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ANOTHER METHOD FOR PADMANABHAM'S TRANSFORMATION FORMULA FOR EXTON'S TRIPLE HYPERGEOMETRIC SERIES X_8

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Abstract. The object of this note is to derive Padmanabham's transformation formula for Exton's triple hypergeometric series X_8 by using a different method from that of Padmanabham's. An interesting special case is also pointed out.

1. Introduction and preliminaries

In 1982, Exton [3] introduced a set of 20 triple hypergeometric series X_1 to X_{20} of which we recall here the definition of X_8:

$$\begin{aligned} & X_8(a,\, b,\, c\,;\, d,\, e,\, f\,;\, x,\, y,\, z) \\ & = \sum_{m,\,n,\,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_n\,(c)_p\,x^m\,y^n\,z^p}{(d)_m\,(e)_n\,(f)_p\,m!\,n!\,p!}, \end{aligned} \tag{1.1}$$

where $(\alpha)_n$ denotes the Pochhammer symbol defined by

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (\alpha \neq 0,\, -1,\, -2,\, \ldots;\; n = 0,\, 1,\, 2,\, \ldots). \tag{1.2}$$

The precise three-dimensional region of convergence of (1.1) is given by Srivastava and Karlsson [9, p. 101, Entry 41a]:

$$4r = (s+t-1)^2, \quad |x| < r, \quad |y| < s, \quad \text{and} \quad |z| < t,$$

where the positive quantities r, s and t are associated radii of convergence. For details about this function and many other three-variables hypergeometric functions, one refers to Srivastava and Karlsson [9].

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Exton [3] gave the following Laplace integral representation of (1.1):

$$\begin{aligned} &X_8(a,\, b,\, c\,;\, d,\, e,\, f\,;\, x,\, y,\, z) \\ &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-u}\, u^{a-1}\, {}_0F_1(-;\, d\,;\, u^2\, x)\, {}_1F_1(b\,;\, e\,;\, u\, y)\, {}_1F_1(c\,;\, f\,;\, u\, z)\, du, \end{aligned} \tag{1.3}$$

provided $\Re(a) > 0$.

It may be remarked in passing that X_8 reduces to Horn's function H_4 when $z \to 0$ and the Appell's function F_2 when $x \to 0$.

Srivastava and Panda [11, p. 423, Eq. (26)] presented a definition of a general double hypergeometric function:

$$\begin{aligned} &F^{p\,:\,q\,;\,k}_{l\,:\,m\,;\,n} \left[\begin{array}{llll} (a_p) & : (b_q) & ; (c_k) & ; \\ (\alpha_l) & : (\beta_m) & ; (\gamma_n) & ; \end{array} \; x,\, y \right] \\ &= \sum_{r,\, s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s}{\prod_{j=1}^{l} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s} \frac{x^r\, y^s}{r!\, s!}, \end{aligned} \tag{1.4}$$

where the several cases of convergence conditions are given in [10, p. 64]. Note that Srivastava and Panda's function (1.4) is more general than the one defined by Kampé de Fériet [5] (*cf.* Appell et Kampé de Fériet [1, p. 150, Eq. (29)].

In 2003, Padmanabham [6] obtained the following result for X_8:

$$\begin{aligned} &X_8(a,\, b,\, b\,;\, d,\, c,\, c\,;\, x,\, -x,\, x) \\ &= F^{2\,:\,2\,;\,0}_{0\,:\,3\,;\,1} \left[\begin{array}{llll} \tfrac{1}{2}a,\ \tfrac{1}{2}a + \tfrac{1}{2} & : b,\ c - b & ;\ ---- & ; \\ ---- & : c,\ \tfrac{1}{2}c,\ \tfrac{1}{2}c + \tfrac{1}{2} & ;\ d & ; \end{array} \; x^2,\, 4x \right] \end{aligned} \tag{1.5}$$

by employing the following result due to Ramanujan [4] involving product of two generalized hypergeometric series:

$${}_1F_1(a\,;\, b\,;\, x) \times {}_1F_1\,(a\,;\, b\,;\, -x) = {}_2F_3 \left(a,\, b - a\,;\, b,\, \frac{1}{2}b,\, \frac{1}{2}b + \frac{1}{2}\,;\, \frac{x^2}{4} \right). \tag{1.6}$$

Consider the special case of (1.5) when $c = 2b$:

$$\begin{aligned} &X_8(a,\, b,\, b\,;\, d,\, 2b,\, 2b\,;\, x,\, -x,\, x) \\ &= F^{2\,:\,1\,;\,0}_{0\,:\,2\,;\,1} \left[\begin{array}{llll} \tfrac{1}{2}a,\ \tfrac{1}{2}a + \tfrac{1}{2} & :\ b & ;\ -- & ; \\ ---- & :\ 2b,\ b + \tfrac{1}{2} & ;\ d & ; \end{array} \; x^2,\, 4x \right]. \end{aligned} \tag{1.7}$$

It is interesting here to point out that, instead of (1.6), the identity (1.7) can also be established by using the following well-known Preece's identity [7] involving product of two generalized hypergeometric series:

$${}_1F_1(a\,;\, 2a\,;\, x) \times {}_1F_1\,(a\,;\, 2a\,;\, -x) = {}_1F_2 \left(a\,;\, 2a,\, a + \frac{1}{2}\,;\, \frac{x^2}{4} \right). \tag{1.8}$$

We will derive Padmanabham's transformation formula (1.5) for Exton's triple hypergeometric series X_8 by using a different method from that of Padmanabham's. For our purpose, we recall here Dixon's theorem [8, p. 250] for the well poised ${}_3F_2(1)$:

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}a,\ b,\ c\\ 1+a-b,\ 1+a-c\end{matrix}\ ;\ 1\right]\\ &= \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)}\\ &\qquad (\Re\,(a-2b-2c)>-2)\,.\end{aligned} \tag{1.9}$$

We also recall the following well-known identities involving the Pochhammer symbol in (1.2) (see [8, p. 6–8]):

$$(\alpha)_{n-p} = \frac{(-1)^p\,(\alpha)_n}{(1-\alpha-n)_p} \quad \text{and} \quad (n-p)! = \frac{(-1)^p\,n!}{(-n)_p}\ \ (\alpha=1); \tag{1.10}$$

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha}{2}+\frac{1}{2}\right)_n; \tag{1.11}$$

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n}; \tag{1.12}$$

$$(\alpha)_m\,(\alpha+m)_n = (\alpha)_{m+n}. \tag{1.13}$$

2. Derivation of (1.5)

Replacing c by b, e and f by c, y by $-x$, and z by x in (1.1), we have

$$\begin{aligned}X_8 &:= X_8(a,\,b,\,b\,;\,d,\,c,\,c\,;\,x,\,-x,\,x)\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+n+p}\,(b)_n\,(b)_p\,(-1)^n\,x^{m+n+p}}{(d)_m\,(c)_n\,(c)_p\,m!\,n!\,p!},\end{aligned}$$

which, upon using $(a)_{2m+n+p} = (a)_{2m}\,(a+2m)_{n+p}$, becomes

$$X_8 = \sum_{m=0}^{\infty}\frac{(a)_{2m}\,x^m}{(d)_m\,m!}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a+2m)_{n+p}\,(b)_n\,(b)_p\,(-1)^n\,x^{n+p}}{(c)_n\,(c)_p\,n!\,p!}. \tag{2.1}$$

By making use of the well-known formal manipulation for double series (for more related formulas, see [2]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A_{k,\,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A_{k,\,n-k}$$

in (2.1), we obtain

$$X_8 = \sum_{m=0}^{\infty}\frac{(a)_{2m}\,x^m}{(d)_m\,m!}\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(a+2m)_n\,(b)_{n-p}\,(b)_p\,(-1)^{n-p}\,x^n}{(c)_p\,(c)_{n-p}\,(n-p)!\,p!}. \tag{2.2}$$

Applying (1.10) and (1.11) to (2.2), we get

$$
\begin{aligned}
X_8 = & \sum_{m=0}^{\infty} \frac{\left(\frac{a}{2}\right)_m \left(\frac{a}{2}+\frac{1}{2}\right)_m (4x)^m}{(d)_m\, m!} \sum_{n=0}^{\infty} \frac{(a+2m)_n\, (b)_n\, (-x)^n}{(c)_n\, n!} \\
& \cdot \sum_{p=0}^{n} \frac{(-n)_p\, (b)_p\, (1-c-n)_p}{(1-n-b)_p\, (c)_p\, p!} \\
= & \sum_{m=0}^{\infty} \frac{\left(\frac{a}{2}\right)_m \left(\frac{a}{2}+\frac{1}{2}\right)_m (4x)^m}{(d)_m\, m!} \sum_{n=0}^{\infty} \frac{(a+2m)_n\, (b)_n\, (-x)^n}{(c)_n\, n!} \\
& \cdot {}_3F_2 \left[\begin{matrix} -n,\ b,\ 1-n-c \\ 1-n-b,\ c \end{matrix} ; 1 \right].
\end{aligned} \tag{2.3}
$$

Applying Dixon's theorem (1.9) to ${}_3F_2(1)$ in (2.3), we obtain

$$
\begin{aligned}
X_8 = & \sum_{m=0}^{\infty} \frac{\left(\frac{a}{2}\right)_m \left(\frac{a}{2}+\frac{1}{2}\right)_m (4x)^m}{(d)_m\, m!} \sum_{n=0}^{\infty} \frac{(a+2m)_n\, (b)_n\, (-x)^n}{(c)_n\, n!} \\
& \cdot \mathcal{A}(b,\, c\,;\, n),
\end{aligned} \tag{2.4}
$$

where, for convenience,

$$
\mathcal{A}(b,\, c\,;\, n) := \frac{\Gamma(c)\, \Gamma(1-b-n)\, \Gamma\left(c-b+\frac{1}{2}n\right)\, \Gamma\left(1-\frac{1}{2}n\right)}{\Gamma(c-b)\, \Gamma\left(c+\frac{1}{2}n\right)\, \Gamma\left(1-b-\frac{1}{2}n\right)\, \Gamma(1-n)}.
$$

By making use of Legendre's duplication formula for the Gamma function:

$$
\Gamma\left(\frac{1}{2}\right) \Gamma\left(2\alpha\right) = 2^{2\alpha-1}\, \Gamma\left(\alpha\right)\, \Gamma\left(\alpha+\frac{1}{2}\right),
$$

we have

$$
\mathcal{A}(b,\, c\,;\, n) = \frac{\Gamma(c)\, \Gamma(1-b-n)\, \Gamma\left(c-b+\frac{1}{2}n\right)}{\Gamma(c-b)\, \Gamma\left(c+\frac{1}{2}n\right)\, \Gamma\left(1-b-\frac{1}{2}n\right)} \cdot \frac{2^n\, \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}n\right)},
$$

from which we see that

$$
\mathcal{A}(b,\, c\,;\, n) = 0 \tag{2.5}
$$

whenever n is an odd positive integer.

Considering (2.5), we can rewrite X_8 in (2.4) as follows:

$$
\begin{aligned}
X_8 = & \sum_{m=0}^{\infty} \frac{\left(\frac{a}{2}\right)_m \left(\frac{a}{2}+\frac{1}{2}\right)_m (4x)^m}{(d)_m\, m!} \sum_{n=0}^{\infty} \frac{(a+2m)_{2n}\, (b)_{2n}\, x^{2n}}{(c)_{2n}\, (2n)!} \\
& \cdot \frac{\Gamma(c)\, \Gamma(1-b-2n)\, \Gamma\left(c-b+n\right)}{\Gamma(c-b)\, \Gamma\left(c+n\right)\, \Gamma\left(1-b-n\right)} \cdot \frac{2^{2n}\, \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)}.
\end{aligned} \tag{2.6}
$$

If we employ (1.11), (1.12), and (1.13) in (2.6), we finally obtain

$$
X_8 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_{m+n} \left(\frac{1}{2}a+\frac{1}{2}\right)_{m+n} (b)_n\, (c-b)_n\, x^{2n}\, (4x)^m}{(c)_n \left(\frac{1}{2}c\right)_n \left(\frac{1}{2}c+\frac{1}{2}\right)_n (d)_m\, n!\, m!},
$$

which, upon using (1.4), leads to the desired formula (1.5).

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