ON REES MATRIX REPRESENTATIONS OF ABUNDANT SEMIGROUPS WITH ADEQUATE TRANSVERSALS

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Abstract. The concepts of $*$-relation of a (Γ-)semigroup and $\overline{\Gamma}$-adequate transversal of a (Γ-)abundant semigroup are defined in this note. Then we develop a matrix type theory for abundant semigroups. We give some equivalent conditions of a Rees matrix semigroup being abundant and some equivalent conditions of an abundant Rees matrix semigroup having an adequate transversal. Then we obtain some Rees matrix representations for abundant semigroups with adequate transversals by the above theories.

Introduction

It is well-known that all abundant semigroups constitute an important class of generalized regular semigroups. An adequate transversal S^0 of an abundant semigroup S is an adequate $*$-subsemigroup of S which for any $x \in S$ there are unique element denoted by x^0 and two idempotents denoted by e_x, f_x such that $x = e_x x^0 f_x$, where $e_x \mathscr{L}^* x^0 \mathscr{R}^* f_x$ ($\mathscr{L}^*, \mathscr{R}^*$ are Green's $*$-relations). Here e_x and f_x are uniquely determined by x. Furthermore, S^0 is multiplicative if $f_x e_y \in E(S^0)$ for any $x, y \in S$.

By the Γ-semigroup T (see [2, 10, 11]) means that for two non-empty sets T and Γ in which an element denoted by x, ν respectively under multiplication

$$x \circ y = x \nu y \in T \text{ satisfying } (x \alpha y)\beta z = x\alpha(y\beta z)$$

for any $x, y, z \in T$ and $\nu, \alpha, \beta \in \Gamma$. A Γ-semigroup T is Γ-commutative, if for any $x, y \in T, \alpha \in \Gamma, x\alpha y = y\alpha x$. Similar to the theory of semigroups, in the theory of Γ-semigroups we have also the well-known correlate concepts. Here we will apply them directly. Clearly, any semigroup T is always a Γ-semigroup for any subset Γ of T or $\Gamma = \{1\}$, where the member 1 is an outer identity. Conversely, a Γ-semigroup T need not be a semigroup in general.

We remark that Rees matrix semigroups have been defined in several slightly different ways. For example, in [9] the matrix semigroup is over an inverse semigroup. In [12] the matrix semigroup is over a monoid with zero. In [6], J. Fountain used some (T_α, T_β)-bisystem $M_{\alpha\beta}$, where $\alpha, \beta \in \Gamma'$ and an outer zero and introduced blocked Rees matrix semigroup $\mu^0(M_{\alpha\beta}; I, \Lambda, \Gamma'; P)$. Here, we say that a Rees matrix semigroup S_μ is a Rees matrix representation of the semigroup S if S_μ is isomorphic to S.

Our purpose in this note is to develop a matrix type theory for abundant semigroups with adequate transversals, that is, to study the conditions of a Rees matrix semigroup being abundant and the conditions of an abundant Rees matrix semigroup having an adequate transversal. The above results such that we may obtain some Rees matrix representations of abundant semigroup with an adequate transversal. We proceed as follows:

We begin in Section 1 by defining $\Gamma*$-relations on a (Γ-)semigroup and discussing their properties. Particularly, the relations between Green's $*$-relations and $\Gamma*$-relations. By these relations, we show that the relations between a Γ-semigroup T and a Rees matrix semigroup over T and obtain some equivalent conditions of a Rees matrix semigroup being abundant. In Section 2, we define the concept of $\bar{\Gamma}$-adequate transversals of Γ-abundant semigroups. Then we show that the relation between $\bar{\Gamma}$-adequate transversals and adequate transversals of Rees matrix semigroups and obtain some equivalent conditions of an abundant Rees matrix semigroup having an adequate transveral. In Section 3, using the results in Sections 1 and 2, for an abundant semigroup S with an adequate transversal S^0, we construct a Rees matrix semigroup S_μ over some subset T of S. Then we prove that S_μ is a (Γ-)Rees matrix representation of given semigroup S.

For terminologies not given in this note the reader is referred to [3, 4, 5, 6, 8, 9, 11, 12].

1. The conditions of a Rees matrix semigroup being abundant

We present first some necessary notation and well-known results. For details consult [6, 9, 11, 12].

Given a semigroup T, non-empty index sets I and Λ and defined a $\Lambda \times I$ matrix $P = (p_{\lambda i})_{\Lambda \times I}$ over T. By [9], we may obtained a Rees matrix semigroup denoted by $S_\mu = \mu(T; I, \Lambda, P)$. It's elements consist of all triples $(x)_{i\lambda}$, where $x \in T, (i, \lambda) \in I \times \Lambda$ with multiplication

$$(1.1) \qquad (\forall (x)_{i\lambda}, (y)_{j\mu} \in S_\mu) \ (x)_{i\lambda}(y)_{j\mu} = (xp_{\lambda j}y)_{i\mu}.$$

In general, S_μ is only a semigroup. Put the set $RS_\mu = \mathscr{R}\mu(T; I, \Lambda, P)$ of all regular elements of S_μ. It was proved in [9] that RS_μ is a regular semigroup if T is regular. This result, we well generalized to abundant semigroups (see Theorem 1.9). Now we consider the converse problem.

Let S be a semigroup with an outer zero. Like [6], we index the set of non-zero $\mathscr{R}^*$-classes of S by I and the set of non-zero $\mathscr{L}^*$-classes of S by Λ, so that we write the $\mathscr{R}^*$-classes as R_i^* $(i \in I)$ and the $\mathscr{L}^*$-classes as L_λ^* $(\lambda \in \Lambda)$. Then we put $H_{i\lambda}^* = R_i^* \bigcap L_\lambda^*$ for $(i, \lambda) \in I \times \Lambda$, so that every non-zero $\mathscr{H}^*$-class of S is some $H_{i\lambda}^*$ and each $H_{i\lambda}^*$ is either empty or a $\mathscr{H}^*$-class. Of course $S \setminus \{0\} = \bigcup \{H_{i\lambda}^*; (i, \lambda) \in I \times \Lambda\}$ we denote this structure express of S by (S, I, Λ).

Further, like [6] we may obtain another structure express denoted by (S, I, Λ, Γ'), where Γ' is the set of non-zero $\mathscr{D}$-classes of S which contain idempotents. We write these $\mathscr{D}$-classes as D_α $(\alpha \in \Gamma')$. Let $I_\alpha = \{i \in I; D_\alpha \bigcap R_i^* \neq \phi\}$, $\Lambda_\alpha = \{\lambda \in \Lambda; D_\alpha \bigcap L_\lambda^* \neq \phi\}$. In general, $I = \bigcup I_\alpha$ and $\Lambda = \bigcup \Lambda_\alpha$. When S satisfies some conditions (for example, S is abundant), they are disjoint union.

Now let the set

(1.2) $\Gamma = \{p_{\lambda i} \in T \mid (i, \lambda) \in I \times \Lambda\}$

be any subset of S determined by the index pair set $I \times \Lambda$. Since S is a semigroup and $\Gamma \subseteq S$, so S becomes a Γ-semigroup. Now let T be a subset with zero of S. In this section we suppose always that T is a Γ-semigroup where the set Γ defined as (1.2). On Γ-semigroup T (or S) we give the following concept.

Definition 1.1. For Γ-semigroup T (resp. semigroup T) the $\Gamma *$-relations on T denoted by $\forall \mathscr{L}_{\Gamma_\lambda}^*$ and $\forall \mathscr{R}_{\Gamma_i}^*$ for $i \in I, \lambda \in \Lambda$ are defined by

$$(a, b \in T) a \mathscr{L}_{\Gamma_\lambda}^* b : (x, y \in T, v, u \in I) \text{ for } \lambda \in \Lambda$$
$$a p_{\lambda v} x = a p_{\lambda u} y \Leftrightarrow b p_{\lambda v} x = b p_{\lambda u} y;$$
$$(a, b \in T) a \mathscr{R}_{\Gamma_i}^* b : (x, y \in T, k, t \in I) \text{ for } i \in I$$
$$x p_{ti} a = y p_{ki} a \Leftrightarrow x p_{ti} b = y p_{ki} b.$$

The following results are clear.

Lemma 1.2. (1) For $\lambda \in \Lambda$ $(i \in I)$, $\mathscr{L}_{\Gamma_\lambda}^* (\mathscr{R}_{\Gamma_i}^*)$ is an equivalence relation on T. We denote the $\mathscr{L}_{\Gamma_\lambda}^*$-class $(\mathscr{R}_{\Gamma_i}^*$-class$)$ by $L^*(\lambda)$ $(\lambda \in \Lambda)$ $(R^*(i)$ $(i \in I))$.
(2) If $a\mathscr{L}^*b$ $(a\mathscr{R}^*b)$ for $a, b \in T$, then $a\mathscr{L}_{\Gamma_\lambda}^* b$ $(a\mathscr{R}_{\Gamma_i}^* b)$ for any $\lambda \in \Lambda$ $(i \in I)$.

By the structure express (S, I, Λ) of S, we know that for $a, b \in S$, if $a, b \in L_\lambda^*$ (R_i^*), then $a\mathscr{L}^*b$ $(a\mathscr{R}^*b)$. Conversely, if $a\mathscr{L}^*b$ $(a\mathscr{R}^*b)$, then there is a member $\lambda \in \Lambda$ $(i \in I)$ such that $a, b \in L_\lambda^*$ (R_i^*). Thus we may think that the Green's $*$-relation $\mathscr{L}^*$ $(\mathscr{R}^*)$ can be written in the form $\mathscr{L}_\lambda^*$ $(a\mathscr{R}_i^*b)$ for some $\lambda \in \Lambda$ $(i \in I)$.

The following we suppose always that the matrix $P = (p_{\lambda i})_{\Lambda \times I}$ over Γ and $S_\mu = \mu(T; I, \Lambda, P)$ is a Rees matrix semigroup over T under the multiplication (1.1).

Lemma 1.3. (1) For any non-zero elements $(a)_{i\lambda}$, $(b)_{j\mu} \in S_\mu$, if $\lambda = \mu$ and $a\mathscr{L}_{\Gamma_\lambda}^* b$, then $(a)_{i\lambda} \mathscr{L}_\lambda^*(b)_{j\mu}$, if $i = j$ and $a\mathscr{R}_{\Gamma_i}^* b$, then $(a)_{i\lambda}\mathscr{R}_i^*(b)_{j\mu}$.
Further, if S_μ is abundant, then the converse case is also true.
(2) We denote the $\mathscr{L}^*$-classes $(\mathscr{R}^*$-classes$)$ of S_μ by L_λ^* (R_i^*). If S_μ is abundant, then

$$(i \in I) \ R_i^* = \{(x)_{i\lambda} \mid x \in R^*(i), \lambda \in \Lambda\};$$
$$(\lambda \in \Lambda) \ L_\lambda^* = \{(x)_{i\lambda} \mid x \in L^*(\lambda), i \in I\}.$$

Proof. (1) Let $0 \neq (a)_{i\lambda}, 0 \neq (b)_{j\mu}$, if $\lambda = \mu$ and $a\mathscr{L}_{\Gamma_\lambda}^* b$, then for $(x)_{vt}, (y)_{uk} \in S_\mu$

$$(a)_{i\lambda}(x)_{vt} = (a)_{i\lambda}(y)_{uk} \Leftrightarrow (ap_{\lambda v}x)_{it} = (ap_{\lambda u}y)_{ik}$$
$$\Leftrightarrow t = k \text{ and } ap_{\lambda v}\mathrm{x} = ap_{\lambda u}\mathrm{y}.$$

Similarly, for the element $(b)_{j\mu} = (b)_{j\lambda}$

$$(b)_{j\lambda}(x)_{vt} = (b)_{j\lambda}(y)_{uk} \Leftrightarrow t = k \text{ and } bp_{\lambda v}x = bp_{\lambda u}y.$$

Thus by $\lambda = \mu$ and $a\mathscr{L}_{\Gamma_\lambda}^* b$ may imply that

$$(a)_{i\lambda}(x)_{vt} = (a)_{i\lambda}(y)_{uk} \Leftrightarrow t = k, ap_{\lambda v}x = ap_{\lambda u}y$$
$$\Leftrightarrow t = k, bp_{\lambda v}x = bp_{\lambda u}y$$
$$\Leftrightarrow (b)_{j\lambda}(x)_{vt} = (b)_{j\lambda}(y)_{uk}.$$

That is $(a)_{i\lambda}\mathscr{L}_\lambda^*(b)_{j\lambda}$. The another result is dual.

If S_μ is abundant, let $e = (c)_{wl} \in L_{(a)_{i\lambda}}^* \bigcap E(S)$, by $e^2 = e$ implies $(a)_{i\lambda}e = (a)_{i\lambda}$ so $l = \lambda$ and if $(b)_{j\mu} \in L_{(a)_{i\lambda}}^* = L_e^*$, similarly $\mu = \lambda$. Thus we obtain $(a)_{i\lambda}\mathscr{L}_\lambda^*(b)_{j\lambda}$, that is, for $(x)_{vt}, (y)_{uk} \in S_\mu$

$$(a)_{i\lambda}(x)_{vt} = (a)_{i\lambda}(b)_{uk} \Leftrightarrow (b)_{j\lambda}(x)_{vt} = (b)_{j\lambda}(y)_{uk}.$$

Computing we may imply that $t = k$ and $a\mathscr{L}_{\Gamma_\lambda}^* b$ for $\lambda \in \Lambda$.

(2) Let $S\mu$ be abundant, by part (1) we know that for $0 \neq (a)_{i\lambda}, 0 \neq (b)_{j\mu} \in S$

$$(a)_{i\lambda}\mathscr{L}^*(b)_{j\mu} \Leftrightarrow \lambda = \mu \text{ and } a\mathscr{L}_{\Gamma_\lambda}^* b.$$

So that $a, b \in L^*(\lambda)$. Conversely, if $a, b \in L^*(\lambda)$, then $(a)_{i\lambda}, (b)_{j\lambda} \in L_\lambda^*$ for any $i, j \in I$. Thus we have that

$$L_\lambda^* = \{(x)_{i\lambda} \mid x \in L^*(\lambda), i \in I\}(\lambda \in \Lambda).$$

The other result is dual. $\qquad\qquad\square$

Corollary 1.4. *For* $(i, \lambda) \in I \times \Lambda, a, b \in T$
(1) *if* $a\mathscr{L}_\lambda^* b$ $(a\mathscr{R}_i^* b)$, *then* $a\mathscr{L}_{\Gamma_\lambda}^* b$ $(a\mathscr{R}_{\Gamma_i}^* b)$.
(2) *if* $a\mathscr{L}_{\Gamma_\lambda}^* b$ $(a\mathscr{R}_{\Gamma_i}^* b)$, *then* $ap_{\lambda i}\mathscr{L}_\lambda^* bp_{\lambda i}$ $(p_{\lambda i}a\mathscr{R}_i^* p_{\lambda i}b)$.

Lemma 1.5. *For any* $p_{\lambda i} \in \Gamma$ *if there are two elements* q_λ *and* r_i *in* S *such that* $p_{\lambda i} = q_\lambda r_i$, *then*
(1) *the mapping* ϕ *defined by*

$$(\forall (a)_{i\lambda} \in S_\mu) \ (a)_{i\lambda}\phi = r_i aq_\lambda$$

is a homomorphism from S_μ *to* S.

(2) *if S_μ is abundant and ϕ is an isomorphism, then*

$$(a)_{i\lambda}\mathscr{L}_\lambda^*(b)_{j\lambda} \ in \ S_\mu \ \Leftrightarrow r_iaq_\lambda \ \mathscr{L}_\lambda^* r_jbq_\lambda \ in \ S$$
$$\Leftrightarrow a\mathscr{L}_{\Gamma_\lambda}^* b \ in \ T,$$
$$(a)_{i\lambda}\mathscr{R}_i^*(b)_{i\mu} in S_\mu \ \Leftrightarrow r_iaq_\lambda \ \mathscr{R}_i^* r_ibq_\mu \ in \ S$$
$$\Leftrightarrow a\mathscr{R}_{\Gamma_i}^* b \ in \ T.$$

Proof. (1) Since S is a semigroup, so ϕ is a mapping from $S\mu$ to S. For $(a)_{i\lambda}, (b)_{j\mu} \in S_\mu$,

$$[(a)_{i\lambda}(b)_{j\mu}]\phi = (ap_{\lambda_j}b)_{i\mu}\phi = r_iap_{\lambda_j}bq_\mu$$
$$= (r_iaq_\lambda)(r_jbq_\mu) = (a)_{i\lambda}\phi \cdot (b)_{j\mu}\phi.$$

(2) By Lemma 1.3, we obtain directly this result. $\qquad\square$

The following concepts are different from that in the theory of Γ-semigroups.

Definition 1.6. An element $a \in T$ is a p_{λ_i}-regular element means that $ap_{\lambda_i}a = a$. The set of all p_{λ_i}-regular elements of T is denoted by $V(p_{\lambda_i})$. Let $a, b \in V(p_{\lambda_i})$ if a and b are p_{λ_i}-commutative (i.e., $ap_{\lambda_i}b = bp_{\lambda_i}a$), then we say that a and b are two p_{λ_i}-commutative regular elements. Let the set

$$CV(p_{\lambda_i}) = \{a \in V(p_{\lambda_i}); \forall x \in V(p_{\lambda_i}), ap_{\lambda_i}x = xp_{\lambda_i}a\},$$

then $CV(p_{\lambda_i})$ is called the center of $V(p_{\lambda_i})$. T is called Γ-abundant if for any $(i, \lambda) \in I \times \Lambda, L^*(\lambda) \bigcap V(p_{\lambda_i}) \neq \phi$ and $R^*(i) \bigcap V(p_{\lambda_i}) \neq \phi$. The Γ-abundant semigroup T is called Γ-adequate, if $V(p_{\lambda_i}) = CV(p_{\lambda_i})$ for any $(i, \lambda) \in I \times \Lambda$. Particularly, for only one $p_{\lambda_i} \in \Gamma$ we have the concepts of p_{λ_i}-abundant and p_{λ_i}-adequate.

Clearly, since S is a semigroup, so the set $V(p_{\lambda_i})$ is the set of inverse elements of the non-zero element p_{λ_i} for $(i, \lambda) \in I \times \Lambda$. T is Γ-abundant (adequate) if and only if for any $p_{\lambda_i} \in \Gamma$, T is p_{λ_i}-abundant (adequate). If T is abundant (adequate), then by Lemma 1.3 and Corollary 1.4, T is necessarily Γ-abundant (adequate), but the converse case is not always true.

Lemma 1.7. *If T is Γ-adequate, then $|L^*(\lambda) \bigcap V(p_{\lambda_i})| = |R^*(i) \bigcap V(p_{\lambda_i})| = 1$ for any $p_{\lambda_i} \in \Gamma$.*

Proof. Let $a_1, a_2 \in L^*(\lambda) \bigcap V(p_{\lambda_i})$ then $0 \neq a_1\mathscr{L}_{\Gamma_\lambda}^* a_2 \neq 0$. By Corollary 1.4, $a_1p_{\lambda_i}\mathscr{L}a_2p_{\lambda_i}$ implies that $a_1 = a_1p_{\lambda_i}a_1\mathscr{L}a_2p_{\lambda_i}a_1 = a_1p_{\lambda_i}a_2\mathscr{L}a_2p_{\lambda_i}a_2 = a_2$ by $V(p_{\lambda_i}) = CV(p_{\lambda_i})$. Since $p_{\lambda_i} \in V(a_k)$ $(k = 1, 2)$, so by $a_1\mathscr{L}a_2$ implies $a_1p_{\lambda_i} = a_2p_{\lambda_i}$. Thus we have that $a_1 = a_1p_{\lambda_i}a_1 = a_2p_{\lambda_i}a_1 = a_1p_{\lambda_i}a_2 = a_2p_{\lambda_i}a_2 = a_2$. Dually, we can show that the other result for any $p_{\lambda i} \in \Gamma$. $\qquad\square$

Lemma 1.8. (1) *A non-zero element $(a)_{i\lambda}$ in S_μ is regular if and only if $a \in Reg(S)$ (the regular element set of S) and $(\exists \ (j, \mu) \in I \times \Lambda)$*

$$p_{\lambda_j}Tp_{\mu_i} \bigcap V(a) \neq \phi.$$

(2) *A non-zero element* $(a)_{i\lambda}$ *in* S_μ *is an idempotent if and only if* $a \in V(p_{\lambda i})$.

(3) *Two idempotents* $(a)_{i\lambda}, (b)_{j\mu}$ *are commutative if and only if* $(i, \lambda) = (j, \mu)$ *and* a, b *are* $p_{\lambda i}$-*commutative regular elements*.

Proof. Here we omit the checking process of part (1) to part (3). □

Theorem 1.9. *The following are equivalent*:

(1) S_μ *is abundant.*

(2) T *is* Γ-*abundant.*

(3) P *is an abundant matrix (i.e., each row and each column of* P *contain a regular element of* S).

Proof. By Definition 1.6, Lemma 1.8, it is easy to show that they are equivalent. We here omit this proof. □

Theorem 1.10. *If* T *is* Γ-*abundant, then the following equivalent*:

(1) S_μ *is adequate.*

(2) T *is* Γ-*adequate where the Gamma set* Γ *denoted by* (1.2) *and satisfies that* $I = \Lambda, |V(p_{ii})| = 1$ *for any* $i \neq I$ *and* $p_{\lambda i} = 0$ *for* $\lambda \neq i$, $i, \lambda \in I$.

Proof. $(1) \Rightarrow (2)$ Let S_μ be adequate, then $E(S_\mu)$ is a semilattice. For $0 \neq (a)_{i\lambda}, 0 \neq (b)_{j\mu} \in E(S_\mu)$ computing we know that $i = j$ and $\lambda = \mu$ and $a, b \in V(p_{\lambda i})$. Since the $\mathscr{L}^*$-($\mathscr{R}^*$-)class $L_\lambda^*(R_i^*)$ of S_μ has only an idempotent, so $|V(p_{\lambda i})| = 1$ and $CV(p_{\lambda i}) = V(p_{\lambda i})$. Suppose that $\lambda \neq i$, and $i \in \Lambda$ (or $\lambda \in I$), then L_i^* has an idempotent $(b)_{ji}$ for some $j \in I$. Since for $(a)_{i\lambda} \in L_i^* \bigcap E(S_\mu)$ and $(b)_{ji} \in L_i^* \bigcap E(S_\mu), (a)_{i\lambda}(b)_{ji} = (b)_{ji}(a)_{i\lambda}$. So we imply that $i = j = \lambda$. It is a contradiction. Thus we obtain that $I = \Lambda$ and $|CV(p_{ii})| = 1$ for any $i \in I$. Since $H_{ii}^* = L_i^* \bigcap R_i^*$ has only an idempotent so by [6, Lemma 1.12] H_{ii}^* is a cancellative monoid with the identity denote by $(e)_{ii}$. So we may write

$$E(S_\mu) = \{(e_i)_{ii} \mid i \in I = \Lambda, e_i \in V(p_{ii}) \text{ and } |V(p_{ii})| = 1\} \bigcup \{0\}.$$

Let $(e_i)_{ii}, (f_j)_{jj} \in E(S_\mu)$, $i \neq j$ by $E(S_\mu)$ is a semilattice we obtain $(e_i p_{ij} f_j)_{ij} = (f_j p_{ji} e_i)_{ji}$ and $i \neq j$, it is necessarily that $e_i p_{ij} f_j = f_j p_{ji} e_i = 0$. Suppose that $p_{ij} \neq 0$. Since T is Γ-abundant so $L^*(j) \bigcap V(p_{ij}) \neq \phi$. Let $a \in L^*(j) \bigcap V(p_{ij})$, by Theorem 1.9 and Lemma 1.8, $(a)_{ij} \in L_i^* \bigcap E(S_\mu)$ and $i \neq j, i, j \in I$. Since $E(S_\mu)$ has above express, it is also a contradiction. So we know that $p_{ij} = 0$. Similarly, we can prove $p_{ji} = 0$ for $i \neq j, i, j \in I$. Concluding we know that Γ satisfies the following conditions: $I = \Lambda, |V(p_{ii})| = 1$ for $i \in I, p_{\lambda i} = 0$ for $\lambda \neq i, i, \lambda \in I$. Finally, since T is Γ-abundant, by the above results we know $V(p_{ii}) = CV(p_{ii})$ for $i \in I$, and $V(p_{\lambda i}) = V(0) = CV(0) = CV(p_{\lambda i})$ for $\lambda \neq i, i, \lambda \in I$. So T is Γ-adequate.

$(2) \Rightarrow (1)$ Suppose that Γ is as required and T is Γ-adequate, then $E(S_\mu)$ can be written in the form

$$E(S_\mu) = \{(e_i)_{ii} | i \in I, e_i \in V(p_{ii}), |V(p_{ii})| = 1\} \bigcup \{0\}.$$

It is easy to check that $E(S_\mu)$ is a semilattice with zero. Thus S_μ is adequate.

$\square$

The above results (see Theorems 1.9 and 1.10) generalized the corresponding results in [6] and [9].

Corollary 1.11. *S_μ contains an adequate subsemigroup S_μ^0 if and only if the matrix $P = (p_{\lambda i})_{\Lambda \times I}$ satisfies conditions*
  1^0 *There is a subset $\bar{\mathbf{I}} \times \bar{\mathbf{I}} \subseteq I \times \Lambda$ (or $\bar{\mathbf{\Lambda}} \times \bar{\mathbf{\Lambda}} \subseteq I \times \Lambda$).*
  2^0 *On $\bar{\mathbf{I}} \times \bar{\mathbf{I}}, |V(p_{\bar{i}\bar{i}})| = 1$ ($\bar{i} \in \bar{\mathbf{I}}$) and $p_{\bar{i}\bar{j}} = 0$ ($\bar{i}, \bar{j} \in \bar{\mathbf{I}}, \bar{i} \neq \bar{j}$).*

2. The conditions of an abundant Rees matrix semigroup having an adequate transversal

In this section the semigroup S and Γ-semigroup T, the sets $I, \Lambda, \Gamma = \{p_{\lambda i} \mid (i, \lambda) \in I \times \Lambda\}$, the matrix $P = (p_{\lambda i})_{\Lambda \times I}$ and the Rees matrix semigroup $S_\mu = \mu(T; I, \Lambda, P)$ are as required in Section 1. We continue to discuss the relations between a $\bar{\mathbf{\Gamma}}$-adequate transversal of T and adequate transversals of S_μ and S. We begin by defining the following concept.

Definition 2.1. Let T^0 be a $\bar{\mathbf{\Gamma}}$-adequate subsemigroup of T, where $\bar{\mathbf{\Gamma}} \subseteq \Gamma$. By Theorem 1.10, we may denote the subset $\bar{\mathbf{\Gamma}}$ of Γ by
(2.1)
$$\bar{\mathbf{\Gamma}} = \{p_{\bar{\lambda}\bar{i}} \in \Gamma \mid (\bar{i}, \bar{\lambda}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}, |V(p_{\bar{i}\bar{i}})| = 1 \text{ for } \bar{i} \in \bar{\mathbf{I}}, p_{\bar{\lambda}\bar{i}} = 0 \text{ if } \bar{\lambda} \neq \bar{i}, \bar{i}, \bar{\lambda} \in \bar{\mathbf{I}}\}.$$

We denote an element of T^0 by x^0. If for any $x \in T$ and $(i, \lambda) \in I \times \Lambda$ there are a unique element $x^0 \in T^0$ and two element $a \in CV(p_{\lambda \bar{i}})$ and $b \in CV(p_{\bar{i}i})$ for a unique $(\bar{i}, \bar{i}) \in \bar{I} \times \bar{I}$, such that x can be uniquely written in the form

$$x = bp_{\bar{i}\bar{i}}x^0 p_{\bar{i}\bar{i}}a,$$

where $b\mathscr{L}^*_{\Gamma_{\bar{i}}}x^0\mathscr{R}^*_{\Gamma_{\bar{i}}}a$. Then T^0 is called a $\bar{\mathbf{\Gamma}}$-adequate transversal of T.

For this concept we have:

Lemma 2.2. (1) *$b\mathscr{R}^*_{\Gamma_{\bar{i}}}x\mathscr{L}^*_{\Gamma_{\bar{i}}}a, 0^0 = 0$, otherwise the element x^0 is uniquely determined by x and (i, λ).*
  (2) *The elements a and b are uniquely determined by x and (i, λ). We denote them by a_x and b_x.*

Proof. (1) By Lemma 1.3 and Definition 2.1 the results may be directly obtained.
  (2) Let $0 \neq x \in T$, suppose that there are $b_1, b_2 \in CV(p_{\bar{i}i})$ such that $b_k\mathscr{L}^*_{\Gamma_{\bar{i}}}x^0$ ($k = 1, 2$) and satisfy all conditions in Definition 2.1. Then $p_{\bar{i}i}b_1 = p_{\bar{i}i}b_2p_{\bar{i}i}b_1$ and $p_{\bar{i}i}b_2 = p_{\bar{i}i}b_1p_{\bar{i}i}b_2$ imply that $p_{\bar{i}i}b_1\mathscr{R}p_{\bar{i}i}b_2$ and $p_{\bar{i}i}b_2 \in V(p_{\bar{i}i}b_1)$. Since for any $(p_{\bar{i}i}b_1)' \in V(p_{\bar{i}i}b_1)$ we have $p_{\bar{i}i}b_1\mathscr{L}(p_{\bar{i}i}b_1)'$. So we obtain that $p_{\bar{i}i}b_1\mathscr{H}p_{\bar{i}i}b_2$. This implies $p_{\bar{i}i}b_1 = p_{\bar{i}i}b_2$ and so $b_1 = b_1p_{\bar{i}i}b_1 = b_1p_{\bar{i}i}b_2 = b_2p_{\bar{i}i}b_1 = b_2$ by $b_1, b_2 \in CV(p_{\bar{i}i})$. The other case is dual. $\square$

By Lemma 2.2, we may denote the sets by

$$A_\Lambda = \bigcup_{\bar{i} \in \bar{\mathbf{I}}} \{a_x \in T \mid \forall x \in T \ \exists \mid a_x \in R^*(\lambda) \bigcap CV(p_{\lambda \bar{i}}), \forall \lambda \in \Lambda \} \bigcup \{0\},$$

$$B_I = \bigcup_{\bar{i} \in \bar{\mathbf{I}}} \{b_x \in T \mid \forall x \in T \ \exists \mid b_x \in L^*(i) \bigcap CV(p_{\bar{i} i}), \ \forall i \in I \ \} \bigcup \{0\}.$$

To our object we omit the discussion for the sets A_Λ and B_I. Furthermore, we introduce the following concept.

Definition 2.3. Let T^0 be a $\bar{\mathbf{\Gamma}}$-adequate transversal of T. If for any $a_x \in A_\Lambda, b_y \in B_I$ and $p_{\lambda i} \in \Gamma$, there is a unique $p_{\bar{i}\bar{i}} \in \bar{\mathbf{\Gamma}}$ such that if $a_x p_{\lambda i} b_y \neq 0$, then

$$a_x p_{\lambda i} b_y \in V(p_{\bar{i}\bar{i}}) \subseteq T^0 \ (\text{i.e., } A_\Lambda \Gamma B_I \subseteq \bigcup_{\bar{i} \in \bar{\mathbf{I}}} V(p_{\bar{i}\bar{i}}) \bigcup \{0\}).$$

Then T^0 is called a Γ multiplicative $\bar{\mathbf{\Gamma}}$-adequate transversal of T.

In the following, T^0 is always a $\bar{\mathbf{\Gamma}}$-adequate transversal of T where $\bar{\mathbf{\Gamma}}$ is given by (2.1).

Lemma 2.4. (1) *Let* $S_\mu^0 = \{(x^0)_{\bar{i}\bar{i}} | x^0 \in T^0, (\bar{i}, \bar{i}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}$. *Then* S_μ^0 *is an adequate transversal of* S_μ.
(2) *In part (1), if* T^0 *is a* Γ *-multiplicative, then* S_μ^0 *is multiplicative.*

Proof. (1) By Theorem 1.9, we know that S_μ is abundant. Let $(x^0)_{\bar{i}\bar{i}}, (y^0)_{\bar{\lambda}\bar{\lambda}} \in S_\mu^0$ then

$$(x^0)_{\bar{i}\bar{i}}(y^0)_{\bar{\lambda}\bar{\lambda}} = (x^0 p_{\bar{i}\bar{\lambda}} y^0)_{\bar{i}\bar{\lambda}} = \left\{ \begin{array}{ll} (x^0 p_{\bar{i}\bar{i}} y^0)_{\bar{i}\bar{i}} & \bar{i} = \bar{\lambda} \\ 0 & \bar{i} \neq \bar{\lambda}. \end{array} \right.$$

Since T^0 is $\bar{\mathbf{\Gamma}}$-semigroup with zero, so $(x^0)_{\bar{i}\bar{i}}(y^0)_{\bar{\lambda}\bar{\lambda}} \in S_\mu^0$. Since $L_{\bar{i}}^* \bigcap S_\mu^0 \neq \phi$ and $R_{\bar{i}}^* \bigcap S_\mu^0 \neq \phi$. Thus S_μ^0 is a $*$-subsemigroup. Clearly, Theorem 1.10 demonstrates that S_μ^0 is adequate. Since T^0 is a $\bar{\mathbf{\Gamma}}$-adequate transversal of T, so for any $x \in T$ and $(i, \lambda) \in I \times \Lambda$, x can be uniquely written in the form $x = b_x p_{\bar{i}\bar{i}} x^0 p_{\bar{i}\bar{i}} a_x$, where $a_x \in A_\Lambda, b_x \in B_I, (\bar{i}, \bar{i}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}$ and $b_x \mathscr{L}_{\Gamma_{\bar{i}}}^* x^0 \mathscr{R}_{\Gamma_{\bar{i}}}^* a_x$. Thus for any $(x)_{i\lambda} \in S_\mu, (x)_{i\lambda}$ can be also uniquely written in the form

$$(x)_{i\lambda} = (b_x p_{\bar{i}\bar{i}} x^0 p_{\bar{i}\bar{i}} a_x)_{i\lambda} = (b_x)_{i\bar{i}}(x^0)_{\bar{i}\bar{i}}(a_x)_{\bar{i}\lambda},$$

where of course $(b_x)_{i\bar{i}} \mathscr{L}_{\bar{i}}^* (x^0)_{\bar{i}\bar{i}} \mathscr{R}_{\bar{i}}^* (a_x)_{\bar{i}\lambda}$. Since $a_x \in A_\Lambda$ and $b_x \in B_I$, so $(a_x)_{\bar{i}\lambda}$ and $(b_x)_{i\bar{i}}$ are uniquely determined by $(x)_{i\lambda}$ and $(a_x)_{\bar{i}\lambda}, (b_x)_{i\bar{i}} \in E(S)$. Thus S_μ^0 is indeed an adequate transversal of S_μ.

(2) Since T^0 is Γ-multiplicative, for any $a_x \in A_\Lambda, b_y \in B_I$ and $p_{\lambda i} \in \Gamma, a_x p_{\lambda i} b_y \in V(p_{\bar{i}\bar{i}})$ for a unique $(\bar{i}, \bar{i}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}$ (or $a_x p_{\lambda i} b_y = 0$), so for any $f_{(x)_{i\lambda}} = (a_x)_{\bar{i}\lambda}$ and $e_{(y)_{j\mu}} = (b_y)_{j\bar{\lambda}}$

$$f_{(x)_{i\lambda}} e_{(y)_{j\mu}} = (a_x)_{\bar{i}\lambda}(b_y)_{j\bar{\lambda}} = (a_x p_{\lambda j} b_y)_{\bar{i}\bar{\lambda}} = \left\{ \begin{array}{ll} (a_x p_{\lambda j} b_y)_{\bar{i}\bar{i}} & \bar{i} = \bar{\lambda} \\ 0 & \bar{i} \neq \bar{\lambda}, \end{array} \right.$$

where $a_x p_{\lambda j} b_y \in V(p_{\bar{i}\bar{i}}) \subseteq T^0$ for $\bar{i} = \bar{\lambda}$, and so $f_{(x)_{i\lambda}} e_{(y)_{j\mu}} \in E(S_\mu^0)$. Thus S_μ^0 is multiplicative. $\square$

Lemma 2.5. *If S_μ^0 is an adequate transversal of S_μ, then*
  (1) *T contains a $\bar{\Gamma}$-adequate transversal T^0 for some $\bar{\Gamma} \subseteq \Gamma$.*
  (2) *If S_μ^0 is multiplicative, then T^0 is Γ-multiplicative.*

Proof. (1) By Theorem 1.9 we know that T is Γ-abundant. Let $(x)_{i\lambda} \in S_\mu^0$ then the element x is belongs to some subset T^0 of T. By Theorem 1.10 we can always denote the set by

$$S_\mu^0 = \{(x^0)_{\bar{i}\bar{\lambda}} | x^0 \in T^0, (\bar{i}, \bar{\lambda}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\},$$

where $(x_{\bar{i}\bar{\lambda}}^0)$ means that $(x)_{i\lambda}^0 \in S_\mu^0$ is uniquely determined by $(x)_{i\lambda}$ and $\bar{\mathbf{I}} \times \bar{\mathbf{I}}$ is some subset of $I \times \Lambda$. Since S_μ^0 is an adequate subsemigroup of S_μ, by Theorem 1.10 T^0 is $\bar{\Gamma}$-adequate subsemigroup of T, so we may suppose that $\bar{\Gamma} = \{p_{\bar{\lambda}\bar{i}} | (\bar{i}, \bar{\lambda}) \in \bar{I} \times \bar{I}\} \subseteq \Gamma$ satisfies $p_{\bar{\lambda}\bar{i}} = \begin{cases} p_{\bar{i}\bar{i}} & \bar{\lambda} = \bar{i} \\ 0 & \bar{\lambda} \neq \bar{i} \end{cases}$ and $|V(p_{\bar{i}\bar{i}})| = 1$ for any $\bar{i} \in \bar{\mathbf{I}}$. This time we may write the set

$$E(S_\mu^0) = \{(a^0)_{\bar{i}\bar{i}} \mid a^0 \in V(p_{\bar{i}\bar{i}}),\ \bar{i} \in \bar{\mathbf{I}} \text{ or } a^0 = 0\}.$$

Since S_μ^0 is an adequate transversal of S_μ, so for any $(x)_{i\lambda} \in S_\mu$ $(x)_{i\lambda}$ can be uniquely written in the form

$$(x)_{i\lambda} = (b)_{i\bar{i}}(x^0)_{\bar{i}\bar{i}}(a)_{\bar{i}\lambda} = (bp_{\bar{i}\bar{i}}x^0p_{\bar{i}\bar{i}}a)_{i\lambda},$$

where $(b)_{i\bar{i}}$ denote $e_{(x)_{i\lambda}}$ and $(a)_{\bar{i}\lambda}$ denote $f_{(x)_{i\lambda}}$. This implies that $x = bp_{\bar{i}\bar{i}}x^0p_{\bar{i}\bar{i}}a$ by the unique property of $e_{(x)_{i\lambda}}$ and $f_{(x)_{i\lambda}}$, we know that $b = b_x \in B_I$ and $a = a_x \in A_\Lambda$, and so that for any $x \in T$, x can be uniquely written in the form $x = b_x p_{\bar{i}\bar{i}} x^0 p_{\bar{i}\bar{i}} a_x$, where $x^0 \in T^0, b_x \in B_I$ and $a_x \in A_\Lambda$ are uniquely determined by x and (i, λ). Thus T^0 is indeed a $\bar{\Gamma}$-adequate transversal of T.

(2) If S_μ^0 is multiplicative in part (1), then for any $f_{(x)_{i\lambda}} = (a_x)_{\bar{i}\lambda}$ and $e_{(y)_{j\mu}} = (b_y)_{j\bar{\lambda}}$, where $a_x \in A_\Lambda, b_y \in B_I$,

$$f_{(x)_{i\lambda}}e_{(y)_{j\mu}} = (a_x)_{\bar{i}\lambda}(b_y)_{j\bar{\lambda}} = (a_x p_{\lambda j} b_y)_{\bar{i}\bar{\lambda}} = \begin{cases} (a_x p_{\lambda j} b_y)_{\bar{i}\bar{i}} & \text{if } \bar{i} = \bar{\lambda} \\ 0 & \text{if } \bar{i} \neq \bar{\lambda} \end{cases} \in E(S_\mu^0).$$

So $a_x p_{\lambda j} b_y \in V(p_{\bar{i}\bar{i}}) \subseteq T^0$ (or $a_x p_{\lambda j} b_y = 0$) for any $a_x \in A_\Lambda, b_y \in B_I$ and $p_{\lambda j} \in \Gamma$, that is, T^0 is a Γ-multiplicative $\bar{\Gamma}$-adequate transversal of T. $\square$

According as the results of Lemmas 2.4 and 2.5 we obtain:

Theorem 2.6. (1) *The Γ-semigroup T contains a $\bar{\Gamma}$-adequate transversal T^0 as in Lemma 2.4 if and only if S_μ contains an adequate transversal $S_\mu^0 = \{(x^0)_{\bar{i}\bar{i}} \mid x^0 \in T^0, (\bar{i}, \bar{i}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}$ with the idempotents semilattice $E(S_\mu^0) = \{(a^0)_{\bar{i}\bar{i}} \mid a^0 \in CV(p_{\bar{i}\bar{i}}), (\bar{i}, \bar{i}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}} \text{ or } a^0 = 0\}$ for some subset $\bar{\mathbf{I}} \times \bar{\mathbf{I}} \subseteq I \times \Lambda$.*
  (2) *T^0 is Γ-multiplicative if and only if S_μ^0 is multiplicative where T^0 and S_μ^0 as that in part (1).*

For the relations between the semigroup S and the Rees matrix semigroup $S_\mu = \mu(T; I, \Lambda, P)$, we have:

Theorem 2.7. *If the mapping ϕ as in Lemma 1.5 is a semigroup isomorphism from S_μ to S, then the following arguments hold.*

(1) S is abundant if and only if and only if T is Γ-abundant if and only if S_μ is abundant.

(2) S contains an adequate transversal S^0 if and only if T contains a $\bar{\Gamma}$-adequate transversal T^0 where $\bar{\Gamma}$ as above required if and only if S_μ contains an adequate transversal $S_\mu^0 = \{(x^0)_{\bar{i}\bar{i}} \mid x^0 \in T^0, \bar{i} \in \bar{\mathbf{I}}\}$.

(3) S^0 is multiplicative if and only if T^0 is Γ-multiplicative if and only if S_μ^0 is multiplicative where S^0, T^0, S_μ^0 are as that in part (2).

Proof. (1) Since ϕ is an isomorphism from S_μ to S, by Lemma 1.5 and Theorem 1.9 we directly obtain part (1).

(2) Let S^0 be an adequate transversal of S. Since ϕ is an isomorphism from S_μ to S and for any $(t)_{i\lambda} \in S_\mu$, where $t \in T$, $(t)_{i\lambda}\phi = r_i t q_\lambda = x$. Let $x = e_x x^0 f_x$ so for $x^0 \in S^0$, $e_x, f_x \in E(S)$. There are a unique element $t^0 \in T^0$ which is a subset of T and two elements $b_t, a_t \in T$. Such that for some $(i, v), (j, \mu), (k, \lambda) \in I \times \Lambda$

$$(b_t)_{iv}\phi = e_x, \quad (t^0)_{j\mu}\phi = x^0, \quad (a_t)_{k\lambda}\phi = f_x.$$

Since $x = e_x x^0 f_x$ where $e_x \mathscr{L}^* x^0 \mathscr{R}^* f_x$ so that

$$x = e_x x^0 f_x = r_i b_t q_v \cdot r_j t^0 q_\mu \cdot r_k a_t q_\lambda = r_i b_t p_{vj} t^0 p_{uk} a_t q_\lambda.$$

Since T is a Γ semigroup, so $t = b_t p_{vj} t^0 p_{uk} a_t \in T$ and such that $(t)_{i\lambda}\phi = x = e_x x^0 f_x = (b_t)_{iv}\phi \cdot (t^0)_{j\mu}\phi \cdot (a_t)_{k\lambda}\phi = [(b_t)_{iv}(t^0)_{j\mu}(a_t)_{k\lambda}]\phi$. This implies that

$$(t)_{i\lambda} = (b_t)_{iv}(t^0)_{j\mu}(a_t)_{k\lambda}.$$

Further, by the fact that S^0 is an adequate transversal of S, like the proof of Lemma 2.5. We may prove that S_μ has an adequate transversal $S_\mu^0 = \{(t^0)_{\bar{i}\bar{i}} \mid t^0 \in T^0, (\bar{i}, \bar{i}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}$, where T^0 is an $\bar{\Gamma}$-adequate transversal of T and $\bar{\Gamma} = \{p_{\bar{i}\bar{\lambda}} \mid (\bar{i}, \bar{\lambda}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}$ as required in (2.1). Here we omit this proof.

Conversely, if T^0 is a $\bar{\Gamma}$-adequate transversal of T by Theorem 2.6, then S_μ^0 as above described is an adequate transversal of S_μ. Since ϕ is a semigroup isomorphism from S_μ to S, so the set

$$S^0 = \{r_{\bar{i}} t^0 q_{\bar{i}} = x^0 \in S \mid \forall\, t^0 \in T^0, (\bar{i}, \bar{i}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}$$

is necessarily an adequate transversal of S. In fact, let $(t)_{i\lambda}\phi = r_i t q_\lambda = x \in S$, by Theorem 2.6

$$(t)_{i\lambda} = (b_t)_{i\bar{i}}(t^0)_{\bar{i}\bar{i}}(a_t)_{\bar{i}\lambda} = (b_t p_{\bar{i}\bar{i}} t^0 p_{\bar{i}\bar{i}} a_t)_{i\lambda}$$

implies that $t = b_t p_{\bar{i}\bar{i}} t^0 p_{\bar{i}\bar{i}} a_t$ and

$$x = r_i t q_\lambda = r_i b_t p_{\bar{i}\bar{i}} t^0 p_{\bar{i}\bar{i}} a_t q_\lambda = (r_i b_t q_{\bar{i}})(r_{\bar{i}} t^0 q_{\bar{i}})(r_{\bar{i}} a_t q_\lambda) = e_x x^0 f_x.$$

Since $b_t \in B_I, a_t \in A_\Lambda$, so

$$e_x^2 = r_i b_t q_{\bar{i}} r_i b_t q_{\bar{i}} = r_i b_t p_{\bar{i}i} b_t q_{\bar{i}} = r_i b_t q_{\bar{i}} = e_x.$$

Similarly, $f_x^2 = f_x$. Since $(b_t)_{i\bar{i}} \mathscr{L}^*(t^0)_{\overline{ii}} \mathscr{R}^*(a_t)_{\bar{i}\lambda}$ and ϕ is an isomorphism by Lemma 1.5(2), so that $e_x \mathscr{L}^* x^0 \mathscr{R}^* f_x$.

(3) By part (2), similar to the proof Lemma 2.5, we may obtain the part (3). $\qquad\square$

In the next section we will prove existence of such a Γ-semigroup T in Theorem 2.7, and so we may give a Rees matrix representation of an abundant semigroup S with the adequate transversal S^0. The following is an example to show the application of Theorem 2.6.

Example 2.8. Consider the monoid T with zero a which is not abundant under multiplication

T	a	b	c	d
a	a	a	a	a
b	a	a	b	b
c	a	b	c	d
d	a	b	d	c

$$P = (p_{\lambda i})_{\Lambda \times I} = \begin{pmatrix} c & d \\ d & d \end{pmatrix}.$$

According to the structure express (T, I, Λ), here let $I = \{1, 2\} = \Lambda$, where $R_1^* = \{c, d\}, R_0^* = \{a\}, R_2^* = \{b\}$ and $L_1^* = \{c, d\}, L_0^* = \{a\}, L_2^* = \{b\}$. Therefore we have $H_{11}^* = \{c, d\}$ is a non-zero group with identity c. H_{22}^* contains no idempotent, $H_{i\lambda}^* = \phi$ $(i \neq \lambda, i, \lambda \in I = \{1, 2\})$. According (i, λ) positions, let $\Gamma = \{p_{\lambda i} \mid (i, \lambda) \in I \times \Lambda\}$ where $p_{\lambda i}$ is as in the above matrix P. Clearly, the matrix P is abundant. By Theorem 1.9 we know that T is Γ-abundant and so that $T_\mu = \mu(T; I, \Lambda, P)$ is also abundant. Computing by Lemma 1.3 and Corollary 1.4, here $L^*(1) = L_1^*, L^*(2) = L_2^*$. Similarly, $R^*(1) = R_1^*, R^*(2) = R_2^*$, and $V(p_{11}) = \{c\}, V(p_{22}) = V(p_{12}) = V(p_{21}) = \{d\}$. We will see that Γ-abundant semigroup T has a $\bar{\Gamma}$-adequate transversal. Put $\bar{\Gamma} = \{p_{11}\} \subsetneq \Gamma$ and $T^0 = T$, then $\bar{\Gamma}$ satisfies the conditions in Theorem 2.6, so by Theorem 2.6, T^0 is a $\bar{\Gamma}$-adequate transversal of T. In fact, for any $x \in T$ and $(i, \lambda) \in I \times \Lambda$, under the sense of Definition 2.1 according (i, λ) position x can be uniquely written in the form

$$
\begin{aligned}
a &= a p_{11} a p_{11} a & &\text{where } a \in A_\Lambda \cap B_I, \ a^0 = a \\
b &= c p_{11} b p_{11} c & &c \in V(p_{11}), \ c \in B_I \cap A_\Lambda, \ b^0 = b \\
&= d p_{11} b p_{11} d & &d \in V(p_{12}) = V(p_{21}), \ d \in B_I \cap A_\Lambda, \ b^0 = b \\
&= c p_{11} b p_{11} d & &c \in V(p_{11}), d \in V(p_{12}), c \in B_I, \ d \in A_\Lambda, \ b^0 = b \\
&= d p_{11} b p_{11} c & &d \in V(p_{21}), \ c \in V(p_{11}), \ d \in B_I, \ c \in A_\Lambda, \ b^0 = b \\
c &= c p_{11} c p_{11} c & &c \in V(p_{11}), \ c \in A_\Lambda \cap B_I, \ c^0 = c \\
&= d p_{11} c p_{11} d & &d \in V(p_{12}) = V(p_{21}), \ d \in A_\Lambda \cap B_I, \ c^0 = c \\
&= c p_{11} d p_{11} d & &c \in V(p_{11}), \ d \in V(P_{12}), \ c \in B_I, \ d \in A_\Lambda, \ c^0 = d \\
&= d p_{11} d p_{11} c & &d \in V(p_{21}), \ c \in V(P_{11}), \ d \in B_I, \ c \in A_\Lambda, \ c^0 = d \\
d &= c p_{11} d p_{11} c & &c \in B_I, \ d \in A_\Lambda, \ d^0 = d \\
&= d p_{11} d p_{11} d & &d \in B_I, \ c \in A_\Lambda, \ d^0 = d
\end{aligned}
$$

$$= dp_{11}cp_{11}c \quad d \in V(p_{21}), \ c \in V(P_{11}), \ d \in B_I, \ c \in A_\Lambda, \ d^0 = c$$
$$= dp_{11}dp_{11}c \quad d \in V(p_{21}), \ c \in V(P_{11}), \ d \in B_I, \ c \in A_\Lambda, \ c^0 = d.$$

By Theorem 2.6 we obtain an adequate transversal T_μ^0 of T_μ as follows

$$T_\mu^0 = \{(x^0)_{11} \mid x^0 \in T^0\}.$$

For any $(x)_{i\lambda} \in T_\mu$, $(x)_{i\lambda}$ can be uniquely written in the form $(x)_{i\lambda} = (b_x)_{i1}$ $(x^0)_{11} (a_x)_{1\lambda}$. Here we omit these expressions. But since there are $a_d = c$ and $b_d = d$ such that $a_d p_{11} b_d = c \cdot c \cdot d = d \notin V(p_{11})$, so T^0 is not Γ-multiplicative by Theorem 2.6 and we know that T_μ^0 is also not multiplicative.

Clearly, since $|T| = 4, |T_\mu| = 13$, so if there is a mapping ϕ as that in Lemma 1.5 then ϕ is impossible an isomorphism from T_μ to T. From this example we may see that for a semigroup T being not abundant, there possible is a set $\bar{\Gamma}$ such that T becomes a $\bar{\Gamma}$-adequate semigroup and such that the Rees matrix semigroup $T_\mu = \mu(T, I, \Lambda, P)$ over T may be an abundant semigroup and it may contain an adequate transversal T_μ^0.

3. Rees matrix representations of an abundant semigroup with an adequate transversal

In this section S is always an abundant semigroup with an adequate transversal S^0. Our aim in this section is to give some Rees matrix representations of S. We begin by blocked Rees matrix semigroups to give a representation of S.

Lemma 3.1. *Let* $0 \neq e \in E(S)$ *and* $a\mathscr{L}^*e$ $(a\mathscr{R}^*e)$. *Then* $a \in Se$ (eS).

Proof. By $a\mathscr{L}^*e$ implies $ae\mathscr{L}e$ and so $a = ae \in Se$. Dually, if $a\mathscr{R}^*e$, then $a \in eS$. $\quad\square$

Lemma 3.2. *Let* $0 \neq x, 0 \neq y \in S$. *If* $xy \neq 0$, *then* $xy \in R_x^* \bigcap L_y^*$.

Proof. Let $e, f \in E(S)$ with $e\mathscr{R}^*x, f\mathscr{L}^*y$, then $x \in eS, y \in Sf$ so that $xy \in eS \bigcap Sf$ and by Lemma 3.1 $xy \in R_e^* \bigcap R_f^* = R_x^* \bigcap R_y^*$. $\quad\square$

Lemma 3.3. *Let* $0 \neq x, 0 \neq y \in S$ *and* $e, f \in E(S)$ *with* $x\mathscr{R}^*e, y\mathscr{L}^*f$. *Then* $yx = 0$ *if and only if* $fe = 0$.

Proof. If $yx = 0$, then $yx = y0$, so that $fx = f0$ and then $fx = 0x$, which gives $fe = 0e = 0$. If $fe = 0$, then $yx = yfex = 0$ by $y = yf$ and $x = ex$. $\quad\square$

By the process of shaping a semigroup into the form of blocked Rees matrix semigroup in [6], we may obtain a blocked Rees matrix representation of S when the condition (M) in [6] holds. By Lemma 3.1 to 3.3 we may show this point. Now suppose that we have shaped S into a blocked set denoted by $S_\mu = \mu(M_{\alpha\beta}; I, \Lambda, \Gamma')$ where Γ' is the set of non-zero $\mathscr{D}$-classes of S which contain idempotents and each $M_{\alpha\beta}$ is a torsion-free (T_α, T_β)-bisystem where T_α, T_β are two cancellative monoid with an outer zero denoted by 0.

Let $\alpha, \beta, \gamma \in \Gamma'$. Suppose that $M_{\alpha\beta}, M_{\beta\gamma}$ are both non-empty. If $a \in M_{\alpha\beta}, b \in M_{\beta\gamma}$, then $a\mathscr{L}^*e_\beta\mathscr{R}^*b$ and it follows from Lemma 3.3 that $ab \neq 0$.

Since $ab \in aS \bigcap Sb \subseteq e_\alpha S \bigcap Se_\gamma$, we have $e_\alpha \mathscr{R}^* ab \mathscr{L}^* e_\gamma$ by Lemma 3.1. Thus $ab \in R^*_{i(\alpha)} \bigcap L^*_{\lambda(\gamma)} = M_{\alpha\gamma}$ and $M_{\alpha\gamma} \neq \phi$. We now define $\varphi_{\alpha\beta\gamma} : M_{\alpha\beta} \bigotimes M_{\beta\gamma} \to M_{\alpha\gamma}$ by $(a \otimes b)\varphi_{\alpha\beta\gamma} = ab$ (see [5, ch. 7]). It is easy to see that this is a well-defined (T_α, T_β)-homomorphism (see [6]) and that the condition (M) in [6] is satisfied.

For each $\alpha \in \Gamma'$ we define the sets as

$$I_\alpha = \{i \in I; \ D_\alpha \bigcap R^*_i \neq \phi\}; \ \Lambda_\alpha = \{\mu \in \Lambda; \ D_\alpha \bigcap L^*_\lambda \neq \phi\}.$$

Since S is an abundant semigroup, so by [6] we know that $I = \bigcup I_\alpha$ and $\Lambda = \bigcup \Lambda_\beta$ are disjoint union. Like [6], we define P as the $\Lambda \times I$ matrix $(p_{\lambda j})$ where for $(\lambda, j) \in \Lambda_\alpha \times I_\beta, p_{\lambda j} = q^\alpha_\lambda r^\beta_j, q^\alpha_\lambda \in H^*_{i(\alpha),\lambda}$ and $r^\beta_j \in H^*_{j,\lambda(\beta)}$ where $q^\alpha_\lambda, r^\beta_j$ are regular elements and $q^\alpha_\lambda r^\alpha_j = r^\alpha_j q^\alpha_\lambda = e_\alpha$ (e_α is the identity of $M_{\alpha\alpha}$). So that $q^\alpha_\lambda \in R^*_{i(\alpha)}, \ r^\beta_j \in L^*_{\lambda(\beta)}$ and hence either $q^\alpha_\lambda r^\beta_j = 0$ or $q^\alpha_\lambda r^\beta_j \in R^*_{i(\alpha)} \bigcap L^*_{\lambda(\beta)} = H^*_{i(\alpha),\lambda(\beta)} = M_{\alpha\beta}$. Thus any non-zero entry in the (α, β)-block of P is a member of $M_{\alpha\beta}$. By [6] we know that $S_\mu = \mu^0(M_{\alpha\beta}; I, \Lambda, \Gamma'; P)$ (see [6]) is a blocked Rees matrix semigroup.

Note, here $S_\mu = \mu^0(M_{\alpha\beta}; I, \Lambda, \Gamma'; P)$ is not necessarily a PA blocked Rees matrix semigroup, that is, the abundant semigroup S_μ need not to satisfy the conditions (U) and (R) in [6], so S_μ is not necessarily a primitive abundant semigroup.

We next show that the bijection $\phi : S_\mu \to S$ given by

$$(3.1) \qquad 0\phi = 0 \text{ and } (a)_{i\lambda}\phi = r^\alpha_i a q^\beta_\lambda ((i, \lambda) \in I_\alpha \times \Lambda_\beta, a \in M_{\alpha\beta})$$

is an isomorphism from S_μ to S. Clearly, $S \backslash \{0\} = \bigcup \{H^*_{i\lambda} \mid (i, \lambda) \in I \times \Lambda\}$ and S is disjoint unions, so it is straightforward to show that ϕ is a bijection and it is also an isomorphism. Thus we have already proved that S_μ is a blocked Rees matrix representation of S. It is such that we may obtain the following representation theorem. It is a generalization of [6, Theorem 3.8].

Theorem 3.4. *Let S be an abundant semigroup with an adequate transversal S^0 then S has a blocked Rees matrix representation $S_\mu = \mu^0(M_{\alpha\beta}; I, \Lambda, \Gamma'; P)$ with an adequate transversal S^0_μ isomorphic to S^0. Furthermore, S^0 is multiplicative if and only if S^0_μ is multiplicative.*

Proof. We first show that S_μ has adequate transversal S^0_μ isomorphic to S^0. For any $x^0 \in S^0$, since ϕ is an isomorphism from S_μ to S, so there is a unique element denoted by $(t_{x^0})_{i\lambda} \in S_\mu$ such that $(t_{x^0})_{i\lambda}\phi = r^\alpha_i t_{x^0} q^\beta_\lambda = x^0$ for $(i, \lambda) \in I_\alpha \times \Lambda_\beta$ and $t_{x^0} \in M_{\alpha\beta}$ for some $\alpha, \beta \in \Gamma'$. Similar to the proof of Theorem 2.7, there is a subset $\bar{\mathbf{I}} \times \bar{\mathbf{I}}$ of $I \times \Lambda$ such that

$$S^0_\mu = \{(t_{x^0})_{\bar{i}\bar{\lambda}} \mid \forall x^0 \in S^0 \ (t_{x^0})_{\bar{i}\bar{\lambda}}\phi = x^0, (\bar{i}, \bar{\lambda}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}.$$

Since ϕ is an isomorphism, S^0_μ is an adequate $*$-subsemigroup of S_μ, and for any $x \in S$ there are a unique element $x^0 \in S^0$ and two idempotents e_x, f_x in

$E(S)$ such that $x = e_x x^0 f_x$, where $e_x \mathscr{L}^* x^0 \mathscr{R}^* f_x$. We denote $x\varphi^{-1}, e_x\phi^{-1}$ and $f_x\phi^{-1}$ by $(t_x)_{i\lambda}, e_{(t_x)_{i\lambda}}$ and $f_{(t_x)_{i\lambda}}$ respectively, then

$$e_{(t_x)_{i\lambda}}, f_{(t_x)_{i\lambda}} \in E(S_\mu) \text{ and } e_{(t_x)_{i\lambda}} \mathscr{R}^* (t_x)_{i\lambda} \mathscr{L}^* f_{(t_x)_{i\lambda}}.$$

The element $(t_x)_{i\lambda}$ of S_μ can be uniquely written in the form

$$(t_x)_{i\lambda} = e_{(t_x)_{i\lambda}} (t_x)_{i\lambda}^0 f_{(t_x)_{i\lambda}} = e_{(t_x)_{i\lambda}} (t_{x^0})_{\bar{i}\bar{\lambda}} f_{(t_x)_{i\lambda}},$$

where $e_{(t_x)_{i\lambda}} \mathscr{L}^* (t_{x^0})_{\bar{i}\bar{\lambda}} \mathscr{R}^* f_{(t_x)_{i\lambda}}$. Like Theorem 2.7 we may write $e_{(t_x)_{i\lambda}} = (b_{t_x})_{i\bar{i}}, f_{(t_x)_{i\lambda}} = (a_{t_x})_{\bar{i}\lambda}$, then $(t_x)_{i\lambda} = (b_{t_x})_{i\bar{i}} (t_{x^0})_{\bar{i}\bar{i}} (a_{t_x})_{\bar{i}\lambda}$. Since $(t_x)_{i\lambda}$ is any element of S_μ, we know that S_μ^0 is an adequate transversal of S_μ. We next show that S^0 is multiplicative if and only if S_μ^0 is also. We denote $y\phi^{-1} = (b_y)_{j\mu}$ for $y \in S$ and $e_y\phi^{-1} = e_{(b_y)_{j\mu}}$, then by ϕ being an isomorphism, $f_x e_y \in E(S^0)$, if and only if $f_{(a_x)_{i\lambda}} e_{(b_y)_{j\mu}} \in E(S_\mu^0)$. It is as required. $\qquad\square$

The blocked Rees matrix semigroup is over (T_α, T_β)-bisystem $M_{\alpha\beta}$ $(\alpha, \beta \in \Gamma')$ where $T_\alpha = M_{\alpha\alpha}, T_\beta = M_{\beta\beta}$ are two cancellative monoid with an outer zero. Using $M_{\alpha\beta} = H_{i(\alpha),\lambda(\beta)}^*$ $(\alpha, \beta \in \Gamma')$, we define the set $T = \bigcup\{M_{\alpha\beta} \mid \alpha, \beta \in \Gamma'\}$. Thus we may think that the blocked Rees matrix semigroup over T. Using here expression that is $S_\mu = \mu(T, I, \Lambda, P)$, where $I \times \Lambda = \bigcup\{I_\alpha \times I_\beta \mid \alpha, \beta \in \Gamma'\}, T = \bigcup\{M_{\alpha\beta} \mid \alpha, \beta \in \Gamma'\}$. But T is not necessarily a semigroup. When $p_{\lambda j} = q_\lambda^\alpha r_j^\beta$, where $q_\lambda^\alpha \in H_{i(\alpha),\lambda}^*$ and $r_j^\beta \in H_{j,\lambda(\beta)}^*$, S_μ is isomorphic to S.

It is clear that S and T are Γ-semigroups, where

(3.2) $$\begin{aligned} \Gamma = \{p_{\lambda j}; \ p_{\lambda j} = q_\lambda^\alpha r_j^\beta \in M_{\alpha\beta}, \ q_\lambda^\alpha \in H_{i(\alpha),\ \lambda}^*, \\ r_j^\beta \in H_{j,\lambda(\beta)}^*, \ \alpha, \beta \in \Gamma', \ (j, \lambda) \in I_\beta \times \Lambda_\alpha\}. \end{aligned}$$

under Γ-operation " $\circ$ " as that for any $x, y \in S, \ p_{\lambda i} \in \Gamma$

(3.3) $$x \circ y = x p_{\lambda i} y = \begin{cases} x p_{\lambda i} y & \text{if } x \in M_{\alpha\beta}, \ y \in M_{\gamma\delta}, p_{\lambda i} \in M_{\beta\gamma}, \\ 0 & \text{otherwise.} \end{cases}$$

So we say that $S_\mu = \mu(T, I, \Lambda, P)$ is a Rees matrix semigroup over Γ-semigroup T. Like Theorem 2.7 we can prove the following representation theorem.

Theorem 3.5. *Let S be an abundant semigroup with an adequate transversal S^0. Then S has a Rees matrix representation $S_\mu = \mu(T; I, \Lambda, P)$ over Γ-semigroup T and the following argument hold.*

(1) *S_μ contains an adequate transversal S_μ^0 may be expressed by*

$$S_\mu^0 = \{(x^0)_{\bar{i}\bar{\lambda}} | x^0 \in T^0, (\bar{i}, \bar{\lambda}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}$$

and S_μ^0 is isomorphic to S^0 where T^0 is a $\bar{\Gamma}$-adequate transversal of T and $\bar{\Gamma}$ as (2.1).

(2) *S^0 is multiplicative if and only if S_μ^0 is multiplicative.*

Proof. We have proved that $S_\mu = \mu(T; I, \Lambda, P)$ is a Rees matrix representation of S. By the proof of Lemma 2.4, we can similarly obtain that S_μ contains an adequate transversal denoted by S_μ^0, and S_μ^0 is isomorphic to S^0. The following we show that S_μ^0 may be expressed as required form. In fact, let $T^0 = \{t^0 \in T \mid \forall x^0 \in S^0, \exists \mid t^0 \in T, \ni x^0 = r_{\bar{i}}^\alpha t^0 q_{\bar{\lambda}}^\beta\}$. Since S^0 is a subsemigroup of S, let $x^0 = r_{\bar{i}}^\alpha t_x^0 q_{\bar{\lambda}}^\beta$, $y^0 = r_{\bar{j}}^\gamma t_y^0 q_{\bar{\mu}}^v \in S^0$, where $(\bar{i}, \bar{\lambda}), (\bar{j}, \bar{\mu}) \in \bar{\mathbf{I}} \times \bar{\mathbf{\Lambda}} \subseteq I \times \Lambda$, then $x^0 y^0 = r_{\bar{i}}^\alpha t_x^0 p_{\bar{\lambda}\bar{j}} t_y^0 q_{\bar{\mu}}^v$ implies that $t_x^0 p_{\bar{\lambda}\bar{j}} t_y^0 \in T^0$. Let $\bar{\mathbf{\Gamma}} = \{p_{\bar{i}\bar{\lambda}} \in \Gamma \mid (\bar{i}, \bar{\lambda}) \in \bar{\mathbf{I}} \times \bar{\mathbf{\Lambda}}\}$, since S^0 is a subsemigroup of S, so T^0 is necessarily a $\bar{\mathbf{\Gamma}}$-subsemigroup of T. Since S^0 is adequate by Theorem 1.10, T^0 is $\bar{\mathbf{\Gamma}}$-adequate and $\bar{\mathbf{I}} = \bar{\mathbf{\Lambda}}$. By Theorem 2.7, T^0 is a $\bar{\mathbf{\Gamma}}$-adequate transversal of T and S_μ^0 may be denoted by

$$S_\mu^0 = \{(t^0)_{\bar{i}\bar{i}} \mid t^0 \in T^0, (\bar{i}, \bar{\lambda}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}$$

which is an adequate transversal of S_μ. Since $\phi|_{S_\mu^0}$ as

$$(\forall (t^0)_{\bar{i}\bar{\lambda}} \in S_\mu^0) \ (t^0)_{\bar{i}\bar{\lambda}}\phi = r_{\bar{i}}^\alpha t^0 q_{\bar{\lambda}}^\beta = x^0 \in S^0$$

is an isomorphism from S_μ^0 to S^0. We complete the proof of part (1). By Theorem 2.7 we know that part (2) holds. $\square$

Our final aim in this section is that given a Γ-Rees matrix representation for semigroup S. This means that taking a some set Γ_1 need not be belong to S and a $\Lambda \times I$ matrix $\rho = (\rho_{\lambda i})_{\Lambda \times I}$ over Γ_1 which is called a Γ-Rees matrix, we can obtain a Γ_1-semigroup T for some subset T of S and the Γ_1-semigroup S. Then taking some set Γ_2 such that we can obtain a Γ_2-Rees matrix semigroup T_μ denoted by $T_\mu = \mu(T; I, \Lambda, \rho)$ over T. Then we will prove that Γ_2-semigroup T_μ is Γ-isomorphic to Γ_1-semigroup S. Since the set $T \subseteq S$ and ρ is over Γ_1, so we call that T_μ is a Γ-Rees matrix representation of S.

Firstly, we recall the concept of Γ-semigroup isomorphism.

Definition 3.6 ([11])**.** Let T_1 be a Γ_1-semigroup and T_2 be a Γ_2-semigroup, a mapping pair denoted by $\phi = (\phi_1, \phi_2)$ from (T_1, Γ_1) to (T_2, Γ_2) as follows

$$\phi_1 : \ T_1 \to T_2 \ , \ \phi_2 : \ \Gamma_1 \to \Gamma_2$$
$$x_1 \mapsto x_2 \ \ \gamma_1 \mapsto \gamma_2$$

If ϕ satisfies that for any $x_1, y_1 \in T_1, \gamma_1 \in \Gamma_1$

$$(x_1 \gamma_1 y_1)\phi = x_1\phi_1 \cdot \gamma_1\phi_2 \cdot y_1\phi_1 = x_2\gamma_2 y_2,$$

then ϕ is called a (Γ_1, Γ_2) homomorphism from T_1 to T_2. If ϕ is a surjection (resp. injection), then ϕ is called a surjection (resp. injection) homomorphism. If ϕ is a bijection, then ϕ is called a (Γ_1, Γ_2)-isomorphism from Γ_1-semigroup T_1 to Γ_2-semigroup T_2.

Note here ϕ is bijective (surjective, injective) means that ϕ_1 and ϕ_2 are bijective (surjective, injective).

Now, we suppose that S is an abundant semigroup with an adequate transversal S^0, then S has the structure express $(S; I, \Lambda, \Gamma')$ as Section 1. We denote

the bijection $x \mapsto xq_\lambda^\beta$ from $H_{i(\alpha),\lambda(\beta)}^*$ onto $H_{i(\alpha),\lambda}^*$ by $\rho_{\lambda(\beta)}$ and denote the bijection $y \mapsto r_i^\alpha y$ from $H_{i(\alpha),\lambda}^*$ onto $H_{i\lambda}^*$ by $\rho_{i(\alpha)}$. Then we obtain a bijection denoted by $\rho_{\lambda(\beta),i(\alpha)} = \rho_{\lambda(\beta)}\rho_{i(\alpha)} : x \mapsto r_i^\alpha xq_\lambda^\beta$ from $H_{i(\alpha)\lambda(\beta)}^*$ onto $H_{i\lambda}^*$. Let $\Gamma_1 = \{\rho_{\lambda(\beta)i(\alpha)} \mid \forall(i,\lambda) \in I \times \Lambda, \alpha,\beta \in \Gamma'\}$, and the Γ-Rees matrix ρ denoted by $\rho = (\rho_{\lambda(\beta)i(\alpha)})_{\Lambda \times I}$. Clearly, Γ_1 is does not belong to S and the matrix ρ is not over S. We denote the bijection $\rho_{i(\alpha)}$ by $x\rho_{\lambda(\beta)}$ for $x \in H_{i(\alpha),\lambda}^*$ and denote the bijection $\rho_{i(\alpha)}$ by $\rho_{i(\alpha)}y$ for $y \in H_{i(\alpha),\lambda}^*$. Then we define an Γ_1-operation "$\circ$" on S as that for any $x, y \in S$, and $\rho_{\lambda(\beta)i(\alpha)} \in \Gamma_1$

$$(3.4) \quad x \circ y = x\rho_{\lambda(\beta)i(\alpha)}y = \begin{cases} x\rho_{\lambda(\beta)}\rho_{i(\alpha)}y & \text{if } x \in H_{i(\alpha),\lambda(\beta)}^*,\ y \in H_{i(\alpha)\lambda}^*, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the Γ_1-operation "$\circ$" satisfies associative law. Thus we obtain a Γ_1-semigroup S.

Let the set $T = \bigcup\{M_{\alpha\beta};\ \alpha,\beta \in \Gamma'\}$. Then T is also a Γ_1-semigroup under the above multiplication (3.4). Let the set T_μ denoted by

$$T_\mu = \mu(T; I, \Lambda, \rho) = \{(x)_{i\lambda} \mid x \in T, (i,\lambda) \in I \times \Lambda\},$$

with multiplication for $(i(\alpha), \lambda(\beta)), (j(\gamma), \mu(v)) \in I \times \Lambda$

$$(3.5) \qquad (x)_{i\lambda}\eta_{ut}(y)_{j\mu} = \begin{cases} (xq_\lambda^\beta \rho_{\lambda(\beta),j(\gamma)} r_j^\gamma y)_{i\mu} & \text{if } (u,t) = (\lambda, j), \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to check that T_μ becomes a Γ_2-semigroup, where $\Gamma_2 = \{\eta_{\lambda i} \mid (i,\lambda) \in I \times \Lambda\}$. We call T_μ a Γ Rees matrix semigroup. If T_μ is Γ-isomorphic to S, we call T_μ is Γ Rees matrix representation of S.

Finally, we define a mapping $\phi = (\phi_1, \phi_2)$ from Γ_2-semigroup T_μ to Γ_1-semigroup S as belows

$$(3.6) \qquad \begin{aligned} \phi_1 : \ &T_\mu \to S, \quad \phi_2 : \ \Gamma_2 \to \Gamma_1 \\ (x)_{i\lambda} &\mapsto r_i^\alpha \rho_{\lambda(\alpha)i(\alpha)} x\rho_{\lambda(\beta)i(\beta)} q_\lambda^\beta \quad \eta_{\lambda i} \mapsto \rho_{\lambda i} \end{aligned}$$

Clearly, ϕ_2 is bijective. Since $r_i^\alpha \rho_{\lambda(\alpha)i(\alpha)} x\rho_{\lambda(\beta)i(\beta)} q_\lambda^\beta = r_i^\alpha q_\lambda^\alpha r_i^\alpha xq_\lambda^\beta r_i^\beta q_\lambda^\beta = r_i^\alpha xq_\lambda^\beta$ on semigroup S, by Theorem 3.5 we know that ϕ_1 is a semigroup isomorphism from $\mu(T; I, \Lambda, P)$ to S. Thus we know that ϕ is a bijection from Γ_2-semigroup T_μ to Γ_1-semigroup S. Let $(x)_{i(\alpha),\lambda(\beta)}, (y)_{j(\gamma)\mu(v)} \in T_\mu, \eta_{ut} \in \Gamma_2$ then

$$\begin{aligned} [(x)_{i\lambda}\ \eta_{ut}\ (y)_{j\mu}]\phi &= \begin{cases} (xq_\lambda^\beta \rho_{\lambda(\beta),j(\gamma)} r_j^\gamma y)_{i\mu}\phi & \text{if } (u,t) = (\lambda, j) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} r_i^\alpha \rho_{\mu(\alpha)i(\alpha)} xq_\lambda^\beta \rho_{\lambda j} r_j^\gamma y\rho_{\mu(v)i(v)} q_\mu^v & \text{if } (u,t) = (\lambda, j) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} r_i^\alpha \rho_{\mu(\alpha)}\rho_{i(\alpha)} xq_\lambda^\beta \rho_{\lambda j} r_j^\gamma y\rho_{\mu(v)}\rho_{i(v)} q_\mu^v & \text{if } (u,t) = (\lambda, j) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$= \begin{cases} r_i^\alpha q_u^\alpha r_i^\alpha x q_\lambda^\beta \rho_{\lambda j} r_j^\gamma y q_\mu^v r_i^v q_\mu^v & \text{if } (u,t) = (\lambda, j) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (r_i^\alpha x q_\lambda^\beta) \rho_{\lambda(\beta), j(\gamma)} (r_j^\gamma y q_\mu^v) & \text{if } (u,t) = (\lambda, j) \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$(x)_{i\lambda} \phi \rho_{ut}(y)_{ju} \phi = \begin{cases} (r_i^\alpha x q_\lambda^\beta) \rho_{\lambda(\beta), j(\gamma)} (r_j^\gamma y q_\mu^v) & \text{if } (u,t) = (\lambda, j), \\ 0 & \text{otherwise.} \end{cases}$$

Thus we obtain that for any $(u,t) \in \Lambda \times I$,

$$[(x)_{i\lambda} \ \eta_{ut}(y)_{j\mu}]\phi = (x)_{i\lambda}\phi\rho_{ut}(y)_{j\mu}\phi.$$

By Definition 3.6, we know that $\phi = (\phi_1, \phi_2)$ is a (Γ_1, Γ_2)-isomorphism from Γ_2-semigroup T_μ to Γ_1-semigroup S, that is, we have:

Theorem 3.7. *Let S be an abundant semigroup with zero, then S is Γ- isomorphic to a Γ-Rees matrix semigroup $T_\mu = \mu(T; I, \Lambda, \rho)$ where $T = \bigcup\{M_{\alpha\beta} \mid \alpha, \beta \in \Gamma'\}$ is a Γ-semigroup and ρ is a Γ-Rees matrix over Γ_1, that is, any abundant semigroup S has a Γ-Rees matrix representation.*

Furthermore, we can prove the following result.

Theorem 3.8. *Let S be an abundant semigroup with zero, if S^0 is an adequate transversal of S, then S is Γ-isomorphic to a Γ-Rees matrix semigroup $T_\mu = \mu(T; I, \Lambda, P)$ with a Γ-adequate transversal T_μ^0 Γ-isomorphic to S^0. Further, T_μ^0 is Γ-multiplicative if and only if S^0 is Γ-multiplicative. That is, any abundant semigroup S with an adequate transversal S^0 has a Γ-Rees matrix representation T_μ such that T_μ has a $\bar{\Gamma}$-adequate transversal T_μ^0 Γ-isomorphic to S^0.*

Proof. By Theorem 3.7, we know that there is a Γ_2-Rees matrix semigroup T_μ such that T_μ is Γ-isomorphic to S by ϕ. Let $T^0 = \{t^0 \in T \mid \forall x^0 \in S^0 \ \exists | \ t^0 \in T, \ r_{\bar{i}}^\alpha, q_{\bar{\lambda}}^\beta \in S, \ni x^0 = r_{\bar{i}}^\alpha t^0 q_{\bar{\lambda}}^\beta\}$. Similar to the proof of Theorem 3.5, T^0 is a $\bar{\Gamma}_1$-subsemigroup of T, under the multiplication (3.4) for some subset

$$(3.7) \qquad \bar{\Gamma}_1 = \left\{ \rho_{\bar{i}\bar{\lambda}} = \begin{cases} \rho_{\bar{i}\bar{\lambda}} & \bar{i} = \bar{\lambda} \\ 0 & \bar{i} \neq \bar{\lambda} \end{cases} \ \forall \bar{i} \in \bar{\mathbf{I}} \ \exists | a \in T, a\rho_{\bar{i}\bar{i}}a = a, \bar{\mathbf{I}} = \bar{\Lambda} \right\}.$$

Further, we put the subset of T_μ as follows

$$T_\mu^0 = \{(t_{\bar{i}\bar{\lambda}}^0) \mid t^0 \in T^0, (\bar{i}, \bar{\lambda}) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}.$$

Since $\phi = (\phi_1, \phi_2)$ given by (3.6) is a Γ-isomorphism from Γ_2-semigroup T_μ to Γ_1-semigroup S, consider $\phi|_{T_\mu^0}$ as belows for any $(t^0)_{\bar{i}\bar{\lambda}} \in T_\mu^0$

$$(t^0)_{\bar{i}\bar{\lambda}} \phi_1 = r_{\bar{i}}^\alpha \rho_{\bar{\lambda}(\alpha), \bar{i}(\alpha)} t^0 \rho_{\bar{\lambda}(\beta), \bar{i}(\beta)} q_{\bar{\lambda}}^\beta (= r_{\bar{i}}^\alpha t^0 q_{\bar{\lambda}}^\beta) \in S^0,$$
$$\eta_{\bar{\lambda}\bar{i}} \phi_2 = \rho_{\bar{\lambda}\bar{i}}.$$

Thus $\phi|_{T_\mu^0}$ is a Γ-isomorphism from $\bar{\pmb{\Gamma}}_{\pmb{2}} = \{\eta_{\bar\lambda\bar i} \in \Gamma_2 \mid (\bar i, \bar\lambda) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}\}$-semigroup T_μ^0 to $\bar{\pmb{\Gamma}}_{\pmb{1}}$-semigroup S^0. Now for any $(t)_{i\lambda} \in T_\mu$ where $t \in T$, $(i, \lambda) \in I \times \Lambda$, let $t = e_t t^0 f_t$ by S^0 is an adequate transversal of S. Then

$$
\begin{aligned}
(t)_{i\lambda}\phi &= r_i^\alpha \rho_{\lambda(\alpha)i(\alpha)} t \rho_{\lambda(\beta)i(\beta)} q_\lambda^\beta \\
&= r_i^\alpha t q_\lambda^\beta \\
&= r_i^\alpha e_t t^0 f_t q_\lambda^\beta \\
&= r_i^\alpha e_t q_{\bar i} r_{\bar i} t^0 q_{\bar i} r_{\bar i} f_t q_\lambda^\beta \ (\because q_{\bar i} r_{\bar i} \text{ is the identity of } T_{\bar\alpha}) \\
&= (r_i^\alpha e_t q_{\bar i}) \rho_{\overline{ii}} (r_{\bar i} t^0 q_{\bar i}) \rho_{\overline{ii}} (r_i f_e q_\lambda^\beta) \\
&= (e_t)_{i\bar i}\phi \rho_{\overline{ii}} (t^0)_{\overline{ii}}\phi \rho_{\overline{ii}} (f_t)_{\bar i\lambda}\phi \ (\text{by } (3.4)) \\
&= [(e_t)_{i\bar i}\eta_{\overline{ii}} (t^0)_{\overline{ii}}\eta_{\overline{ii}} (f_t)_{\bar i\lambda}]\phi \ (\text{by } \Gamma \text{ isomorphism } \phi)
\end{aligned}
$$

for some $(\bar i, \bar i) \in \bar{\mathbf{I}} \times \bar{\mathbf{I}}$. So we have that

$$
(t)_{i\lambda} = (e_t)_{i\bar i}\eta_{\overline{ii}} (t^0)_{\overline{ii}}\eta_{\overline{ii}} (f_t)_{\bar i\lambda},
$$

where $(t^0)_{\overline{ii}} \in T_\mu^0$ and $(e_t)_{i\bar i}\mathscr{L}_{\Gamma_{\bar i}}^* (t^0)_{\overline{ii}}\mathscr{R}_{\Gamma_{\bar i}}^* (f_t)_{\bar i\lambda}$ by Lemma 1.3. Thus by Definition 2.1 and Definition 3.6 we know that T_μ^0 is an $\bar{\pmb{\Gamma}}$adequate transversal of T_μ and T_μ^0 is Γ-isomorphic to S^0. The remanent proofs are omitted. We complete the proof of this theorem. $\qquad\square$

Finally, we use an example to conclude this note, at the same time to illustrate the application of Theorem 3.8.

Example 3.9. Let M be a regular idempotent generated semigroup with zero and having a multiplicative semilattice transversal $M^0 = \{a, e\}$. M is not orthodox with Caley table as below.

M	a	b	c	d	e
a	a	b	a	b	e
b	a	b	e	e	e
c	c	d	c	d	e
d	c	d	e	e	e
e	e	e	e	e	e

$L_1 = \{a, c\}$, $L_2 = \{b, d\}$

$R_1 = \{a, b\}$, $R_2 = \{c, d\}$

$H_{11} = \{a\}$, $H_{12} = \{b\}$

$H_{21} = \{c\}$, $H_{22} = \{d\}$ $(H_{22}^2 = \{e\})$.

Under structure express $(M, I, \Lambda,)$ of M, computing we let the mappings

$$\rho_\lambda : \ H_{11} \to H_{1\lambda}, \ x\rho_\lambda = xq_\lambda, \text{ where } q_\lambda = \begin{cases} a & \lambda = 1 \\ b & \lambda = 2 \end{cases}$$

$$\rho_i : \ H_{1\lambda} \to H_{i\lambda}, \ \rho_i y = r_i y, \text{ where } r_\lambda = \begin{cases} a & i = 1 \\ c & i = 2. \end{cases}$$

Further, let $\rho_{\lambda i} = \rho_\lambda \rho_i$ for $(i, \lambda) \in I \times \Lambda$ and the Γ_1-Rees matrix $\rho = (\rho_{\lambda i})_{\Lambda \times I}$, where $\Gamma_1 = \{\rho_{\lambda i} \mid (i, \lambda) \in I \times \Lambda\}$. Let $\Gamma_2 = \{\eta_{\lambda i} \mid (i, \lambda) \in I \times \Lambda\}$ where $\eta_{\lambda i}$ as required in (3.5). Then M becomes a Γ_1-semigroup under the multiplication (3.4), $M_\mu = \mu(H_{11}, I, \Lambda, \rho)$ becomes a Γ_2-semigroup under the multiplication

(3.5). By Theorem 3.8, M is Γ-isomorphic to the Γ_2-Rees matrix semigroup M_μ over Γ_1.

Nextly. Let $N \bigcup \{0\}$ be cancellative monoid of natural number with an outer zero under multiplication. Using the above set $I \times \Lambda$ take the matrix $\theta = (1)_{\Lambda \times I}$. It is easy to show that the Rees semigroup $K = \mu(N, I, \Lambda, \theta)$ is an abundant semigroup with a multiplicative transversal $K^0 = \{(n)_{11} \mid \forall n \in N \bigcup \{0\}\}$. We denote the element of K by $x_{i\lambda}$ for $(i, \lambda) \in I \times \Lambda$. Computing we have that $E(K) = \{1_{11}, 1_{12}, 1_{21}, 1_{22}\} \bigcup \{0\}$ and the mappings

$$\rho'_\lambda : H^*_{11} \to H^*_{1\lambda}, \ x_{11}\rho'_\lambda = x_{11}1_{1\lambda} = x_{1\lambda}, \text{ where } q'_\lambda = \begin{cases} 1_{11} & \lambda = 1 \\ 1_{12} & \lambda = 2 \end{cases} ;$$

$$\rho'_i : H^*_{1\lambda} \to H^*_{i\lambda}, \ \rho'_i y_{1\lambda} = 1_{i1}y_{1\lambda} = y_{i\lambda}, \text{ where } r'_\lambda = \begin{cases} 1_{11} & i = 1 \\ 1_{21} & i = 2. \end{cases}$$

Let $\rho'_{\lambda i} = \rho'_\lambda \rho'_i$ for $(i, \lambda) \in I \times \Lambda$ and the Γ'_1-Rees matrix $\rho' = (\rho'_{\lambda i})_{\Lambda \times I}$, where $\Gamma'_1 = \{\rho'_{\lambda i} \mid (i, \lambda) \in I \times \Lambda\}$. Let $\Gamma'_2 = \{\eta'_{\lambda i} \mid (i, \lambda) \in I \times \Lambda\}$, where $\eta'_{\lambda i}$ as required in (3.5). Then K becomes a Γ'_1-semigroup under the multiplication (3.4), $K_\mu = \mu(H^*_{11}, I, \Lambda, \rho')$ becomes a Γ'_2-semigroup under the multiplication (3.5). Similar to M, by Theorem 3.8, K is Γ-isomorphic to the Γ'_2-Rees matrix semigroup K_μ over Γ'_1.

Let $S = M \times K, S^0 = M^0 \times K^0$ be two direct product sets. Let $\Gamma^*_1 = (\Gamma_1, \Gamma'_1) = \{(\rho_{\lambda i}, \rho'_{\lambda i}) \mid (i, \lambda) \in I \times \Lambda\}$ and $\Gamma^*_2 = (\Gamma_2, \Gamma'_2) = \{(\eta_{\lambda i}, \eta'_{\lambda i}) \mid (i, \lambda) \in I \times \Lambda\}$. Under the multiplications of (3.4) and (3.5), we define the following multiplications of S and $M_\mu \times K_\mu$.

$$(\forall \ (x, n_{i\lambda}), (y, m_{j\mu}) \in S) \ (x, n_{i\lambda}) \circ (y, m_{j\mu}) = (x, n_{i\lambda})(\rho_{ut}, \rho'_{ut})(y, m_{j\mu})$$
$$= (x\rho_{ut}y, n_{i\lambda}\rho'_{ut}m_{j\mu});$$

$$(\forall \ ((x)_{i\lambda}, (n_{ut})_{i\lambda}), ((y)_{j\mu}, (m_{vk})_{j\mu}) \in M_\mu \times K_\mu$$

$$((x)_{i\lambda}, (n_{ut})_{i\lambda}) \circ ((y)_{j\mu}, (m_{vk})_{j\mu}) = ((x)_{i\lambda}, (n_{ut})_{i\lambda})(\eta, \eta')((y)_{j\mu}, (m_{vk})_{j\mu})$$
$$= ((x)_{i\lambda}\eta(y)_{j\mu}), (n_{ut})_{i\lambda}\eta'(m_{vk})_{j\mu}),$$

where (η, η') denote some $(\eta_{\lambda i}, \eta'_{\lambda i})$ for $(i, \lambda) \in I \times \Lambda$. Thus $S = M \times K$ becomes a Γ^*_1-semigroup and $M_\mu \times K_\mu$ becomes a Γ^*_2-semigroup. Further, using $\rho_{\lambda i}, \rho'_{\lambda i}$ we define the following bijection $\phi = (\phi_1, \phi_2)$

$$\phi_1 : M_\mu \times K_\mu \ \to \ S, \ ((x)_{i\lambda}, (n_{ut})_{i\lambda}) \mapsto (r_i\rho_{\lambda i}x\rho_{\lambda i}q_\lambda, r'_i\rho'_{\lambda i}n_{ut}\rho'_{\lambda i}q'_\lambda),$$
$$\phi_2 : \ \Gamma^*_2 \to \Gamma^*_1, \qquad (\eta_{\lambda i}, \eta'_{\lambda i}) \mapsto (\rho_{\lambda i}, \rho'_{\lambda i}).$$

It is easy to check that ϕ is a Γ-isomorphism from Γ^*_2-semigroup $M_\mu \times K_\mu$ to Γ^*_1-semigroup S. By Theorem 3.8 we know that $M_\mu \times K_\mu$ has a Γ^*_2-multiplicative adequate transversal $M^0_\mu \times K^0_\mu$ where M^0_μ (K^0_μ) is the Γ_2-$(\Gamma'_2$-) multiplicative adequate transversal of M_μ (K_μ) and $M^0_\mu \times K^0_\mu$ is Γ-isomorphic to S^0. Computing we know that M^0_μ and K^0_μ may be described by

$$M^0_\mu = \{ \ (a)_{11}, (e)\}, \ K^0_\mu = \{ \ (n_{11})_{11} \mid n_{11} \in K^0\}.$$

Note that under the multiplication (1.1) and the multiplication of direct product S is indeed an abundant semigroup with a multiplicative adequate transversal S^0. Therefore concluding above results we may say that the abundant semigroup S has a Γ-Rees matrix representation $M_\mu \times K_\mu$ with a Γ-multiplicative $\bar{\Gamma}$-adequate transversal $M_\mu^0 \times K_\mu^0$ Γ-isomorphic to S^0.

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