + (1 - \beta_n)\phi(x_n, T_i{}^n x_n) \\
&= (1 - \beta_n)\phi(x_n, T_i{}^n x_n).
\end{aligned}$$

Since $\lim_{n\to\infty} \beta_n = 1$ and $\{x_n\}$ is bounded, we have $\phi(x_n, z_{n,i}) \to 0$ as $n \to \infty$. By Lemma 2.1, we have that $x_n - z_{n,i} \to 0$ as $n \to \infty$. And

$$\begin{aligned}
\|x_n - T_i{}^n x_n\| &\leq \|x_n - T_i{}^n z_{n,i}\| + \|T_i{}^n z_{n,i} - T_i{}^n x_n\| \\
&\leq \|x_n - T_i{}^n z_{n,i}\| + L_i\|z_{n,i} - x_n\|.
\end{aligned}$$

So, we have that $x_n - T_i^n x_n \to 0$ as $n \to \infty$. Noting that $x_n \to p$ as $n \to \infty$, we have $T_i^n x_n \to p$ as $n \to \infty$. Observe that

$$\|T_i^{n+1} x_n - p\| \leq \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - p\|. \tag{3.6}$$

Observe that

$$\begin{aligned}\|T_i^{n+1} x_n - T_i^n x_n\| &\leq \|T_i^{n+1} x_n - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| \\ &\leq (L_i + 1)\|x_{n+1} - x_n\| + \|T_i^{n+1} x_{n+1} - x_{n+1}\| \\ &\quad + \|x_n - T_i^n x_n\|,\end{aligned}$$

so that $T_i^{n+1} x_n - T_i^n x_n \to 0$ as $n \to \infty$. This together with (3.6), we have $T_i^{n+1} x_n \to p$ as $n \to \infty$, this is, $T_i T_i^n \to p$. Since T_i is continuous, we have $T_i p = p$ for all $i \in I$, which implies that $p \in F$.

By using the definition of Q_n and noting the fact that $F \subset Q_n$, we have

$$\langle x_n - q, Jx_0 - Jx_n \rangle \geq 0, \quad \forall q \in F. \tag{3.7}$$

Taking the limit of (3.7) yields

$$\langle p - q, Jx_0 - Jp \rangle \geq 0, \quad \forall q \in F.$$

At this point, in view of Lemma 2.2, we see that $p = \Pi_F x_0$. This completes the proof.

Remark 3.2. The boundedness assumption on C in Theorem 3.1 can be weaken to $\sup\{\|p\| : p \in F\} < \infty$.

Corollary 3.3. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and $\{T_i\}_{i \in I} : C \to C$ be a family of uniformly L_i-Lipschitzian and quasi-ϕ-asymptotically nonexpansive mappings such that $F = \bigcap_{i \in I} F(T_i) \neq \emptyset$, where I is an index set. Assume that $R = \sup\{\|p\| : p \in F\} < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \beta_n = 1$. Define a sequence $\{x_n\}$ in C by the following algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_{n,i} = J^{-1}(\beta_n Jx_n + (1-\beta_n) JT_i^n x_n), \\ y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT_i^n z_{n,i}), \\ C_{n,i} = \{v \in C : \phi(v, y_{n,i}) \leq \phi(v, x_n) + \eta_{n,i}\}, \\ C_n = \bigcap_{i \in I} C_{n,i}, \\ Q_0 = C, \\ Q_n = \{v \in Q_{n-1} : \langle x_n - v, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{3.8}$$

where $\eta_{n,i} = (1-\alpha_n)(k_{n,i} - 1)(k_{n,i}(1-\beta_n) + 1)(R + \|x_n\|)^2$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Remark 3.4. $T : C \to C$ is said to be asymptotically regular on C, if for any bounded subset $\tilde{C}$ of C, there holds the following inequality:

$$\lim_{n\to\infty} \sup\{\|T^{n+1}x - T^n x\| : x \in \tilde{C}\} = 0.$$

If $\{T_i\}_{i\in I}$ is asymptotically regular on C, from (3.6) and $\{x_n\}$ is bounded, we can easily have that ${T_i}^{n+1}x_n - {T_i}^n x_n \to 0$ as $n \to \infty$. So the uniformly Lipschitz continuity of T_i in Theorem 3.1 can be replaced by the asymptotic regularity of T_i.

In the sprit of Theorem 3.1, we can prove the following strong convergence theorem.

Theorem 3.5. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and $\{T_i\}_{i\in I} : C \to C$ be a family of uniformly L_i-Lipschitzian and quasi-ϕ-nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$, where I is an index set. Let $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \beta_n = 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_{n,i} = J^{-1}(\beta_n Jx_n + (1-\beta_n)J{T_i}^n x_n), \\ y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J{T_i}^n z_{n,i}), \\ C_{n,i} = \{v \in C : \phi(v, y_{n,i}) \le \phi(v, x_n)\}, \\ C_n = \bigcap_{i\in I} C_{n,i}, \\ Q_0 = C, \\ Q_n = \{v \in Q_{n-1} : \langle x_n - v, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n} x_0, \end{cases} \tag{3.9}$$

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. Following the proof lines of Theorem 3.1, we have conclusions:

(1) F is a nonempty closed convex subset of C;
(2) C_n and Q_n are closed convex sets for all $n \ge 0$;
(3) $F \subset C_n \cap Q_n$ for all $n \ge 0$;
(4) $\lim_{n\to\infty} \|x_n - x_0\|$ exists;
(5) $\{x_n\}$ is a Cauchy sequence;
(6) $x_n - x_{n+1} \to 0$ as $n \to \infty$;
(7) $\forall i \in I$, $x_n - {T_i}^n x_n \to 0$ as $n \to \infty$.

The continuousness property of T_i together with (5) and (7) implies that $\{x_n\}$ converges strongly to a common fixed point p of $\{T_i\}_{i\in I}$. As shown in Theorem 3.1, $p = \Pi_F x_0$. This completes the proof. □

Theorem 3.6. *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E, and $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-asymptotically nonexpansive mappings such that*

$F = \bigcap_{i\in I} F(T_i) \neq \emptyset$, *where* I *is an index set. Let* $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* [0,1]. *Define a sequence* $\{x_n\}$ *in* C *by the following algorithm*:

$$\text{(3.10)}\qquad \begin{cases} x_0 \in H \ \text{chosen arbitrarily}, \\ C_{1,i} = C, \quad C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \quad x_1 = \Pi_{C_1} x_0, \\ z_{n,i} = J^{-1}(\beta_n J x_n + (1-\beta_n) J {T_i}^n x_n), \\ y_{n,i} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J {T_i}^n z_{n,i}), \\ C_{n+1,i} = \{v \in C_{n,i} : \phi(v, y_{n,i}) \leq \phi(v, x_n) + \xi_{n,i}\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\xi_{n,i} = (1-\alpha_n)(k_{n,i}-1)(k_{n,i}(1-\beta_n)+1)M$, $M \geq \phi(p, x_n)$ *for all* $p \in F$, $x_n \in C$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

Proof. First, we show that C_n is closed and convex for all $n \geq 0$. It suffices to show that, for any fixed but arbitrary $i \in I$, $C_{n,i}$ is closed and convex for every $n \geq 0$. This can be proved by induction. It is obvious that $C_{1,i} = C$ is closed and convex. Assume that $C_{n,i}$ is closed and convex for some $n \in N$. Observe that the set

$$C_{n+1,i} = \{v \in C : \phi(v, y_{n,i}) \leq \phi(v, x_n) + \xi_{n,i}\}$$

is identical to the set

$$D_{n,i} = \{v \in C : 2\langle v, Jx_n - Jy_{n,i}\rangle \leq \|x_n\|^2 - \|y_{n,i}\|^2 + \xi_{n,i}\},$$

and $D_{n,i}$ is closed and convex. Since $C_{n+1,i}$ is the intersection of $C_{n,i}$ and $D_{n,i}$, $C_{n+1,i}$ is closed and convex. Then, for all $n \geq 0$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}} x_0$ is well defined. Next, we prove $F \subset C_n$ for all $n \geq 0$. It suffices to show that $F \subset C_{n,i}$ for every $i \in I$. We prove this by induction. It is obvious that $F \subset C_{1,i} = C$. Assume that $F \subset C_{n,i}$ for some $n \in N$. For $\forall p \in F \subset C_{n,i}$, as shown in Theorem 3.1, we have that $F \subset C_{n+1,i}$. Then, for all $n \geq 0$, $F \subset C_n$. Since $x_n = \Pi_{C_n} x_0$ and $C_{n+1} \subset C_n$ and $x_{n+1} \in C_{n+1}$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \tag{3.11}$$

for all $n \geq 0$. Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. Further, by Lemma 2.3 we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0)$$

for all $p \in F \subset C_n$ and for all $n \geq 0$. Therefore $\{\phi(x_n, x_0)\}$ and $\{x_n\}$ are bounded. This together with (3.11) ensures that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, we see that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for all $m \geq n$. By Lemma 2.3, we have, for any positive integer $m \geq n$, that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_0) \leq \phi(x_m, x_0) - \phi(x_n, x_0). \tag{3.12}$$

Taking the limit in (3.12) yields

$$\lim_{n\to\infty} \phi(x_m, x_n) = 0.$$

By Lemma 2.1, we have that $x_m - x_n \to 0$ as $n, m \to \infty$, hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is a closed subset of E, we can assume that $x_n \to p \in C$ as $n \to \infty$. Finally, we show that $p = \Pi_F x_0$.

Since $x_n = \Pi_{C_n} x_0$ and $F \subset C_n$, from Lemma 2.2, we have

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in F. \tag{3.13}$$

Taking the limit of (3.13) yields

$$\langle p - w, Jx_0 - Jp\rangle \geq 0, \quad \forall w \in F.$$

At this point, in view of Lemma 2.2, we see that $p = \Pi_F x_0$. This completes the proof. □

Remark 3.7. The boundedness assumption on C in Theorem 3.6 can be weaken to $\sup\{\|p\| : p \in F\} < \infty$.

In the sprit of Theorem 3.6, we have the following strong convergence theorem.

Theorem 3.8. *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E, and $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$, where I is an index set. Let $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ C_{1,i} = C, \quad C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \quad x_1 = \Pi_{C_1} x_0, \\ z_{n,i} = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_i^n x_n), \\ y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_i^n z_{n,i}), \\ C_{n+1,i} = \{v \in C_{n,i} : \phi(v, y_{n,i}) \leq \phi(v, x_n) + \xi_{n,i}\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.10}$$

where $\xi_{n,i} = (1-\alpha_n)(k_{n,i}-1)(k_{n,i}(1-\beta_n)+1)M$, $M \geq \phi(p, x_n)$ for all $p \in F$, $x_n \in C$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Remark 3.9. In theorems above, if one takes $I = \{1, 2, \ldots, N\}$, $I = \{1, 2, \ldots\}$ and $I = R^+$, respectively, then one can obtain strong convergence theorem for a finite, countable infinite of family and nonlinear semigroup of asymptotically nonexpansive mappings, respectively.

References

[1] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications, in: A. G. Kartsatos (Ed.)*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, 15–50.

[2] Ya. I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operator equations in Banach spaces*, Panamer. Math. J. **4** (1994), 39–54.

[3] M. Y. Carlos and H. K. Xu, *Strong convergence of the CQ method for fixed point iteration process*, Nonlinear Anal. **64** (2006), 2240–2411.

[4] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[5] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

[6] Y. Haugazeau, *Sur les inéquations variationnelles et la minimisation de fonctionnelles convexes*, Thése, Université de Paris, Paris, France.

[7] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.

[8] T. H. Kim and H. K. Xu, *Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups*, Nonlinear Analysis **64** (2006), 1140–1152.

[9] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semi-groups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[10] J. Schu, *Iteration construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), 407–413.

[11] Y. F. Su and X. L. Qin, *Strong convergence of modified Ishikawa iterations for nonlinear mappings*, Proc. Indian Acad. Sci.(Math.Sci.) **117** (2007), 97–107.

[12] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama-Publishers, 2000.

[13] H. Y. Zhou, Y. J. Cho, and S. M. Kang, *A new iterative algorithm for approximating common fixed points for asymptotically nonexpansive mappings*, Fixed Point Theory and Applications **2007** (2007), doi:10.1155/2007/64874.

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