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Variable sampling interval control charts for variance-covariance matrix†

Duk-Joon Chang[1] · Jae-Kyoung Shin[2]

Department of Statistics, Changwon National University

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Abstract

Properties of multivariate Shewhart and EWMA (Exponentially Weighted Moving Average) control charts for monitoring variance-covariance matrix of quality variables are investigated. Performances of the proposed charts are evaluated for matched fixed sampling interval (FSI) and variable sampling interval (VSI) charts in terms of average time to signal (ATS) and average number of samples to signal (ANSS). Average number of swiches (ANSW) of the proposed VSI charts are also investigated.

Keywords: ANSS, ANSW, ATS, FSI, VSI.

1. Introduction

The quality of a product is often characterized by joint levels of several correlated variables rather than a single variable. When the quality variables are correlated, one can obtain better sensitivity by using multivariate control chart than separate control charts for each of the process parameters.

Control charts are usually used for monitoring quality variables from a process to detect any shifts in the production process and eliminate assignable causes in the parameters of the distribution of these quality variables. One wishes to detect any departure from a satisfactory state as quickly as possible and identify which assignable causes are responsible for the deviation.

The ability of a control chart to detect process changes is determined by the length of time required for the chart to signal. The expected time to signal is simply the product of the average number of samples to signal (ANSS) and the length of the fixed sampling interval. Therefore the ANSS can be thought of as the expected time to signal in FSI procedures.

Recent years, applications of VSI procedure have become quite frequent. In VSI procedures we can use a finite number of sampling interval $d_1, d_2, \cdots, d_\eta$, and the choice of any sampling interval depends on sampling interval function $d(\underline{x})$ when $\underline{X}_i = \underline{x}$ is observed. Reynolds (1989) showed that the use of two sampling intervals spaced as apart as possible

† This research is financially supported by Changwon National University in 2008.
[1] Corresponding author: Professor, Department of Statistics, Changwon National University, Changwon 641-773, Korea. E-mail: djchang@changwon.ac.kr
[2] Professor, Department of Statistics, Changwon National University, Changwon 641-773, Korea.

is optimal in VSI procedures. In this paper, we also consider VSI procedures with two sampling time intervals d_1 and d_2 ($d_1 < d_2$). Amin and Letsinger (1991) described general procedures for VSI scheme and examined switching behavior and runs rules for switching between different sampling intervals on univariate $\bar{X}$ -chart.

The original work on multivariate control chart was introduced by Hotelling (1947). Alt (1984) reviewed much of the study on multivariate control charts. Woodall and Ncube (1985) considered a single multivariate CUSUM procedure for monitoring the means of multivariate normal process. Lucas and Saccucci (1990) evaluated the properties of an EWMA scheme to monitor mean of normally distributed procee, and a robust EWMA by using Markov chain approach. Many authors concluded that the properties of EWMA schemes are similar to those of CUSUM schemes (see, Chang and Kwon, 2002; Kwon and Chang, 2006; Reynolds *et al.*, 1990; Vargas *et al.*, 2004).

Most of the studies on multivariate control charts have been concentrated on monitoring mean vector of multivariate normal process. Often, the shifts in the components of dispersion matrix for the related quality variables are often important.

In this paper, we investigate the properties of multivariate VSI control charts for monitoring dispersion matrix Σ in terms of ATS and ANSS where the target process mean vector $\underline{\mu}$ remained known constant. And we also investigate the ANSW of the proposed chart.

2. Description of some control procedures

Suppose that p quality characteristics of interest represented by the random vector $\underline{X} = (X_1, X_2, \cdots, X_p)'$ and $\underline{X}$ has a multivariate normal distribution with mean vector $\underline{\mu}$ and dispersion matrix Σ. We obtain a sequence of random vectors $\underline{X}_1, \underline{X}_2, \cdots$ to judge the state of the process where $\underline{X}_t = (\underline{X}'_{t1}, \cdots, \underline{X}'_{tn})'$ is an observation vector of each sampling time t and $\underline{X}_{tj} = (X_{tj1}, \cdots, X_{tjp})'$. Let $\underline{\theta}_0 = (\underline{\mu}_0, \Sigma_0)$ be the known target process parameters for $\underline{\theta} = (\underline{\mu}, \Sigma)$ of related multiple quality variables, where $\underline{\theta}_0$ is represented as

$$\underline{\mu_0} = \begin{bmatrix} \mu_{10} \\ \mu_{20} \\ \vdots \\ \mu_{p0} \end{bmatrix} \text{ and } \Sigma_0 = \begin{bmatrix} \sigma_{10}^2 & \rho_{120}\sigma_{10}\sigma_{20} & \cdots & \rho_{1p0}\sigma_{10}\sigma_{p0} \\ & \sigma_{20}^2 & \cdots & \rho_{2p0}\sigma_{20}\sigma_{p0} \\ & & \ddots & \vdots \\ & Sym & & \sigma_{p0}^2 \end{bmatrix}.$$

2.1. Evaluating sample statistic

The general multivariate statistical quality control chart can be considered as a repetitive tests of significance where each quality characteristic is defined by p quality variables $X_1, X_2, \cdots, X_p$. Therefore, we can obtain a sample statistic for momitoring variance-covariance matrix Σ by using the likelihood ratio test (LRT) statistic for testing $H_0 : \Sigma = \Sigma_0$ vs $H_1 : \Sigma \neq \Sigma_0$ where target mean vector of the quality variables $\underline{\mu}_0$ is known. The regions above the upper control limit (UCL) corresponds to the LRT rejection region. For the i th sample, likelihood ratio λ_i can be expressed as

$$\lambda_i = n^{-\frac{np}{2}} \cdot A_i\Sigma_0^{-1}|^{\frac{n}{2}} \cdot \exp\left[-\frac{1}{2}tr(\Sigma_0^{-1}A_i) + \frac{1}{2}np\right].$$

Let TV_i be $-2\ln\lambda_i$. Then

$$TV_i = tr(A_i\Sigma_0^{-1} - n\ln|A_i| + n\ln|\Sigma_0| + np\ln n - np \tag{2.1}$$

and, we use the LRT statistic TV_i as the sample statistic for Σ. If the sample statistic TV_i plots above the UCL, the process is deemed out-of-control state and assignable causes are sought.

2.2. ATS and ANSW of VSI procedure

In FSI control chart, $t_i + 1 - t_i$, the length of sampling interval between sampling times, is constant for all $i(i = 0, 1, \cdots)$. But for a VSI chart, the sampling times are random variables and $t_i + 1 - t_i$ is a function of chart statistic and depends on the past sample informations $X_1, X_2, \cdots, X_i$. For VSI charts, the time required for the chart to signal is not a constant multiple of the run length. To evaluate the performance of a VSI control chart, it is necessary to obtain time and number of samples separately. Therefore, we use ATS and ANSS for evaluating and comparing the properties of the FSI and VSI charts.

For VSI chart, the sampling times are random variables and the sampling interval depends on the past sample informations of $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_i$. Reynolds (1989) investigated the theoretical aspects of a VSI one- and two-sided Shewhart charts.

To implement two sampling interval VSI control scheme, the in-control region is divided into 2 regions I_1, I_2 where I_i is the region in which the sampling interval d_i is used $(i = 1, 2)$. In this paper, we assume that the VSI chart is started at time 0 and the interval used before the first sample, is a fixed constant, say d_0. Then the ANSS and ATS can be expressed as

$$ANSS = 1 + \psi_1 + \psi_2 \text{ and } ATS = d_0 + d_1\psi_1 + d_2\psi_2, \tag{2.2}$$

where ψ_i is the expected number of samples before the signal.

The VSI procedures have been shown to be more efficient when compared to the corresponding FSI procdures with respect to the ATS. But, frequent switching between the different sampling intervals d_1 and d_2 can be a complicating factor in the application of control charts with VSI procedures. Therefore, it is necessary to define the number of swiches (NSW) as the number of swiches made from the start of the process until the chart signals, and let ANSW be the expected value of the NSW.

The ANSW can be obtained as follows

$$ANSW = ARL \cdot P(swich). \tag{2.3}$$

And, the probability of swich is given by

$$P(swich) = P(d_1) \cdot P(d_2|d_1) + P(d_2) \cdot (d_1|d_2), \tag{2.4}$$

where $P(d_i)$ is the probability of using sampling interval d_i, and $P(d_i|d_j)$ is the conditional probability of using sampling interval d_i in the current sample given that the sampling interval d_j $(d_i ned_j)$ was used in the previous sample.

3. Shewhart control chart

The Shewhart chart has a good ability to detect large changes in monitored parameter quickly and is easy to implement the process. However, the Shewhart chart is slow to detect small or moderate changes of the parameters.

The control limits for the FSI Shewhart chart based on the LRT statistic TV_i would be set by using percentage point of TV_i, and the chart signals whenever

$$TV_i \geq h_{TV(S)}. \tag{3.1}$$

And for VSI Shewhart chart based on TV_i, suppose that the sampling interval;

$$\begin{aligned} d_1 \text{ is used when } TV_i &\in (g_{TV(S)}, h_{TV(S)}], \\ d_2 \text{ is used when } TV_i &\in (0, g_{TV(S)}], \end{aligned}$$

where $g_{TV(S)} <= h_{TV(S)}$ and $d_1 < d_2$.

Since it is difficult to obtain the exact distribution of LRT statistic TV_i, the design parameters $g_{TV(S)}$ and $h_{TV(S)}$ can be obtained to satisfy a desired ATS and ANSS by simulation.

4. EWMA control chart

For FSI EWMA chart based on LRT statistic TV_t can be constructed as

$$Y_{TV,t} = (1-\lambda)Y_{TV,t-1} + \lambda TV_t, \tag{4.1}$$

where $Y_{TV,0} = \omega(\omega \geq 0)$ and $0 < \lambda \leq 1$. This chart signals whenever $Y_{TV,t} > h_{TV}$. When the smoothing constant λ is 1, this EWMA chart changes to Shewhart chart.

Since it is difficult to obtain the exact distribution of chart statistic $Y_{TV,t}$, the performances of this chart can be evaluated by simulation when the parameters of the process are on-target or changed. And the process parameter h_{TV} can be obtained to satisfy a specified ANSS.

And for VSI EWMA chart based on TV_i, suppose that the sampling interval;

$$\begin{aligned} d_1 \text{ is used when } Y_{TV,i} &\in (g_{TV}(E), h_{TV}(E)], \\ d_2 \text{ is used when } Y_{TV,i} &\in (0, g_{TV}(E)], \end{aligned}$$

where $g_{TV(E)} <= h_{TV(E)}$ and $d_1 < d_2$. The design parameters $g_{TV(E)}$ and $h_{TV(E)}$ can be obtained to satisfy a desired ATS and ANSS by simulation.

5. Concluding remarks

The properties of proposed charts are determined by the choice of the process parameters λ, h, g, d_1, d_2. For the purposes of comparison and evaluation of different FSI and VSI charts, all charts being considered were set up so that the ANSS and ATS are 200.0 when $\underline{\mu} = \underline{\mu}_0$, $d_0 = 1$ and the sample size for each variable was five for $p = 3$ and 4. For simplicity in our computation, we assume that the target mean vector $\underline{\mu}_0 = \underline{0}'$, all diagonal and off-diagonal elements of Σ_0 are 1 and 0.3, respectively. And we let that the sampling interval of unit time $d = 1$ in FSI chart and two sampling intervals used as $d_1 = 0.1$ and $d_2 = 1.9$ in VSI chart. After the smoothing constants of the proposed EWMA charts have been determined, the design parameters $g's$ and $h's$ and the ANSS, ATS and ANSW values were calculated by simulation with 10,000 runs. And, for VSI chart, the amount of switching for the different charts can also be compared.

Since the performance of the charts depends on the components of Σ, it is not possible to investigate all of the different ways in which Σ could change. Hence, we consider the following typical types of shifts for comparison in the process parameters.

1. V_i: σ_{10} of Σ_0 is increased to $[1 + (4i - 3)/10]$.

2. C_i: ρ_{120} and ρ_{210} of Σ_0 are changed to $[0.3 + (2i - 1)/10]$.

3. (V_i, C_i) for $i = 1, 2, 3$.

4. S_i: Σ_0 is changed to $c_i\Sigma_0$ where $c_i = [1 + (3i - 2)/10]^2$.

The properties and comparison of the proposed procedures are given in Table 5.1 through Table 5.3. From the numerical results, we found the following properties. VSI schemes are more efficient than FSI schemes in terms of ATS.

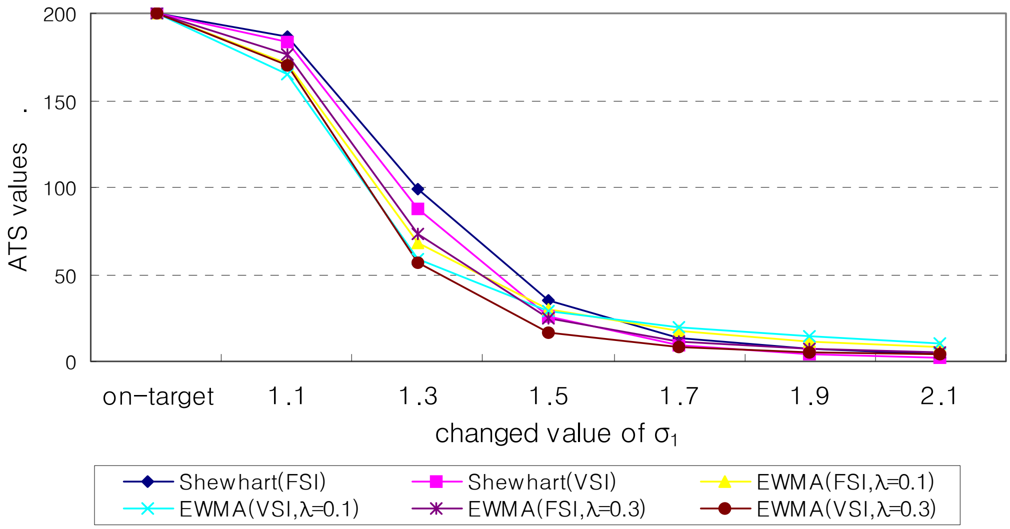

Figure 5.1 ATS for the proposed charts ($p = 3$)

The results in Table 5.2 and Table 5.3 show that smaller smoothing constant λ is more efficient in detecting small shifts of the parameters in terms of ANSS and ATS.

And, we also found that the EWMA procedures exhibit far fewer switches when compared to the Shewhart procedure. Also, it was established that smaller values of λ for EWMA procedures will reduce the ANSW between two sampling intervals, respectively. The optimal selection of λ depends on the size of the shifts to be detected quickly.

Table 5.1 Properties of the Shewhart chart for Σ

types of shift	$p=3$ ANSS	$p=3$ ATS	$p=3$ ANSW	$p=4$ ANSS	$p=4$ ATS	$p=4$ ANSW
in-control	200.0	200.0	100.12	200.0	200.0	99.94
V_1	186.2	183.2	93.24	192.3	190.2	96.09
V_2	35.2	26.2	16.21	68.7	54.5	32.52
V_3	6.9	4.1	2.66	13.6	8.0	5.38
C_1	191.4	189.1	95.8	194.6	193.0	97.20
C_2	133.7	116.6	65.58	157.8	144.1	78.19
C_3	56.8	31.2	21.25	90.0	61.2	39.26
(V_1,C_1)	181.2	177.0	90.71	188.9	185.8	94.36
(V_2,C_2)	30.4	21.4	13.52	58.3	44.0	26.99
(V_3,C_3)	5.5	2.7	176	10.3	5.1	3.49
S_1	164.1	156.9	82.00	176.1	169.2	87.88
S_2	18.3	10.8	7.22	29.0	16.7	11.27
S_3	3.5	1.8	1.08	4.4	1.9	1.21
S_4	1.7	1.1	0.04	1.7	1.1	0.45

Table 5.2 Properties of the EWMA chart for $\Sigma(p=3)$

types of shift	$\lambda=0.1$ ANSS	$\lambda=0.1$ ATS	$\lambda=0.1$ ANSS	$\lambda=0.2$ ANSS	$\lambda=0.2$ ATS	$\lambda=0.2$ ANSW	$\lambda=0.3$ ANSS	$\lambda=0.3$ ATS	$\lambda=0.3$ ANSW
in-control	200.0	200.0	27.46	200.1	200.0	40.48	200.0	200.0	50.29
V_1	170.7	164.7	23.08	173.0	166.6	34.84	176.1	170.4	44.25
V_2	29.5	29.0	3.26	24.9	18.6	4.21	24.8	16.5	5.34
V_3	11.8	14.0	2.13	8.1	7.5	2.11	6.9	5.4	2.05
C_1	178.8	174.1	24.24	181.3	176.4	36.64	183.9	179.6	46.21
C_2	77.8	64.6	8.71	84.2	63.8	15.29	93.7	71.6	21.93
C_3	24.4	23.7	2.48	21.3	13.6	2.89	23.3	11.3	3.58
(V_1,C_1)	157.9	150.4	21.10	162.8	154.3	32.70	167.4	159.6	41.97
(V_2,C_2)	25.9	26.1	2.92	21.2	15.9	3.57	20.9	13.6	4.39
(V_3,C_3)	9.7	11.7	2.02	6.5	6.2	1.95	5.5	4.3	1.82
S_1	132.2	121.3	17.25	138.0	125.0	27.36	143.5	131.0	35.72
S_2	18.0	19.6	2.38	13.5	10.8	2.56	12.7	8.2	2.78
S_3	7.6	9.2	1.97	4.9	4.8	1.84	4.1	3.3	1.61
S_4	4.4	5.4	1.76	2.9	2.8	1.39	2.3	1.9	1.03

Table 5.3 Properties of the EWMA chart for $\Sigma(p=4)$

types of shift	$\lambda=0.1$ ANSS	$\lambda=0.1$ ATS	$\lambda=0.1$ ANSS	$\lambda=0.2$ ANSS	$\lambda=0.2$ ATS	$\lambda=0.2$ ANSW	$\lambda=0.3$ ANSS	$\lambda=0.3$ ATS	$\lambda=0.3$ ANSW
in-control	200.0	200.0	27.29	200.0	200.0	40.39	200.0	200.0	50.28
V_1	181.1	177.0	24.49	184.0	179.5	37.03	186.0	181.9	46.71
V_2	44.6	41.6	4.51	41.7	30.6	6.85	44.5	30.5	9.60
V_3	17.7	20.8	2.32	12.5	11.2	2.46	11.2	8.2	2.63
C_1	186.4	183.0	25.23	189.2	185.7	38.15	190.0	186.9	47.76
C_2	104.0	89.9	12.34	113.9	93.6	21.54	123.4	102.8	29.73
C_3	37.7	34.4	3.26	36.9	23.0	4.86	42.7	22.8	7.34
(V_1,C_1)	172.6	166.6	23.17	176.6	170.1	35.45	179.7	173.7	45.09
(V_2,C_2)	38.3	36.7	3.78	34.6	25.1	5.46	36.9	24.0	7.59
(V_3,C_3)	14.5	17.6	2.14	9.9	9.2	2.18	8.6	6.4	2.20
S_1	142.4	131.9	18.50	149.5	136.9	29.65	155.6	143.6	38.73
S_2	21.9	24.1	2.46	16.9	13.2	2.70	16.4	10.1	3.13
S_3	9.4	12.0	2.01	6.0	6.1	1.97	4.9	4.1	1.85
S_4	5.5	7.2	1.86	3.4	3.6	1.68	2.7	2.4	1.33

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