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AN IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF STRONGLY PSEUDOCONTRACTIVE MAPPINGS

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ABSTRACT. The purpose of this paper is to establish a strong convergence of an implicit iteration process with errors to a common fixed point for a finite family of continuous strongly pseudocontractive mappings. The results presented in this paper extend and improve the corresponding results of References [2, 6, 11-12].

1. INTRODUCTION

From now onward, we assume that K is a nonempty closed convex subset of a real Banach space E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 \text{ and } \|f^*\| = \|x\|\}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality mapping by j.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called strongly accretive if there exists a constant $0 < k < 1$ such that, for each $x, y \in D(T)$, there is a $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \geq k\|x-y\|^2. \tag{1}$$

Definition 1.2. A mapping T with domain $D(T)$ and range $R(T)$ in E is called strongly pseudocontractive if for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a constant $0 < k < 1$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq (1-k)\|x-y\|^2. \tag{2}$$

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It is known that T is strongly pseudocontractive if and only if $(I-T)$ is strongly accretive.

In 2001, Xu and Ori [11] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \cdots, N\}$), with $\{\alpha_n\}$ a real sequence in $(0,1)$, and an initial point $x_0 \in K$:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_{N+1} x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad \text{for all } n \geq 1, \tag{3}$$

where $T_n = T_{n \pmod N}$ (here the $\bmod N$ function takes values in I). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family of nonexpansive mappings defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [12], Zhou and Chang studied the weak and strong convergence of this implicit process to a common fixed point for a finite family of nonexpansive mappings. More precisely, they proved the following result.

Theorem 1.1 ([12, Theorem 3]). *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N semi-compact nonexpansive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ (here $F(T_i)$ denotes the set of fixed points of T_i). Suppose that $x_0 \in K$ and $\{\alpha_n\} \subset (b,c)$ for some b, $c \in (0,1)$. Then the sequence $\{x_n\}$ defined by the implicit iteration process* (3) *converges strongly to a common fixed point in F.*

Definition 1.3 ([2]). A family $\{T_i : i \in I\}$ of N self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy condition (A) on K if there is a nondecreasing function $f : [0,\infty] \to [0,\infty]$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in K$

$$\max_{1 \le i \le N} \|x - T_i x\| \ge f(d(x, F)).$$

In [2], Chidume and Shahzad studied the strong convergence of the implicit process (3) to a common fixed point for a finite family of nonexpansive mappings. They proved the following results.

Theorem 1.2 ([2, Theorem 3.2]). *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N nonexpansive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that $\{T_i : i \in I\}$ satisfies condition (A). Let $\{\alpha_n\}_{n \ge 1} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process* (3). *Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$.*

Theorem 1.3 ([2, Theorem 3.3]). *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N nonexpansive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that one of the mappings in $\{T_i : i \in I\}$ is semi-compact. Let $\{\alpha_n\}_{n \ge 1} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process* (3). *Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$.*

In [6], Oslilike proved the following theorem

Theorem 1.4. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N strictly pseudocontractive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:*

(i) $0 < \alpha_n < 1$,

(ii) $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$,

(iii) $\sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (3). *Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if $\lim_{n \to \infty} d(x_n, F) = 0$.*

Inspired and motivated by the above facts, we suggest the following implicit iteration process with errors and define the sequence $\{x_n\}$ as follows

$$x_n = a_n x_{n-1} + b_n T_n x_n + c_n u_n, \quad \text{for all} \quad n \geq 1, \tag{4}$$

where $T_n = T_{n \ (\text{mod } N)}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ for each $n \in \mathbb{N}$ and $\{u_n\}$ is a bounded sequence in K.

Remark 1.1. In (4), if we take $0 \leq a_n = 1 - b_n \leq 1$, then it takes the form

$$x_n = a_n x_{n-1} + (1 - a_n) T_n x_n + c_n u_n. \tag{5}$$

By putting $c_n = 0$ in (5), it reduces to (3). So there is no need to discuss the implicit iteration processes (3) and (4) separately.

Remark 1.2. Observe that if $T : K \to K$ is a continuous strongly pseudocontractive mapping, then for every fixed $u \in K$ and $t \in (0,1)$, the mapping $S_t : K \to K$ defined for all $x \in K$ by

$$S_t x = tu + (1-t)Tx,$$

satisfies

$$\langle S_t x - S_t y, j(x-y) \rangle \leq (1-t)\|x-y\|^2, \quad \text{for all} \quad x, y \in K.$$

It follows that S_t is a strongly pseudocontractive mapping. Since S_t is also continuous, S_t has a unique fixed point x_t in K by [3, Corollary 2], i.e.,

$$x_t = tu + (1-t)Tx_t.$$

Thus the implicit iteration process (3) is defined in K for the continuous strongly pseudocontractive self-mappings of a nonempty convex subset K of a Banach space provided that $\alpha_n \in (0,1)$ for all $n \geq 1$.

The purpose of this paper is to study the strong convergence of implicit iteration process (4) to a common fixed point for a finite family of continuous strongly pseudocontractive mappings in real Banach spaces. The results presented in this paper extend and improve the corresponding results of References [2, 6, 11-12].

2. Main Results

In this section we study the convergence of Algorithm (4). For this purpose, we need the following results.

Lemma 2.1 ([10]). *Let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any x, $y \in E$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle, \quad \textit{for all} \quad j(x+y) \in J(x+y). \tag{6}$$

The following lemma is proved in [8].

Lemma 2.2. *If there exists a positive integer N such that for all $n \geq N$, $n \in \mathbb{N}$*

$$\rho_{n+1} \leq (1-\alpha_n)\rho_n + b_n,$$

then

$$\lim_{n\to\infty} \rho_n = 0,$$

where $\alpha_n \in [0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $b_n = o(\alpha_n)$.

Theorem 2.1. *Let $\{T_1, T_2, \cdots, T_N\} : K \to K$ be N continuous strongly pseudocontractive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (4) satisfying $\sum_{n=1}^{\infty} b_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $c_n = o(b_n)$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \cdots, T_N\}$.*

Proof. Since each T_i is strongly pseudocontractive, then there exists $k_i \in (0,1)$ such that

$$\langle T_i x - T_i y, j(x-y)\rangle \leq (1-k_i)\|x-y\|^2, \quad i = 1, 2, \cdots, N.$$

Let $k = \min_{1\leq i\leq N}\{k_i\}$. Then

$$\langle T_i x - T_i y, j(x-y)\rangle \leq (1-k)\|x-y\|^2, \quad i = 1, 2, \cdots, N. \tag{7}$$

We know that the mappings $\{T_1, T_2, \cdots, T_N\}$ have a common fixed point in k, say w, then the fixed point set $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is nonempty. We will show that w is the unique fixed point of each T_i. Suppose there exists $w_1 \in F$. Then from (7),

$$\|w - w_1\|^2 = \langle w - w_1, j(w-w_1)\rangle = \langle T_i w - T_i w_1, j(w - w_1)\rangle \leq (1-k)\|w-w_1\|^2.$$

Since $k \in (0,1)$, it follows that $\|w - w_1\|^2 \leq 0$, which implies the uniqueness.

Set $M := \sup_{n\geq 1}\|u_n - w\|$. We will prove that $\{x_n\}$ is bounded. Indeed, from (4) we have

$$\begin{aligned}
&\|x_n - w\|^2 \\
&= \langle x_n - w, j(x_n - w)\rangle \\
&= \langle a_n x_{n-1} + b_n T_n x_n + c_n u_n - w, j(x_n - w)\rangle \\
&= \left\langle (1-b_n)\left(\frac{a_n}{1-b_n}x_{n-1} + \frac{c_n}{1-b_n}u_n\right) + b_n T_n x_n - w, j(x_n - w)\right\rangle
\end{aligned}$$

$$= \left\langle (1-b_n)\left(\frac{a_n}{1-b_n}(x_{n-1}-w)+\frac{c_n}{1-b_n}(u_n-w)\right)+b_n(T_nx_n-w), j(x_n-w)\right\rangle$$
$$= (1-b_n)\left\langle\left(\frac{a_n}{1-b_n}(x_{n-1}-w)+\frac{c_n}{1-b_n}(u_n-w), j(x_n-w)\right)\right\rangle$$
$$+ b_n(T_nx_n-w, j(x_n-w))$$
$$\leq (1-b_n)\left\|\frac{a_n}{1-b_n}(x_{n-1}-w)+\frac{c_n}{1-b_n}(u_n-w)\right\|\|x_n-w\|+b_n(1-k)\|x_n-w\|^2,$$

implies

$$\|x_n-w\| \leq (1-b_n)\left\|\frac{a_n}{1-b_n}(x_{n-1}-w)+\frac{c_n}{1-b_n}(u_n-w)\right\|+(1-k)b_n\|x_n-w\|,$$

and consequently, we obtain

$$\begin{aligned}\|x_n-w\| &\leq \frac{(1-b_n)}{1-(1-k)b_n}\left\|\frac{a_n}{1-b_n}(x_{n-1}-w)+\frac{c_n}{1-b_n}(u_n-w)\right\| \\ &\leq \left\|\frac{a_n}{1-b_n}(x_{n-1}-w)+\frac{c_n}{1-b_n}(u_n-w)\right\| \\ &\leq \frac{a_n}{1-b_n}\|x_{n-1}-w\|+\frac{c_n}{1-b_n}\|u_n-w\| \\ &\leq \frac{a_n}{1-b_n}\|x_{n-1}-w\|+\frac{c_n}{1-b_n}M \\ &\leq \max\{\|x_{n-1}-w\|, M\}.\end{aligned}$$

Now the induction yields

$$\|x_n-w\| \leq \max\{\|x_0-w\|, M\}, \quad n\geq 1.$$

So, from the above discussion, we can conclude that the sequence $\{x_n\}$ is bounded. Set $M_1 := \sup_{n\geq 1}\|x_n-w\|+M$. From Lemma 2.1 and (4), we have

$$\begin{aligned}&\|x_n-w\|^2 \\ &= \|a_nx_{n-1}+b_nT_nx_n+c_nu_n-w\|^2 \\ &= \|a_n(x_{n-1}-w)+b_n(T_nx_n-w)+c_n(u_n-w)\|^2 \\ &\leq a_n^2\|x_{n-1}-w\|^2+2\langle b_n(T_nx_n-w)+c_n(u_n-w), j(x_n-w)\rangle \\ &\leq (1-b_n)^2\|x_{n-1}-w\|^2+2b_n\langle T_nx_n-w, j(x_n-w)\rangle+2c_n\langle u_n-w, j(x_n-w)\rangle \\ &\leq (1-b_n)^2\|x_{n-1}-w\|^2+2b_n(1-k)\|x_n-w\|^2+2M_1^2c_n,\end{aligned}$$

implies

$$\|x_n-w\|^2 \leq \frac{(1-b_n)^2}{1-2(1-k)b_n}\|x_{n-1}-w\|^2+2M_1^2\frac{c_n}{1-2(1-k)b_n}. \tag{8}$$

Let

$$A_n = (1 - b_n)^2,$$
$$B_n = 1 - 2(1-k)b_n,$$

and consider

$$\beta_n = 1 - \frac{A_n}{B_n} = 1 - \frac{(1-b_n)^2}{1-2(1-k)b_n} = \frac{b_n(2k - b_n)}{1-2(1-k)b_n} \geq b_n(2k - b_n). \tag{9}$$

By $\lim_{n\to\infty} b_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $b_n \leq \frac{k}{2}$, which implies that $1 - 2(1-k)b_n \geq 1 - k(1-k)$ and consequently from (9), we get $\beta_n \geq kb_n$. Thus from (8), we obtain

$$\|x_n - w\| \leq (1 - kb_n)\|x_{n-1} - w\| + 2M_1^2 \frac{1}{1-k(1-k)} c_n. \tag{10}$$

With the help of Lemma 2.2 and using the fact that $\sum_{n=1}^{\infty} b_n = \infty$ and $c_n = o(b_n)$, we obtain

$$\lim_{n\to\infty} \|x_n - w\| = 0.$$

Consequently $x_n \to w \in F$ and this completes the proof. □

Corollary 2.1. *Let $\{T_1, T_2, \cdots, T_N\} : K \to K$ be N continuous strongly pseudo-contractive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process* (3) *satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, $\lim_{n\to\infty}(1-\alpha_n) = 0$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \cdots, T_N\}$.*

Remark 2.1. Theorem 2.1 and Corollary 2.1 extend and improve Theorems 1.1-1.4 in the following directions:

- We do not need the assumption semi-compact as in Theorem 1.1 and Theorem 1.3;
- We do not need the "condition A" as in Theorem 1.2;
- We do not need the assumption lim as in Theorem 1.4.

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