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Estimations in a skewed uniform distribution

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Abstract

We obtain a skewed uniform distribution by a uniform distribution, and evaluate its coefficient of skewness. And we obtain the approximate maximum likelihood estimator (AML) and moment estimator of skew parameter in the skewed uniform distribution. And we compare simulated mean squared errors (MSE) of those estimators, and also compare MSE of two proposed reliability estimators in two independent skewed uniform distributions each with different skew parameters.

Keywords: Approximate ML, reliability, right-tail probability, skewed uniform distribution.

1. Introduction

Many authors had studied estimation and characterization in a uniform symmetric distribution about origin, whose distribution was used widely in engineering applications in Johnson *et al.* (1994).

Azzalini (1985) studied a class of distributions which includes the normal ones. Azzalini and Capitanio (1999) studied the multivariate skewed normal distribution. Woo (2007) studied reliability in half-triangle distributions and skewness of an induced skewed distribution, Son and Woo (2007) studied the approximate maximum likelihood estimator (AML) of a skew parameter in a skewed Laplace. Ali *et al.* (2008) studied skew reflected distributions generated by reflected gamma kernel.

Balakrishnan and Cohen (1991) proposed method of finding AML of a scale parameter in several distributions. Han and Kang (2006) studied AML of parameters in several distributions with multiple censored samples.

Let X and Y be two independent identical distributed continuous random variables with the probability density function(pdf) $f(x) = F'(x)$ which is symmetric about origin.

Then, for any $\alpha \in R^1$, $1/2 = P\{X - \alpha Y \le 0\} = \int_{-\infty}^{\infty} f(t)F(\alpha t)dt$.
And hence, a skewed density is as given in Azzalini (1985) by:

$$f(z;\alpha) \equiv 2f(z)F(\alpha z). \tag{1.1}$$

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The density $f(z;\alpha)$ becomes a skewed density of a random variable Z, and parameter α is called skew parameter of the skewed distribution. Especially if $\alpha = 0$, then $f(z;0)$ becomes the original symmetric density.

In this paper, we obtain a skewed uniform distribution from a uniform distribution, and evaluate its coefficient of skewness. And we obtain AML and moment estimator of skew parameter in the skewed uniform distribution, and compare simulated MSE's of those estimators. And we compare simulated MSE's of two defined estimators of reliability in two independent skewed uniform distributions each with two different skew parameters and common known parameter of the origin distribution. And we introduce a skewed uniform generated by a Laplace kernel, and hence we observe the skewness by evaluating its coefficient of skewness.

2. A skewed uniform distribution

Let X and Y be two independent uniform random variables with the pdf $f(x) = F'(x) = 1/(2\theta)$ $x \in [-\theta,\ \theta]$, $\theta > 0$. Then from the density (1.1), for a real number α,

$$\frac{1}{2} = \begin{cases} P(X \le \alpha Y) = \int_{-\theta}^{\theta} f(x)F(\alpha x)dx, & \text{if}|\alpha| \le 1 \\ P((-Y) \le \frac{1}{\alpha}(-X)) = \int_{-\theta}^{\theta} f(x)F(\frac{1}{\alpha}x)dx, & \text{if}\alpha > 1 \\ P(Y \le \frac{1}{\alpha}X) = \int_{-\theta}^{\theta} f(x)F(\frac{1}{\alpha}x)dx, & \text{if}\alpha < -1. \end{cases} \tag{2.1}$$

Define

$$\alpha^* = \begin{cases} \alpha, & \text{if}|\alpha| \le 1 \\ 1/\alpha, & \text{if}|\alpha| > 1. \end{cases}$$

Then $|\alpha^*| \le 1$ and $f(z;\alpha^*) \equiv 2f(z)\cdot(\alpha^* \cdot z)$ is a skewed uniform density and α^* is called skew parameter of the distribution. The density $f(z;\alpha^*)$ becomes a skewed density of a random variable Z. From the preceding process of obtaining a skewed uniform, since α^* is a monotone function of α, inference on α is equivalent to inference on α^* ($|\alpha^*| < 1$) (McCool, 1991). Therefore, from now on we consider inference on α only if $|\alpha| \le 1$. Throughout this paper except Section 3, let us assume $|\alpha| \le 1$.

From the density (2.1), the cdf of the skew random variable Z is as given in Azzalini (1985) by: For any real number α,

$$F(z;\alpha) = 2\int_{-\infty}^{z} f(t)\int_{-\infty}^{\alpha\cdot t} f(s)dsdt. \tag{2.2}$$

From the density (2.1) and the cdf of uniform, we obtain a skewed uniform density as below:

$$f(z;\alpha) = \frac{\alpha}{2\theta^2}z + \frac{1}{2\theta}, \quad \text{if } |z| \le \theta, |\alpha| \le 1. \tag{2.3}$$

Since the density (2.3) is a segment of line, the density (2.3) is skewed to the right when $-1 \le \alpha < 0$, and skewed to the left when $0 < \alpha \le 1$.

From the cdf (2.2) and uniform density over $[-\theta,\ \theta]$, the cdf of the skewed uniform random variable Z is given by:

$$F(z;\alpha) = \frac{\alpha}{4\theta^2}(z^2-\theta^2) + \frac{1}{2\theta}(z+\theta), \quad \text{if } |z| \le \theta. \tag{2.4}$$

From the density (2.3), k-th moment of the skewed uniform random variable Z is obtained as the following:

$$\text{For } |\alpha| \le 1, \quad E(Z^k;\alpha) = \frac{\theta^k}{2}\Big[\frac{1-(-1)^k}{k+2}\cdot\alpha + \frac{1-(-1)^{k+1}}{k+1}\Big]. \tag{2.5}$$

From the density (2.3) and (2.5), we observe the following:

Fact 2.1 For the skewed uniform density (2.3)
(a) The density (2.3) is skewed to the right when $-1 \le \alpha < 0$, and skewed to the left when $0 < \alpha \le 1$. (b) The larger $|\alpha|$ has, the more skewness has.

2.1. Estimation of skew parameter

Here we consider estimation of skew parameter α in the skewed uniform density (2.3) with $|\alpha| \le 1$.

Assume $Z_1, Z_2, \cdots, Z_n$ be iid random variables each with the density (2.3) having $\theta = 1$. Then, by method of finding AML of parameter in a distribution in Balakrishnan and Cohen (1991), from log-likelihood function of α, we obtain the following AML $\hat{\alpha}$ of α as follow:

For $Z_1 = z_1, Z_2 = z_2, \ldots, Z_n = z_n$, the likelihood function is

$$f(\alpha; z_1,\ldots,z_n) = \frac{1}{2^n}\prod_{i=1}^{n}(\alpha\cdot z_i + 1) \text{ and } \frac{d\ln f(\alpha; z_1,\ldots,z_n)}{d\alpha} = \sum_{i=1}^{n}\frac{z_i}{1+\alpha\cdot z_i} = 0.$$

Define $h(\alpha|z_i) \equiv \dfrac{z_i}{1+\alpha\cdot z_i}$. By taking 1st 2 terms of Taylor series for function $h(\alpha|z_i)$, then $h(\alpha|z_i) \approx h(c|z_i) + h'(c|z_i)(\alpha - c)$, for real number c. We obtain AML $\hat{\alpha}(c)$ of α as below:

$$\hat{\alpha}(c) \equiv \hat{\alpha} = c - \sum_{i=1}^{n} h(c|Z_i) \Big/ \sum_{i=1}^{n}\frac{d}{dc}h(c|Z_i). \tag{2.6}$$

And from the density (2.3) with $\theta = 1$, we obtain moment estimator (MME) $\tilde{\alpha}$ of α ($|\alpha| \le 1$):

$$\tilde{\alpha} = 3\sum_{i=1}^{n} Z_i/n. \tag{2.7}$$

Remark 2.1 As AML $\hat{\alpha}(\alpha)$ performs better in the sense of simulated MSE than AML $\hat{\alpha}(c)$ for $c \ne \alpha$ (see Son and Woo, 2007), we simulate MSE's of AML $\hat{\alpha}(\alpha)$ for given true value α.

We first consider the following process of generating distribution numbers to simulate MSE's of two estimators, AML and MME of skew parameter α when the pdf (2.3) has $\theta = 1$.

Process of generating distribution numbers

1. We choose random numbers u from a uniform distribution over (0,1).
2. For given $0 < u < 1$, by incremental search method in Deitel, et al (2003), we choose number $y = G^{-1}(u)$, which G(y) is the cdf (2.4) with $\theta = 1$, where number of simulations is 10,000.

From the process of generating distribution numbers, we simulate MSE's of AML $\hat{\alpha}$ in (2.6) and MME $\tilde{\alpha}$ in (2.7) of skew parameter α when the cdf (2.4) has $\theta = 1$.

Table 2.1 provides simulated MSE's of AML $\hat{\alpha}$ and MME $\tilde{\alpha}$ in the skewed uniform distribution for $\theta = 1$, when $n = 10(10)30$, $\alpha = c = \pm 1.0, \pm 0.5$. We observe the following from Table 2.1:

Table 2.1 MSE of AML and MME of skew parameter in the skewed uniform density (2.3) with $\theta = 1$

(i) $\alpha = 1.0 = c$			(ii) $\alpha = -1.0 = c$		
n	AML	MME	n	AML	MME
10	0.008049	0.019250	10	0.009823	0.010754
20	0.002781	0.010853	20	0.001847	0.010147
30	0.000797	0.010135	30	0.000720	0.009632

(iii) $\alpha = 0.5 = c$			(iv) $\alpha = -0.5 = c$		
n	AML	MME	n	AML	MME
10	0.009548	0.027532	10	0.007894	0.027175
20	0.005725	0.009995	20	0.006052	0.014119
30	0.000603	0.009002	30	0.000516	0.006529

Fact 2.2 For the density (2.3) with $\theta = 1$, AML $\hat{\alpha}$ performs better than MME $\tilde{\alpha}$ in the sense of MSE when $c = \alpha = \pm 1.0,\ \pm 0.5$ and $n = 10(10)30$.

2.2. Estimation of reliability

In this Section, we consider estimation of right-tail probability and reliability in a skewed uniform having the density (2.3) with skewed parameter α ($|\alpha| \le 1$). First we consider estimation of right-tail probability in the skewed uniform. From the cdf (2.4) of skewed uniform variable Z, right-tail probability $R(t;\alpha) = P(Z > t)$ of Z is given by:

$$R(t;\alpha) = 1 - \frac{\alpha}{4\theta^2}(z^2 - \theta^2) - \frac{1}{2\theta}(z + \theta), \quad \text{if } |z| \le \theta. \tag{2.8}$$

Let $\theta = 1$ in the density (2.3). Since $R(t;\alpha)$ in (2.8) is a monotone function of α , inference on $R(t;\alpha)$ is equivalent to inference on α (McCool, 1991). When we replace AML $\hat{\alpha}$ and MME $\tilde{\alpha}$ instead of α in $R(t;\alpha)$, we observe the following from Fact 2.2:

Fact 2.3 If the density (2.3) has $\theta = 1$ and $|\alpha| \le 1$. then estimator $\hat{R}(t;\alpha) \equiv R(t;\hat{\alpha})$ performs better than $\tilde{R}(t;\alpha) \equiv R(t;\tilde{\alpha})$ in the sense of MSE when $\alpha = \pm 1.0, \pm 0.5$ and n is 10(10)30.

Here we consider estimation of reliability in independent skewed uniforms each with different skew parameters and common known θ. Let Z and W be independent skewed uniform random variables, which they have the density (2.3) each with common $\theta = 1$ and different skew parameters α_1 and $\alpha_2(|\alpha_i| \leq 1)$, respectively. Then from the density (2.3) and the cdf (2.4), we obtain reliability $P(Z < W)$:

For $|\alpha_i| \leq 1$, i=1 and 2,

$$R(\alpha_1, \alpha_2) \equiv P(Z < W) = \frac{1}{6}\eta + \frac{1}{2}, \quad \text{where} \eta \equiv \alpha_2 - \alpha_1. \tag{2.9}$$

Especially if Z and W are identical random variables with $\alpha_1 = \alpha_2$, then it is no wonder that the reliability is 1/2. Since $R(\alpha_1, \alpha_2)$ in (2.9) is a monotone function of η, inference on $R(\alpha_1, \alpha_2)$ is equivalent to inference on η (McCool, 1991). It's sufficient for us to consider estimation of η instead of estimating $R(\alpha_1, \alpha_2)$.

Assume $Z_1, Z_2, \ldots, Z_n$ and $W_1, W_2, \ldots, W_m$ be two independent random samples each with the densities $f(z; \alpha_1)$ and $f(w; \alpha_2)$ in (2.3), which they have common $\theta = 1$ and different skew parameters α_1 and α_2 ($|\alpha_i| \leq 1$), respectively.

Then from using AML $\hat{\alpha}$ in (2.6) and MME $\tilde{\alpha}$ of α in (2.7), the following proposed estimators of η are defined by:

$$\hat{\eta} \equiv \hat{\alpha_2} - \hat{\alpha_1} \text{and } \tilde{\eta} \equiv \tilde{\alpha_2} - \tilde{\alpha_1}, \tag{2.10}$$

where $\hat{\alpha_1} = c_1 - \sum_{i=1}^{n} h(c_1|Z_i)/\sum_{i=1}^{n} h'(c_1|Z_i)$, $\hat{\alpha_2} = c_2 - \sum_{i=1}^{m} h(c_2|W_i)/\sum_{i=1}^{m} h'(c_2|W_i)$, $\tilde{\alpha_1} = 3\sum_{i=1}^{n} Z_i/n$, and $\tilde{\alpha_2} = 3\sum_{i=1}^{m} W_i/m$.

Through simulations of MSE's of estimators $\hat{\eta}$ and $\tilde{\eta}$ by the process of generating distribution numbers as in Section 2.1 when the cdf (2.4) has $\theta = 1$, Table 2.2 provides simulated MSE's of estimators $\hat{\eta}$ and $\tilde{\eta}$ in the skewed uniform distribution with common $\theta = 1$ and different skew parameters α_1 and α_2 ($|\alpha_i| \leq 1$), when n and m are 10(10)30 and $(\alpha_1, \alpha_2) = (c_1, c_2) = (1.0.5), (-1, -0.5), (1-1), (-0.5, 0.5)$. From Table 2.2 and equivalence between inferences of $R(\alpha_1, \alpha_2)$ and η in McCool (1991), we observe the following:

Fact 2.4 For the densities (2.3) with common $\theta = 1$ and different skew parameters α_1 and $\alpha_2(|\alpha_i| \leq 1)$, $\hat{R} = R(\hat{\alpha}_1, \hat{\alpha_2})$ performs better in the sense of MSE than $\tilde{R} = R(\tilde{\alpha}_1, \tilde{\alpha_2})$, when $(\alpha_1, \alpha_2) = (c_1, c_2) = (1.0.5), (-1, -0.5), (1, -1), (-0.5, 0.5)$, and $n(m) = 10(10)30$.

3. A skewed uniform distribution generated by a Laplace kernel

Let X and Y be two independent identical distributed continuous random variables each with the density $f(x) = F'(x)$ of X and the density $g(x) = G'(x)$ of Y, which they are symmetric distributions about origin. Then the following density is as given by Ali et al (2008),

$$f(z; \alpha) \equiv 2f(z)G(\alpha z), \text{for any } \alpha \in R^1, \tag{3.1}$$

which is a skewed density generated by a symmetric kernel distribution.

When $f(x) = \dfrac{1}{2\theta}$, if $|x| \leq \theta$, is a uniform density, and a Laplace distribution is given as:

$$G(x) = \frac{1}{2}(1 + sgn(x)(1 - e^{-|x|/\beta})), \; x \in R^1, \quad \beta > 0,$$

Table 2.2 MSE of $\hat{\eta}$and$\tilde{\eta}$ in two independent skewed uniform densities (2.3) with common known $\theta = 1$ and two different skew parameters α_1 and α_2

(i) $\alpha_1 = 1 = c_1, \alpha_2 = 0.5 = c_2, \eta = -0.5$				(ii) $\alpha_1 = -1 = c_1,\ \alpha_2 = -0.5 = c_2, \eta = 0.5$			
n	m	$\hat{\eta}$	$\tilde{\eta}$	n	m	$\hat{\eta}$	$\tilde{\eta}$
10	10	0.002195	0.004529	10	10	0.002078	0.004487
20	0.001883	0.003109	20	0.001893	0.003117		
30	0.001377	0.002891	30	0.001583	0.002848		
20	10	0.001983	0.003800	20	10	0.001825	0.003780
20	0.001665	0.002441	20	0.001709	0.002465		
30	0.001552	0.001934	30	0.001548	0.001991		
30	10	0.001773	0.002359	30	10	0.001745	0.003428
20	0.001549	0.002071	20	0.001636	0.002115		
30	0.001377	0.001673	30	0.001424	0.001713		

(iii) $\alpha_1 = 1 = c_1,\ \alpha_2 = -1 = c_2, \eta = -2.0$				(iv) $\alpha_1 = -0.5 = c_1,\ \alpha_2 = 0.5 = c_2, \eta = 1.0$			
n	m	$\hat{\eta}$	$\tilde{\eta}$	n	m	$\hat{\eta}$	$\tilde{\eta}$
10	10	0.002504	0.003729	10	10	0.001834	0.005239
20	0.002320	0.002773	20	0.001425	0.003848		
30	0.002033	0.002655	30	0.001370	0.003557		
20	10	0.002214	0.003000	20	10	0.001569	0.004219
20	0.002049	0.002628	20	0.001174	0.002720		
30	0.001932	0.002301	30	0.001104	0.002340		
30	10	0.001963	0.002722	30	10	0.001282	0.003627
20	0.001893	0.002415	20	0.001135	0.002366		
30	0.001720	0.002012	30	0.001036	0.001976		

where

$$sgn(x) = \begin{cases} 1, & \text{if } \ x > 0 \\ -1, & \text{if } \ x < 0, \end{cases}$$

then from (3.1), a skewed density is given by:

$$f(z;\alpha) = \frac{1}{2\theta}[1 + sgn(\alpha z)(1 - e^{-|\alpha z|/\beta})], \quad \text{if } |z| \le \theta, \tag{3.2}$$

which is a skewed uniform density generated by a Laplace kernel, whose random variable is denoted by Z.

From the density (3.2) we consider the cdf of Z and k-th moment of Z as follows:

(a) The cdf of Z is given by:

$$F(z;\alpha) = 1 - I(\alpha) + \frac{z}{\theta}I(\alpha z) + \frac{\beta}{2\alpha\theta}(e^{-|\alpha z|/\beta} - e^{-|\alpha|\theta/\beta}),$$

where

$$I(x) = \begin{cases} 1, & \text{if } \ x > 0 \\ 0, & \text{if } \ x < 0. \end{cases}$$

(b) From formula 8.352(1) in Gradshteyn and Ryzhik (1965, p.940), k-th moment of Z is given by:

For $\alpha > 0$,

$$E(Z^k;\alpha) = \frac{1}{k+1}\theta^k + \frac{(-1)^k - 1}{2\theta}(\frac{\beta}{\alpha})^{k+1} \cdot \gamma(k+1, \alpha\frac{\theta}{\beta}), \tag{3.3}$$

where $\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt$, $a > 0$ in Gradshteyn and Ryzhik (1965), and we have known the following in Ali and Woo (2006) as below:

$$E(Z^k;\alpha) = (-1)^k E(Z^k;-\alpha), \quad \text{for} \quad \alpha < 0. \tag{3.4}$$

From the moment (3.3) and the result (3.4), Table 3.1 provides mean, variance, and skewness for the skewed uniform distribution generated by a Laplace kernel, which it has the density (3.2) with $\theta = 1$, when $\alpha = \pm 1/8, \pm 1/4, \pm 1/2, \pm 1, \pm 2, \pm 4, \pm 8$. And hence from Table 3.1, we observe the following:

Table 3.1 Mean, variance, and skewness of a skewed uniform generated by a Laplace kernel, whose density (3.2) has known $\theta = 1$ and skew parameter α. (signs preserve in its order for each row)

α	mean	variance	skewness
$\pm$ 1/8	$\pm$ 0.03978	0.33175	$\mp$ 0.08325
$\pm$ 1/4	$\pm$ 0.76016	0.32755	$\mp$ 0.15999
$\pm$ 1/2	$\pm$ 0.13918	0.31396	$\mp$ 0.29527
$\pm$ 1	$\pm$ 0.23576	0.27775	$\mp$ 0.50198
$\pm$ 2	$\pm$ 0.35150	0.20978	$\mp$ 0.71002
$\pm$ 4	$\pm$ 0.44322	0.13669	$\mp$ 0.63895
$\pm$ 8	$\pm$ 0.48442	0.09867	$\mp$ 0.27311

Fact 3.1 When the density (3.2) has $\theta = 1$, the density (3.2) is skewed to the left when $\alpha > 0$, but the density is skewed to the right when $\alpha < 0$.

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