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A NEW VECTOR QUASI-EQUILIBRIUM-LIKE PROBLEM

Byung-Soo Lee

Abstract. In this paper, we consider the existence of solutions to some generalized vector quasi-equilibrium-like problem under a c-diagonal quasi-convexity assumptions, but not monotone concepts. For an example, in the proof of Theorem 1, the c-diagonally quasi-convex concepts of a set-valued mapping was used but monotone condition was not used. Our problem is a new kind of equilibrium problems, which can be compared with those of Hou et al. [4].

1. Introduction and preliminaries

Equilibrium problems are important parts in nonlinear analysis connecting with variational inequality problems, optimization problems, complementarity problems, fixed point theorems and so on [1, 4, 5, 6, 7, 10].

In consideration the existences of solutions to equilibrium problems and its applications, the continuity, the monotonicity or the convexity of the core mappings are very meaningful.

In 2003, Ansari and Bazan [1] considered, so called, generalized vector quasi-equilibrium problems for condensing set-valued mappings in Hausdorff topological vector spaces;
Find $\bar{x} \in K$ such that

$$\bar{x} \in A(\bar{x}); \;\; F(\bar{x}, y) \not\subseteq -\operatorname{int} C(x) \;\text{ for all }\; y \in A(\bar{x}).$$

In 2003, Hou, Yu, and Chen [4] considered the following vector quasi-equilibrium problems for set-valued mappings under c-diagonal quasi-convexity assumptions;
Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$f(\bar{x}, \bar{y}, u) \not\subseteq -\operatorname{int} C(\bar{x}) \;\text{ for }\; u \in S(\bar{x}), \tag{1.1}$$

where E, W and Z are Hausdorff topological vector spaces, X and Y are nonempty subsets of E and W, respectively. And $S : X \to 2^X$, $T : X \to 2^Y$,

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$f : X \times Y \times X \to 2^Z$ are set-valued mappings and $C : X \to 2^Z$ is a set-valued mapping with nonempty interior of $C(x)$, int $C(x)$ for all $x \in X$.

Let $L(E, Z)$ be the space of all continuous linear mappings from E to Z, which is equipped with a σ-topology. Then $L(E, Z)$ is also a Hausdorff topological vector space [8].

Putting Y to be a nonempty subset of $L(E, Z)$ instead of W in (1.1), and adding a set-valued mapping $A : L(E, Z) \to 2^{L(E,Z)}$ and a single-valued mapping $\eta : X \times X \to E$, we obtain a new vector quasi-equilibrium-like problem (VQELP) for set-valued mappings as follows;
Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$\langle w, \eta(u, \bar{x})\rangle + f(\bar{x}, \bar{y}, u) \not\subseteq -\text{ int } C(\bar{x}) \text{ for } u \in S(\bar{x}) \text{ and } w \in A(\bar{y}). \tag{1.2}$$

In this paper, we consider the existence of solutions to (VQELP) with the c-diagonal quasi-convexity, but not monotone concepts.

2. Quasi-equilibrium-like problem

The concept of the c-diagonal quasi-convexity was introduced by Hou et. al. [4]. The following lemmas are essential to prove our result.

Definition 1. Let X be a convex subset of a topological vector space E and Z be a topological vector space. Let $C : X \to 2^Z$ and $f : X \times X \to 2^Z$ be set-valued mappings. f is said to be c-diagonally quasi-convex in the second argument if for some $x_i \in M$,

$$f(x, x_i) \not\subseteq -\text{ int } C(x),$$

for $x \in coM$, the convex hull of a finite subset $M = \{x_i : i = 1, 2, \dots, n\}$ of X, i.e., $x = \sum_{i=1}^n \alpha_i x_i$ with $\alpha_i \geq 0$ $(i = 1, \dots, n)$ and $\sum_{i=1}^n \alpha_i = 1$.

A set-valued mapping $T : X \to 2^Y$ is said to have open lower sections if its fibers

$$T^-(y) = \{x \in X : y \in T(x)\} \quad \text{is open in } X \text{ for every } y \in Y.$$

Lemma 1 ([9]). *Let X and Y be topological spaces, and S, $T : X \to 2^Y$ be set-valued mappings with open lower sections. Then*

(i) *A set-valued mapping $F : X \to 2^Y$ defined by $F(x) = coS(x)$, for each $x \in X$, has open lower sections.*

(ii) *A set-valued mapping $G : X \to 2^Y$ defined by $G(x) = S(x) \cap T(x)$, for each $x \in X$, has open lower sections.*

Lemma 2 ([2]). *Let X_i $(i \in I)$ be a nonempty compact convex subset of a Hausdorff topological vector space E_i $(i \in I)$ and $E = \prod_{i\in I} E_i$, $X = \prod_{i\in I} X_i$, where I is an index set. If set-valued mappings $T_i : X \to 2^{X_i}$ $(i \in I)$ have convex set-values and $X = \bigcup_{x_i\in X_i}$ int $T_i^-(x_i)$, then there exists $\bar{x}(:= (\bar{x}_i)_{i\in I}) \in X$ such that $\bar{x} \in \prod_{i\in I} T_i(\bar{x})$, i.e., $\bar{x}_i \in T_i(\bar{x})$, where $\bar{x}_i$ is a projection of $\bar{x}$ onto X_i.*

Browder fixed point theorem [3]. Let X be a nonempty compact convex subset of a Hausdorff topological vector space E. Suppose that $F : X \to 2^X$ is a set-valued mapping with nonempty convex set-values and open lower sections, then F has a fixed point.

Theorem 1. *Let E and Z be Hausdorff topological vector spaces, and X be a nonempty compact convex subset of E. Let Y be a nonempty compact convex subset of a Hausdorff topological vector space $L(E, Z)$ equipped with a σ-topology. Let $A : L(E, Z) \to 2^{L(E,Z)}$, $f : X \times Y \times X \to 2^Z$, $S : X \to 2^X$ and $T : X \to 2^Y$ be set-valued mappings, $\eta : X \times X \to E$ be a single-valued mapping and $C : X \to 2^Z$ be a set-valued mapping with* int $C(x) \neq \emptyset$ *for $x \in X$.*

Let the following conditions be satisfied;

(i) *S and T have nonempty convex set-values and open lower sections,*

(ii) *for $y \in Y$ and $x \in coM$, where $M = \{x_i : i = 1, 2, \ldots, n\}$ is any finite subset of X, a set-valued mapping $\ell : X \to 2^Z$ defined by, for each $u \in X$,*

$$\ell(u) = \langle w, \eta(u, x)\rangle + f(x, y, u) \quad \text{for some } w \in A(y)$$

is c-diagonally quasi-convex, i.e.,

$$\ell(x_i) = \langle w, \eta(x_i, x)\rangle + f(x, y, x_i) \not\subseteq - \text{ int } C(x) \quad \text{for some } w \in A(y),$$

(iii) *for $u \in X$, a set*

$$K = \{(x, y) \in X \times Y : \langle w, \eta(u, x)\rangle + f(x, y, u) \subseteq - \text{ int } C(x) \text{ for } w \in A(y)\}$$

is open.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ satisfying

$$\langle w, \eta(u, \bar{x})\rangle + f(\bar{x}, \bar{y}, u) \not\subseteq - \text{ int } C(\bar{x})$$

for $u \in S(\bar{x})$ and $w \in A(\bar{y})$.

Proof. Define a set-valued mapping $P : X \times Y \to 2^X$ by, for $(x, y) \in X \times Y$,

$$P(x, y) = \{u \in X : \langle w, \eta(u, x)\rangle + f(x, y, u) \subseteq - \text{ int } C(x) \text{ for } w \in A(y)\},$$

then

(a) for all $(x, y) \in X \times Y$, $x \notin coP(x, y)$.

If not, there exists a point $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in coP(\bar{x}, \bar{y})$. Hence there exists a finite subset $M = \{x_i : i = 1, 2, \ldots, n\} \subset P(\bar{x}, \bar{y})$ for $\bar{x} \in coM$ such that

$$\langle w, \eta(x_i, \bar{x})\rangle + f(\bar{x}, \bar{y}, x_i) \subseteq - \text{ int } C(\bar{x}) \text{ for } w \in A(\bar{y}),$$

which contradicts condition (ii).

(b) P has open lower sections.

In fact, for each $u \in X$,

$$P^-(u) = \{(x, y) \in X \times Y : \langle w, \eta(u, x)\rangle + f(x, y, u) \subseteq - \text{ int } C(x) \text{ for } w \in A(y)\}$$

is open by condition (iii).

Define a set-valued mapping $G : X \times Y \to 2^X$ by, for $(x, y) \in X \times Y$,

$$G(x, y) = coP(x, y) \cap S(x),$$

then

(c) G has open lower sections.

In fact, by condition (i) S has open lower sections and, by (b) and Lemma 1(i) coP has open lower sections, hence by Lemma 1(ii) G has open lower sections.

Put $W = \{(x, y) \in X \times Y : G(x, y) \neq \emptyset\}$.

(d) If $W = \emptyset$, then by the definition of $G(x, y)$,

$$P(x, y) \cap S(x) = \emptyset \quad \text{for } (x, y) \in X \times Y.$$

On the other hand, since X is a compact convex subset of E and S is a convex set-valued mapping, by Browder fixed point theorem from condition (i) there exists $\bar{x} \in S(\bar{x})$. Since $T(\bar{x}) \neq \emptyset$, by taking $\bar{y} \in T(\bar{x})$ we have

$$P(\bar{x}, \bar{y}) \cap S(\bar{x}) = \emptyset.$$

Hence for $u \in S(\bar{x})$, $u \notin P(\bar{x}, \bar{y})$, i.e., for $u \in S(\bar{x})$,

$$\langle w, \eta(u, \bar{x})\rangle + f(\bar{x}, \bar{y}, u) \not\subseteq -\text{ int } C(\bar{x}) \quad \text{for } w \in A(\bar{y}).$$

(e) If $W \neq \emptyset$, then we define a set-valued mapping $M : X \times Y \to 2^X$ by, for each $(x, y) \in X \times Y$

$$M(x, y) = \begin{cases} G(x, y), & (x, y) \in W, \\ S(x), & (x, y) \in X \times Y \backslash W. \end{cases}$$

Then for each $(x, y) \in X \times Y$, $M(x, y)$ is convex, and for $x \in X$, $M^-(x) = G^-(x) \cup (S^-(x) \times Y)$ is open.

Define a set-valued mapping

$$H : X \times Y \to 2^{X \times Y}$$

by, for each $(x, y) \in X \times Y$, $H(x, y) = (M(x, y), T(x))$, where $H_1(x, y) = (M(x, y))$ and $H_2(x, y) = T(x)$, then by the assumption on T in condition (i) and the properties of $M(x, y)$, H satisfies conditions of Lemma 2, i.e.,

$$\begin{aligned} X \times Y &= \bigcup_{\substack{x \in X \\ y \in Y}} H^-(x, y) \\ &= \bigcup_{\substack{x \in X \\ y \in Y}} (M^-(x) \cup T^-(y)) \\ &= \left(\bigcup_{x \in X} M^-(x)\right) \bigcup \left(\bigcup_{y \in Y} T^-(y)\right) \end{aligned}$$

$$= \bigcup_{x\in X} \left(G^-(x) \cup (S^-(x) \times Y)\right) \bigcup \left(\bigcup_{y\in Y} T^-(y)\right)$$
$$= \left(\bigcup_{x\in X} \text{int } H_1^-(x)\right) \bigcup \left(\bigcup_{y\in Y} \text{int } H_2^-(y)\right).$$

Hence, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $(\bar{x}, \bar{y}) \in H(\bar{x}, \bar{y})$. Thus we obtain $(\bar{x}, \bar{y}) \notin W$.

In fact, if $(\bar{x}, \bar{y}) \in W$, then $G(\bar{x}, \bar{y}) \neq \emptyset$. Hence $\bar{x} \in coP(\bar{x}, \bar{y}) \cap S(\bar{x})$, so that $\bar{x} \in coP(\bar{x}, \bar{y})$, which is a contradiction to (a). Since $(\bar{x}, \bar{y}) \in X \times Y \backslash W$, $(\bar{x}, \bar{y}) \in H(\bar{x}, \bar{y}) = (M(\bar{x}, \bar{y}), T(\bar{x})) = (S(\bar{x}), T(\bar{x}))$, and $G(\bar{x}, \bar{y}) = \emptyset$. Consequently, $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$, $coP(\bar{x}, \bar{y}) \cap S(\bar{x}) = \emptyset$. Thus

$$P(\bar{x}, \bar{y}) \cap S(\bar{x}) = \emptyset.$$

Hence for $u \in S(\bar{x})$,

$$\langle w, \eta(u, \bar{x}) + f(\bar{x}, \bar{y}, u)\rangle \not\subseteq -\text{ int } C(\bar{x}) \text{ for } w \in A(\bar{y}). \qquad \square$$

Remark. Problem (1.2) can be compared with Hou et al.'s problem (1.1) in the point of view that the term $\langle w, \eta(u, \bar{x})\rangle$ is added and Y is a subset of a particular Hausdorff topological vector space $L(E, Z)$ instead of an arbitrary Hausdorff topological vector space W.

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Basic Science Research Center
Kyungsung University
Busan 608-736, Korea
E-mail address: bslee@ks.ac.kr