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A NOTE ON DIFFERENCE SEQUENCES

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ABSTRACT. It is well known that for a sequence $\mathbf{a} = (a_0, a_1, \ldots)$ the general term of the dual sequence of $\mathbf{a}$ is $a_n = c_0 {n \atop 0} + c_1 {n \atop 1} + \cdots + c_n {n \atop n}$, where $\mathbf{c} = (c_0, \ldots, c_n$ is the dual sequence of $\mathbf{a}$. In this paper, we find the general term of the sequence $(c_0, c_1, \ldots)$ and give another method for finding the inverse matrix of the Pascal matrix.

And we find a simple proof of the fact that if the general term of a sequence $\mathbf{a} = (a_0, a_1, \ldots)$ is a polynomial of degree p in n, then $\Delta^{p+1}\mathbf{a} = \mathbf{0}$.

Let $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ be a sequence of numbers. The *difference sequence* $\Delta\mathbf{a} = (\Delta a_0, \Delta a_1, \Delta a_2, \ldots)$ is defined by $\Delta a_i = a_{i+1} - a_i$, $i \geq 0$. For $p = 0, 1, 2, \ldots$, the *pth-order difference sequence* $\Delta^p\mathbf{a} = (\Delta^p a_0, \Delta^p a_1, \Delta^p a_2, \ldots)$ of $\mathbf{a}$ is defined inductively by $\Delta^p\mathbf{a} = \Delta(\Delta^{p-1}\mathbf{a})$ where $\Delta^0\mathbf{a} = \mathbf{a}$. The infinite matrix

$$A_{\mathbf{a}} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ \Delta a_0 & \Delta a_1 & \Delta a_2 & \cdots \\ \Delta^2 a_0 & \Delta^2 a_1 & \Delta^2 a_2 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

is called the *difference matrix* of $\mathbf{a}$ and the sequence $(a_0, \Delta a_0, \Delta^2 a_0, \ldots)$ is called the *dual sequence* of $\mathbf{a}$. Note that if $A_{\mathbf{a}} = [a_{ij}]$, then $a_{i-1,j+1} - a_{i-1,j} = a_{ij}$.

Let α and β be subsets of $\{1, 2, \ldots\}$. For a given matrix A, $A[\alpha|\beta]$ denotes the submatrix of A using rows numbered α and columns numbered β.

The following two theorems are well known facts.

Theorem A ([1, Theorem 8.2.2]). *Let $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ be a sequence and let $\mathbf{c} = (c_0, c_1, c_2, \ldots)$ be the dual sequence of $\mathbf{a}$. Then*

$$a_n = c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_n\binom{n}{n}$$

for each $n = 0, 1, 2, \ldots$.

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By the Theorem A, if we know the general term of the dual sequence of $\mathbf{a}$, then we can find the general term of $\mathbf{a}$, and the relationship between the sequence $\mathbf{a}$ and its dual sequence is

$$\mathbf{a}^T = P\mathbf{c}^T \tag{1}$$

where P is the matrix which is defined by

$$p_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise} \end{cases}$$

and is called the *Pascal matrix.*

Theorem B ([1, Theorem 8.2.1]). *If the general term of a sequence* $\mathbf{a} = (a_0, a_1, \ldots)$ *is a polynomial of degree* p *in* n, *then* $\Delta^{p+1}\mathbf{a} = \mathbf{0}$.

From now on, we will find the general term of the dual sequence of a sequence $\mathbf{a}$, and give the simple proof of the Theorem B.

Theorem 1. *Let* $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ *be a sequence and let* $\mathbf{c} = (c_1, c_1, c_2, \ldots)$ *be a dual sequence of* $\mathbf{a}$. *Then*

$$c_n = a_n\binom{n}{0} - a_{n-1}\binom{n}{1} + a_{n-2}\binom{n}{2} + \cdots + (-1)^n a_0\binom{n}{n}.$$

Proof. We prove this theorem by induction on n.

If $n = 0$, it is clear because of $c_0 = a_0$ and $\binom{0}{0} = 1$. So we assume that the theorem holds for $n-1$. Then, by the definition of the difference matrix of $\mathbf{a}$ and the induction hypothesis,

$$\begin{aligned} c_n &= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}a_{n-k} - \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}a_{n-k-1} \\ &= \binom{n-1}{0}a_n + \sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}a_{n-k} \\ &\quad - \left(\sum_{k=0}^{n-2}(-1)^k\binom{n-1}{k}a_{n-k-1} + (-1)^{n-1}\binom{n-1}{n-1}a_0\right) \\ &= \binom{n-1}{0}a_n + \sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}a_{n-k} + \sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k-1}a_{n-k} \\ &\quad + (-1)^n\binom{n-1}{n-1}a_0 \end{aligned}$$

$$= a_n + \sum_{k=1}^{n-1} (-1)^k \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) a_{n-k} + (-1)^n a_0$$

$$= a_n + \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} a_{n-k} + (-1)^n a_0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{n-k}.$$

□

To prove the Theorem B, we use the induction on the index of the sequence $\Delta^{p+1}\mathbf{a}$. Let $\Delta^{p+1}\mathbf{a} = (b_0, b_1, \ldots)$. Since $b_0 = 0$, the case of $k = 0$ is trivial. Assume that it is true for all $k \leq n-1$. Let $\alpha = \{p+1, \ldots, p+n\}$ and $\beta = \{1, \ldots, n\}$. Consider the submatrix $A_{\mathbf{a}}[\alpha|\beta]$ of $A_{\mathbf{a}}$. If $b_n = a$, then, by the definition of $A_{\mathbf{a}}$ and induction hypothesis, $EA_{\mathbf{a}}[\alpha|\beta]$ is a upper triangular matrix where $E = [e_{ij}]$ is a permutation matrix with

$$e_{ij} = \begin{cases} 1 & \text{if } i+j=n \\ 0 & \text{otherwise.} \end{cases}$$

Since the first column of $A_{\mathbf{a}}[\alpha|\beta]$ is $\mathbf{0}$, a is also 0. Hence $b_n = 0$ and so the Theorem B is proved. □

Remark. By Theorem 1, we know that

$$c_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{n-k}, \quad n = 0, 1, 2, \ldots.$$

If we put $t = n - k$, then

$$c_n = \sum_{t=0}^{n} (-1)^{n-t} \binom{n}{n-t} a_t.$$

So we know that the relationship between a sequence $\mathbf{a}$ and the dual sequence $\mathbf{c}$ of $\mathbf{a}$ is

$$\mathbf{c}^T = Q\mathbf{a}^T \tag{2}$$

where the matrix $Q = [q_{ij}]$ is defined by

$$q_{ij} = \begin{cases} (-1)^{i-j}\binom{i}{i-j} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

By the equation (1) and (2), $\mathbf{a}^T = PQ\mathbf{a}^T$ for all sequence $\mathbf{a}$. Hence Q is the inverse matrix of the Pascal matrix P.

This is a simple proof for finding the inverse matrix of the Pascal matrix.

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