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DERIVATIONS ON SUBTRACTION ALGEBRAS

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Abstract. In this paper, we introduce the notions of a derivation and a generalized derivation determined by a derivation for a complicated subtraction algebra. We give some related properties and equivalent conditions which derivations hold.

1. Introduction

B. M. Schein [12] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [14] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun, H. S. Kim, and E. H. Roh [7] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [6], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. In [9], Y. B. Jun, Y. H. Kim, and K. A. Oh introduced the notion of complicated subtraction algebras and investigated some related properties. In [10], Y. H. Kim and H. S. Kim showed that a subtraction algebra is equivalent to an implicative BCK-algebra, and a subtraction semigroup is a special case of a BCI-semigroup. In [4], Çeven and Öztürk introduced some additional concepts on subtraction algebras, so called subalgebra, bounded subtraction algebra and union of subtraction algebras and investigated some related properties. The properties of derivations play an important role in many fields such as the theory of rings, near rings and Banach algebras ([11], [2], [3], [5]).

In this paper, we introduce the notions of a derivation and a generalized derivation for a subtraction algebra and discuss some related properties.

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2. Preliminaries

An algebra $(X; -)$ with a single binary operation "$-$" is called a *subtraction algebra* if for all $x, y, z \in X$ the following conditions hold:
(S1) $x - (y - x) = x$,
(S2) $x - (x - y) = y - (y - x)$,
(S3) $(x - y) - z = (x - z) - y$.
The subtraction determines an order relation on X: $a \leq b \iff a - b = 0$, where $0 = a - a$ is an element that doesn't depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sence of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to induced order. Here $a \wedge b = a - (a - b)$ and the complement of an element $b \in [0, a]$ is $a - b$.

In a subtraction algebra, the following are true [7, 8]:
(a1) $(x - y) - y = x - y$,
(a2) $x - 0 = x$ and $0 - x = 0$,
(a3) $(x - y) - x = 0$,
(a4) $x - (x - y) \leq y$,
(a5) $(x - y) - (y - x) = x - y$,
(a6) $x - (x - (x - y)) = x - y$,
(a7) $(x - y) - (z - y) \leq x - z$,
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$,
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$,
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$,
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$,
(a12) $(x - y) - z = (x - z) - (y - z)$.

Definition 2.1 ([7]). A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies
(1) $0 \in A$,
(2) $(\forall x \in X)(\forall y \in A)(x - y \in A \implies x \in A)$.

Lemma 2.2 ([8]). *An ideal A of a subtraction algebra X has the following property*:
$$(\forall x \in X)(\forall y \in A)(x \leq y \implies x \in A).$$

Definition 2.3 ([9]). Let X be a subtraction algebra. For any $a, b \in X$, let $G(a, b) = \{x \in X : x - a \leq b\}$. X is said to be *complicated subtraction algebra* (*c-subtraction algebra*) if for any $a, b \in X$ the set $G(a, b)$ has the greatest element.

Note that $0, a, b \in G(a, b)$. The greatest element of $G(a, b)$ is denoted $a + b$.

Proposition 2.4 ([9]). *If X is a c-subtraction algebra, then for all $x, y, z \in X$,*
(i) $x \leq x + y$, $y \leq x + y$,
(ii) $x + 0 = x = 0 + x$,
(iii) $x + y = y + x$,

(iv) $x \leq y \Longrightarrow x + z \leq y + z$,
(v) $x \leq y \Longrightarrow x + y = y$,
(vi) *$x + y$ is the least upper bound of x and y.*

Definition 2.5 ([4])**.** Let X be a subtraction algebra. X is called a *bounded subtraction algebra* if there is an element 1 of X satisfying $x \leq 1$ for all x in X.

3. Derivations in subtraction algebra

The following definition introduces the notion of derivation for a subtraction algebra.

Definition 3.1. Let X be a c-subtraction algebra and $d : X \longrightarrow X$ be a function. We call d a *derivation* on X if it satifies the following condition:

$$d(x \wedge y) = (dx \wedge y) + (x \wedge dy).$$

We often abbreviate $d(x)$ to dx.

Now we give some examples and present some properties for the derivations in subtraction algebra.

Example 3.2. Let $X = \{0, a, b, c\}$ be a c-subtraction algebra with the following Cayley table [9]:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Define a function d on X by

$$dx = \begin{cases} 0, & \text{if } x \in \{0, b\} \\ c, & \text{if } x \in \{a, c\}. \end{cases}$$

Then we can see that d is not a derivation on X since $d(c \wedge b) = db = 0 \neq (dc \wedge b) + (c \wedge db) = (c \wedge b) + (c \wedge 0) = b + 0 = b$.

Example 3.3. Define a function d on X in Example 3.2 by

$$dx = \begin{cases} 0, & \text{if } x \in \{0, b\} \\ a, & \text{if } x \in \{a, c\}. \end{cases}$$

Then it is seen that d is a derivation on X.

Proposition 3.4. *In a c-subtraction algebra X, the following properties hold*:

(i) *$dx = dx \wedge x \leq x$ for all $x \in X$,*
(ii) *$d(G(a, b)) \subseteq G(a, b)$,*
(iii) *$G(da, db) \subseteq G(a, b)$,*
(iv) *If I is an ideal of a subtraction algebra X, then $d(I) \subseteq I$.*

Proof. (i) $dx = d(x \wedge x) = (dx \wedge x) + (x \wedge dx) = x \wedge dx \le x$ (by S2 and a4).

(ii) For all $z \in G(a, b)$, we have $z - a \le b$. From (i), we have $dz \le z$ and $dz - a \le z - a \le b$ by (a9). Hence we obtain $dz \in G(a, b)$.

(iii) For all $z \in G(da, db)$, $z - da \le db \le b$. Hence we have $z - b \le da \le a$ or $z - a \le b$. Then we get $z \in G(a, b)$.

(iv) For all $x \in I$, we know that $dx \le x$. That is, $dx - x = 0 \in I$. From the definition of an ideal of X, we have $dx \in I$. □

Corollary 3.5. *In a c-subtraction algebra X, the following properties hold*:

(i) $d(a + b) \le a + b$,
(ii) $da + db \le a + b$,
(iii) $d^2x = dx$.

Proof. (i) Trivial from Proposition 3.4(i).

(ii) Since the greatest element of $G(da, db)$ is $da+db$ and the greatest element of $G(a, b)$ is $a+b$, then we have $da+db \le a+b$, by applying Proposition 3.4(iii).

(iii)
$$\begin{aligned} d^2x &= d(dx) \\ &= d(dx \wedge x) \\ &= (d^2x \wedge x) + (dx \wedge dx) \quad (\text{since } d^2x \le dx \le x) \\ &= d^2x + dx = dx. \end{aligned}$$
□

Theorem 3.6. *Let X be a c-subtraction algebra and d be a derivation on X. Then*

$$dx \wedge dy \le d(x \wedge y) \le dx + dy.$$

Proof. Since $dy \le y$, we have $dx-y \le dx-dy$ and $dx-(dx-dy) \le dx-(dx-y)$ or $dx \wedge dy \le dx \wedge y$. Similarly, since $dx \le x$ we obtain $dx \wedge dy \le dy \wedge x$. Then we get $dx \wedge dy \le dx \wedge y + dy \wedge x = d(x \wedge y)$. Furthermore, since $dx \wedge y \le dx$ and $dy \wedge x \le dy$ by (a4), we obtain $d(x \wedge y) = dx \wedge y + dy \wedge x \le dx + dy \wedge x \le dx + dy$. Then we have $d(x \wedge y) \le dx + dy$. □

Definition 3.7. Let X be a subtraction algebra and d be a derivation on X. If $x \le y$ implies $dx \le dy$ for all $x, y \in X$, d is called an *isotone derivation.*

Proposition 3.8. *Let X be a c-subtraction algebra and d be a derivation on X. Then the following hold*: *for all $x, y \in X$,*

(i) *If $d(x \wedge y) = dx \wedge dy$, then d is an isotone derivation,*
(ii) *If $d(x + y) = dx + dy$, then d is an isotone derivation.*

Proof. (i) Let $x \le y$. Then by (a4), we get $dx = d(x \wedge y) = dx \wedge dy \le dy$.

(ii) Let $x \le y$. Then since $x+y = y$ from Proposition 2.4(v), $dy = d(x+y) = dx + dy$. Hence we get $dx \le dy$. □

Proposition 3.9. *Let X be a bounded c-subtraction algebra and let 1 be the greatest element of X and d be a derivation on X. Then the following hold*: *for all $x, y \in X$,*

(i) *if $x \le d1$, then $dx = x$,*

(ii) *if* $x \geq d1$, *then* $d1 \leq dx$,
(iii) *if* $x \leq y$ *and* $dy = y$, *then* $dx = x$.

Proof. Since $dx = d(x \wedge 1) = (dx \wedge 1) + (x \wedge d1) = dx + (x \wedge d1)$, we have $x \wedge d1 \leq dx$. Then,

(i) if $x \leq d1$, since $x - d1 = 0$, we have $x = x - (x - d1) = x \wedge d1 \leq dx$ and $dx = x$ by Proposition 3.4(i).
(ii) if $x \geq d1$, since $d1 - x = 0$, we have $d1 = d1 - (d1 - x) = d1 \wedge x = x \wedge d1 \leq dx$.
(iii) if $x \leq y$, since $x = x \wedge y$, we have $dx = d(x \wedge y) = (dx \wedge y) + (x \wedge dy) = dx + (x \wedge y) = dx + x = x$. □

Theorem 3.10. *Let* X *be a c-subtraction algebra and* d *be a derivation on* X. *Then the following conditions are equivalent*:

(i) d *is the identity derivation,*
(ii) $d(x + y) = (dx + y) \wedge (x + dy)$,
(iii) d *is one-to-one,*
(iv) d *is onto.*

Proof. The proof is similar to Theorem 3.17 in [13]. □

Definition 3.11. Let X be a c-subtraction algebra and $d : X \longrightarrow X$ be a derivation. A function $D : X \longrightarrow X$ is called a *generalized derivation* determined by d if

$$D(x \wedge y) = Dx \wedge y + x \wedge dy$$

holds for all $x, y \in X$.

Example 3.12. Let $X = \{0, a, b, c\}$ be a c-subtraction algebra with the following Cayley table [9]:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Define a function D on X by

$$Dx = \begin{cases} 0, & x = 0 \\ a, & x = a \\ b, & x = b \text{ or } c. \end{cases}$$

Also, define a function d on X by

$$dx = \begin{cases} 0, & x = 0 \text{ or } c \\ a, & x = a \\ b, & x = b. \end{cases}$$

Then we can see that d is a derivation on X and D is a generalized derivation determined by d.

Example 3.13. Let $X = \{0, a, b, c\}$ be a c-subtraction algebra with the following Cayley table [9]:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Define a function D on X by

$$Dx = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a \\ c, & \text{if } x = b \text{ or } c. \end{cases}$$

Also, define a function d on X

$$dx = \begin{cases} 0, & \text{if } x = 0 \text{ or } c \\ a, & \text{if } x = a \\ b, & \text{if } x = b. \end{cases}$$

Then we can see that d is a derivation on X. But D is not a generalized derivation determined by d. Because $D(b \wedge a) = D0 = 0 \neq Db \wedge a + b \wedge da = (c \wedge a) + (b \wedge a) = a + 0 = a$.

Proposition 3.14. *In a c-subtraction algebra X, the following properties hold*:

(a) $D(dx) \leq dx \leq x$ *for all* $x \in X$,
(b) *If* $Dx = 0$, *then* $dx = 0$ *for all* $x \in X$.

Proof. (a) Since

$$\begin{aligned} D(dx) &= D(dx \wedge dx) \\ &= D(dx) \wedge dx + dx \wedge d^2x \\ &\leq dx + dx \text{ (since } d^2x \leq dx \leq x \text{ and } D(dx) - (D(dx) - dx) \leq dx) \\ &= dx, \end{aligned}$$

we have $D(dx) \leq dx \leq x$ for all $x \in X$ by Proposition 3.4(i).

(b) Straightforward. □

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