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The exponentiated extreme value distribution†

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Abstract

This paper deals with properties of the exponentiated extreme value distribution. We derive the approximate maximum likelihood estimators of the scale parameter and location parameter of the exponentiated extreme value distribution based on multiply Type-II censored samples. We compare the proposed estimators in the sense of the mean squared error for various censored samples.

Keywords: Approximate maximum likelihood estimator, exponentiated extreme value distribution, multiply type-II censored sample.

1. Introduction

The idea of exponentiated distribution was introduced by Gupta *et al.* (1998), who discussed a new family of distributions termed as exponentiated exponential (EE) distribution. Gupta and Kundu (1999) introduced the three-parameter generalized exponential (GE) distribution which include location, scale, and shape parameters and studied the theoretical properties of this family. Gupta and Kundu (2000a, 2000b) studied extensively several properties of the GE distribution. Cancho and Bolfarine (2001) considered the two-parameter EE distribution and studied some of its properties. They also compared its properties with the Weibull and gamma distribution, and presented some numerical experimental results. Kang and Park (2005) obtained the approximate maximum likelihood estimator (AMLE) for the exponentiated exponential distribution based on multiply Type-II censored samples when shape parameter is known. Kang (2005) proposed some estimators for the extreme value distribution based on multiply Type-II censored samples. Kundu *et al.* (2005) considered the problem of discriminating between the log-normal and GE distributions of using the ratio of the maximized likelihood. Recently, Ali *et al.* (2007) studied the properties of some new

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exponentiated distributions. Han *et al.* (2007) derived the AMLEs of the scale parameter and location parameter in an exponentiated logistic distribution based on multiply Type-II censored samples. And they also proposed the estimators of the reliability function using the proposed AMLEs.

The exponentiated extreme value (EEV) distribution has the following cumulative distribution function (cdf)

$$F(x;\theta,\sigma,\lambda)=\left[1-\exp\left\{-e^{(x-\theta)/\sigma}\right\}\right]^{\lambda}, \tag{1.1}$$
$$-\infty<x<\infty,\ \ -\infty<\theta<\infty,\ \sigma>0,\ \lambda>0,$$

where θ, σ, and λ are the location, scale and shape parameters respectively.

If X has the distribution function (1.1), then the corresponding probability density function (pdf) is

$$f(x;\theta,\sigma,\lambda)=\frac{\lambda}{\sigma}\exp\left\{-e^{(x-\theta)/\sigma}+\frac{x-\theta}{\sigma}\right\}\left[1-\exp\left\{-e^{(x-\theta)/\sigma}\right\}\right]^{\lambda-1}, \tag{1.2}$$
$$-\infty<x<\infty,\ \ -\infty<\theta<\infty,\ \sigma>0,\ \lambda>0.$$

We denote the EEV distribution with shape parameter λ, scale parameter σ, and location parameter θ as EEV(λ,σ,θ) .

In the special case when $\lambda=1$, the distribution is the extreme value distribution. The extreme value distributions have been used in the analysis of data concerning floods, extreme sea levels, and air pollution problems.

Figure 1.1 shows that the peakedness of the distribution increases as λ increases, and the distribution also shifts to the right as λ increases.

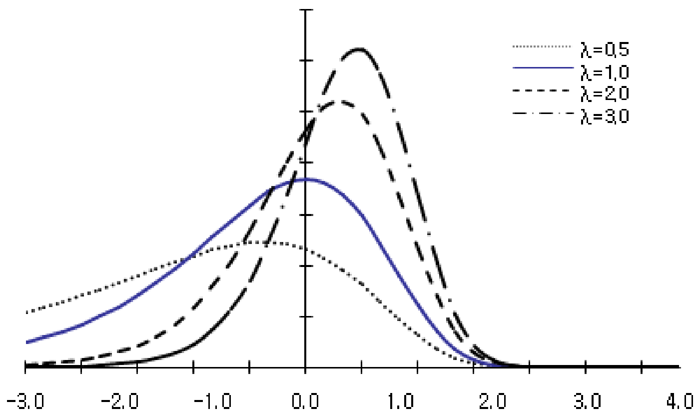

Figure 1.1 The pdfs of EEV distribution with $\sigma=1.0$ and $\theta=0.0$, when $\lambda=0.5,1.0,2.0,3.0$

If $X \sim \text{EEV}(\lambda,\sigma,\theta)$, the survival function and hazard function are given by

$$S(x;\lambda,\sigma,\theta) = 1 - F(x;\lambda,\sigma,\theta) = 1 - \left[1 - \exp\left\{-e^{(x-\theta)/\sigma}\right\}\right]^{\lambda}, \tag{1.3}$$

$$\begin{aligned} h(x;\lambda,\sigma,\theta) &= \frac{f(x;\lambda,\sigma,\theta)}{S(x;\lambda,\sigma,\theta)} \\ &= \frac{\frac{\lambda}{\sigma}\exp\left\{-e^{(x-\theta)/\sigma} + \frac{x-\theta}{\sigma}\right\}\left[1-\exp\left\{-e^{(x-\theta)/\sigma}\right\}\right]^{\lambda-1}}{1-\left[1-\exp\left\{-e^{(x-\theta)/\sigma}\right\}\right]^{\lambda}}. \end{aligned} \tag{1.4}$$

It is easily seen that the hazard function $h(x;\lambda,\sigma,\theta)$ is an increasing function of x for all $\lambda > 0$ (see Figure 1.2).

In this paper, we introduce the three-parameter EEV distribution and studied the theoretical properties of this distribution. We also derive the AMLEs of the scale parameter σ and the location parameter θ based on multiply Type-II censored sample and compare the proposed estimators in the sense of the mean squared error (MSE) for various censored samples.

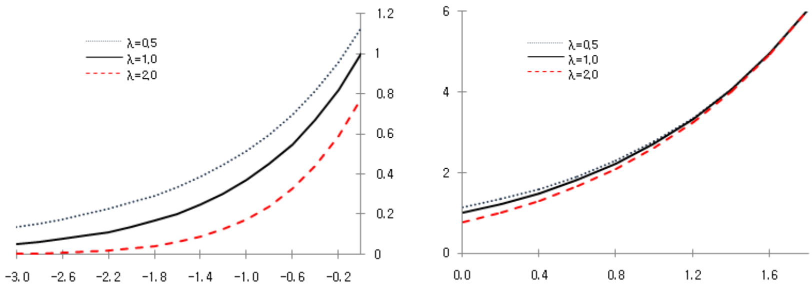

Figure 1.2 The hazard functions of EEV distribution with $\sigma = 1.0$ and $\theta = 0.0$, when $\lambda = 0.5, 1.0, 2.0$

2. Moments

For simplicity and clarity here we assume $\theta = 0$ and $\sigma = 1$ and develop the results for $\text{EEV}(\lambda) = \text{EEV}(\lambda, 1, 0)$, since if $Y \sim \text{EEV}(\lambda)$ then $\theta + \sigma Y = Z \sim \text{EEV}(\lambda,\sigma,\theta)$.

The corresponding moment generating function of Y, is given by

$$\begin{aligned} M_Y(t) = E(e^{tY}) &= E\left(e^{t(Z-\theta)/\sigma}\right) \\ &= \frac{\lambda}{\sigma}\int_{-\infty}^{\infty} e^{t(Z-\theta)/\sigma}\exp\left\{-e^{(z-\theta)/\sigma}\right\}e^{(z-\theta)/\sigma}\left[1-\exp\left\{-e^{(z-\theta)/\sigma}\right\}\right]^{\lambda-1}dz \\ &= \lambda\int_{-\infty}^{\infty} e^{ty}\exp\left\{-e^{y}+y\right\}\left[1-\exp\left\{-e^{y}\right\}\right]^{\lambda-1}dy \end{aligned} \tag{2.1}$$

$$= \lambda \int_{-\infty}^{\infty} \exp\left\{-e^{y}\right\} e^{(t+1)y} \left[1 - \exp\left\{-e^{y}\right\}\right]^{\lambda-1} dy$$

$$= \lambda \sum_{i=1}^{\lambda-1} \begin{pmatrix} \lambda - 1 \\ i \end{pmatrix} (-1)^{i} \int_{0}^{\infty} x^{t} e^{-(i+1)x} dx.$$

Hence, the $M_Z(t) = e^{\theta t} M_Y(\sigma t) = e^{\theta t} \lambda \sum_{i=0}^{\lambda-1} (-1)^i \begin{pmatrix} \lambda - 1 \\ i \end{pmatrix} (i+1)^{-(\sigma t+1)} \Gamma(\sigma t+1), \quad \sigma t > -1.$

Differentiating $\ln M(t)$ and evaluating at $t = 0$, we get the mean and variance of $\mathrm{EEV}(\lambda, \sigma, \theta)$ as

$$E(Z) = \theta - C\sigma + \frac{\sigma \sum_{i=0}^{\lambda-1} (-1)^{i+1} \begin{pmatrix} \lambda - 1 \\ i \end{pmatrix} \ln(i+1)/(i+1)}{\sum_{i=0}^{\lambda-1} (-1)^{i+1} \begin{pmatrix} \lambda - 1 \\ i \end{pmatrix} /(i+1)} \tag{2.2}$$

and

$$\begin{aligned} Var(Z) =& \sigma^2 \frac{\pi^2}{6} + \frac{\sum_{i=0}^{\lambda-1} (-1)^{i} \begin{pmatrix} \lambda - 1 \\ i \end{pmatrix} \left[\sigma \ln(i+1)\right]^2 /(i+1)}{\sum_{i=0}^{\lambda-1} (-1)^{i} \begin{pmatrix} \lambda - 1 \\ i \end{pmatrix} /(i+1)} \\ & - \left[\frac{\sum_{i=0}^{\lambda-1} (-1)^{i+1} \begin{pmatrix} \lambda - 1 \\ i \end{pmatrix} \sigma \ln(i+1)/(i+1)}{\sum_{i=0}^{\lambda-1} (-1)^{i} \begin{pmatrix} \lambda - 1 \\ i \end{pmatrix} /(i+1)} \right]^2, \end{aligned} \tag{2.3}$$

where $C = 0.57721566490$ is Euler's constant.

3. Maximum likelihood estimators

In this section, we obtain the maximum likelihood estimators (MLEs) of three parameter. We consider the three-parameter EEV distribution, and for sake of simplicity we reparametrize $\beta = 1/\sigma$.We denote the MLEs of λ, β, θ as $\tilde{\lambda}$, $\tilde{\beta}$, $\tilde{\theta}$, respectively.

Let $X_1, X_2, \cdots, X_n$ be a random sample from $\mathrm{EEV}(\lambda, 1/\beta, \theta)$.

The log-likelihood function is

$$\begin{aligned} \ln L(x; \lambda, \beta, \theta) =& n \ln(\lambda) + n \ln(\beta) + \sum_{i=1}^{n} \left\{ -e^{(x_i - \theta)\beta} + (x_i - \theta)\beta \right\} \\ & + (\lambda - 1) \sum_{i=1}^{n} \ln \left[1 - \exp\left\{ e^{(x_i - \theta)\beta} \right\} \right]. \end{aligned} \tag{3.1}$$

The derivative with respect to λ, β, and θ and equating to 0. Hence, the likelihood equations are given by normal equation as

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \ln\left[1 - \exp\left\{-e^{(x_i-\theta)\beta}\right\}\right] = 0, \tag{3.2}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \beta} =& \frac{n}{\beta} + \sum_{i=1}^{n}\left\{-(x_i-\theta)e^{(x_i-\theta)\beta} + (x_i-\theta)\right\} \\ &+ (\lambda-1)\sum_{i=1}^{n}\frac{(x_i-\theta)\exp\left\{-e^{(x_i-\theta)\beta}+(x_i-\theta)\beta\right\}}{1-\exp\left\{-e^{(x_i-\theta)\beta}\right\}} = 0,\end{aligned} \tag{3.3}$$

$$\frac{\partial \ln L}{\partial \theta} = \sum_{i=1}^{n}\left\{\beta e^{(x_i-\theta)\beta}\right\} - n\beta - (\lambda-1)\beta\sum_{i=1}^{n}\frac{\exp\left\{-e^{(x_i-\theta)\beta}+(x_i-\theta)\beta\right\}}{1-\exp\left\{-e^{(x_i-\theta)\beta}\right\}} = 0. \tag{3.4}$$

From the equation (3.2), we obtain the maximum likelihood estimator of λ as follows;

$$\tilde{\lambda} = -\frac{n}{\sum_{i=1}^{n}\ln\left[1-\exp\left\{-e^{(x_i-\tilde{\theta})\tilde{\beta}}\right\}\right]}. \tag{3.5}$$

Since the likelihood equations are very complicated, the equations do not admit some explicit solutions. So we can obtain the maximum likelihood estimates by using the numerically methods. We evaluated the mean squared errors of using the bisection method. These values are given in Table 4.2.

4. Approximate maximum likelihood estimators

Let us assume that the multiply Type-II censored sample from a sample of size n is

$$X_{a_1:n} < X_{a_2:n} < \cdots < X_{a_s:n}, \tag{4.1}$$

where $1 \le a_1 < a_2 < \ldots < a_s \le n$.

$$a_0 = 0,\ a_{s+1} = n+1,\ F(x_{a_0:n}) = 0,\ F(x_{a_{s+1}:n}) = 1. \tag{4.2}$$

The likelihood function based on the multiply Type-II censored sample (4.1) can be written as

$$L = n!\prod_{j=1}^{s} f(x_{a_j:n})\prod_{j=1}^{s+1}\frac{\left[F(x_{a_j:n}) - F(x_{a_{j-1}:n})\right]^{a_j-a_{j-1}-1}}{(a_j-a_{j-1}-1)!}. \tag{4.3}$$

The random variable $Z_{i:n} = (X_{i:n}-\theta)/\sigma$ has a standard exponentiated extreme value distribution with pdf and cdf;

$$f(z) = \lambda\exp\{-e^z+z\}\left[1-\exp\{-e^z\}\right]^{\lambda-1}, F(z) = \left[1-\exp\{-e^z\}\right]^{\lambda},\ \ -\infty < z < \infty.$$

Since the $f(z)$ and $F(z)$ satisfy as

$$f'(z) = f(z)\left[1 - e^z + \frac{f(z)}{F(z)}\left(1-\frac{1}{\lambda}\right)\right], \tag{4.4}$$

we obtain the estimating equations as

$$\begin{aligned}\frac{\partial lnL}{\partial\sigma} =& -\frac{1}{\sigma}\left[s+(a_1-1)\frac{f(z_{a_1:n})}{F(z_{a_1:n})}z_{a_1:n}-(n-a_s)\frac{f(z_{a_s:n})}{1-F(z_{a_s:n})}z_{a_s:n}\right. \\ &+\sum_{j=1}^{s}z_{a_j:n}-\sum_{j=1}^{s}e^{z_{a_j:n}}z_{a_j:n}+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\frac{f(z_{a_j:n})}{F(z_{a_j:n})}z_{a_j:n} \\ &\left.+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(z_{a_j:n})z_{a_j:n}-f(z_{a_{j-1}:n})z_{a_{j-1}:n}}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}\right] \\ =&0.\end{aligned} \tag{4.5}$$

Since the likelihood equations is very complicated, the equation (4.5) does not admit an explicit solution for σ.

Let $\xi_i = F^{-1}(p_i) = \ln\left[-\ln\left(1-p_i^{1/\lambda}\right)\right]$, where $p_i = i/(n+1)$, $q_i = 1-p_i$, λ is known. Further, we may expand the following functions in Taylor series around the points ξ_{a_1}, ξ_{a_s}, ξ_{a_j} and ($\xi_{a_{j-1}}$, ξ_{a_j}) respectively.

First, we can approximate these functions by

$$\frac{f(z_{a_s:n})}{1-F(z_{a_s:n})}z_{a_s:n} \simeq \alpha_1+\beta_1 Z_{a_s:n}, \tag{4.6}$$

$$\frac{f(z_{a_j:n})}{F(z_{a_j:n})}z_{a_j:n} \simeq \kappa_{1j}+\delta_{1j} Z_{a_j:n}, \tag{4.7}$$

$$e^{z_{a_j:n}}z_{a_j:n} \simeq \kappa_{2j}+\delta_{2j} Z_{a_j:n}, \tag{4.8}$$

$$\frac{f(z_{a_j:n})z_{a_j:n}-f(z_{a_{j-1}:n})z_{a_{j-1}:n}}{F(z_{a_j:n})-F(z_{a_{j-1}:n})} \simeq \alpha_{1j}+\beta_{1j}Z_{a_j:n}+\gamma_{1j}Z_{a_{j-1}:n}, \tag{4.9}$$

where

$$\begin{aligned}\alpha_1 &= -\frac{\xi_{a_s}^2}{q_{a_s}}\left[f'(\xi_{a_s})+\frac{f^2(\xi_{a_s})}{q_{a_s}}\right],\ \beta_1 = \frac{1}{q_{a_s}}\left[f(\xi_{a_s})+\xi_{a_s}f'(\xi_{a_s})+\frac{f^2(\xi_{a_s})}{q_{a_s}}\xi_{a_s}\right], \\ \kappa_{1j} &= \frac{-\xi_{a_j}^2}{p_{a_j}}\left[f'(\xi_{a_j})-\frac{f^2(\xi_{a_j})}{p_{a_j}}\right],\ \delta_{1j} = \frac{1}{p_{a_j}}\left[f(\xi_{a_j})+\xi_{a_j}f'(\xi_{a_j})-\frac{f^2(\xi_{a_j})}{p_{a_j}}\xi_{a_j}\right], \\ \kappa_{2j} &= -e^{\xi_{a_j}}{\xi_{a_j}}^2, \delta_{2j} = e^{\xi_{a_j}}\left[1+\xi_{a_j}\right],\ \alpha_{1j} = K_j^2-\frac{\xi_{a_j}^2 f'(\xi_{a_j})-\xi_{a_{j-1}}^2 f'(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}, \\ \beta_{1j} &= \frac{1}{p_{a_j}-p_{a_{j-1}}}\left[(1-K_j)f(\xi_{a_j})+\xi_{a_j}f'(\xi_{a_j})\right], \\ \gamma_{1j} &= -\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[(1-K_j)f(\xi_{a_{j-1}})+\xi_{a_{j-1}}f'(\xi_{a_{j-1}})\right], \\ K_j &= \frac{\xi_{a_j}f(\xi_{a_j})-\xi_{a_{j-1}}f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}.\end{aligned}$$

By substituting the equations (4.6), (4.7), (4.8), and (4.9) into the equation (4.5), we can derive an estimator of σ as follows;

$$\hat{\sigma}_1 = \frac{-B_1 + C_1\hat{\theta}}{A_1}, \tag{4.10}$$

where

$$\begin{aligned}
A_1 =& s + (a_1 - 1)\kappa_{11} - (n - a_s)\alpha_1 - \sum_{j=1}^{s}\kappa_{2j} + \left(1 - \frac{1}{\lambda}\right)\sum_{j=1}^{s}\kappa_{1j} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\alpha_{1j}, \\
B_1 =& (a_1 - 1)\delta_{11}X_{a_1:n} - (n - a_s)\beta_1 X_{a_s:n} + \sum_{j=1}^{s}X_{a_j:n} - \sum_{j=1}^{s}\delta_{2j}X_{a_j:n} \\
& + \left(1 - \frac{1}{\lambda}\right)\sum_{j=1}^{s}\delta_{1j}X_{a_j:n} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_{1j}X_{a_j:n} + \gamma_{1j}X_{a_{j-1}:n}), \\
C_1 =& (a_1 - 1)\delta_{11} - (n - a_s)\beta_1 + s - \sum_{j=1}^{s}\delta_{2j} \\
& + \left(1 - \frac{1}{\lambda}\right)\sum_{j=1}^{s}\delta_{1j} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_{1j} + \gamma_{1j}).
\end{aligned}$$

Second, we can approximate these functions by

$$\frac{f(z_{a_s:n})}{1 - F(z_{a_s:n})} \simeq \alpha_2 + \beta_2 Z_{a_s:n}, \tag{4.11}$$

$$\frac{f(z_{a_j:n})}{F(z_{a_j:n})} \simeq \kappa_{3j} + \delta_{3j} Z_{a_j:n}, \tag{4.12}$$

$$e^{z_{a_j:n}} \simeq \kappa_{4j} + \delta_{4j} Z_{a_j:n}, \tag{4.13}$$

where

$$\begin{aligned}
\alpha_2 &= \frac{1}{q_{a_s}}\left[f(\xi_{a_s}) - \xi_{a_s}f'(\xi_{a_s}) - \frac{f^2(\xi_{a_s})}{q_{a_s}}\xi_{a_s}\right], \;\; \beta_2 = \frac{1}{q_{a_s}}\left[f'(\xi_{a_s}) + \frac{f^2(\xi_{a_s})}{q_{a_s}}\right], \\
\kappa_{3j} &= \frac{1}{p_{a_j}}\left[f(\xi_{a_j}) - \xi_{a_j}f'(\xi_{a_j}) + \frac{f^2(\xi_{a_j})}{p_{a_j}}\xi_{a_j}\right], \;\; \delta_{3j} = \frac{1}{p_{a_j}}\left[f'(\xi_{a_j}) - \frac{f^2(\xi_{a_j})}{p_{a_j}}\right], \\
&\kappa_{4j} = e^{\xi_{a_j}}\left[1 - \xi_{a_j}\right], \delta_{4j} = e^{\xi_{a_j}}.
\end{aligned}$$

By substituting the equations (4.9), (4.11), (4.12), and (4.13) into the equation (4.5), we can derive another estimator of σ as follows;

$$\hat{\sigma_2} = \frac{-B_2 + \sqrt{B_2^2 - 4A_2C_2}}{2A_2}, \tag{4.14}$$

where

$$
\begin{aligned}
A_2 =& s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\alpha_{1j},\\
B_2 =& (a_1-1)\kappa_{31}X_{a_1:n}-(n-a_s)\alpha_2 X_{a_s:n}+\sum_{j=1}^{s}X_{a_j:n}-\sum_{j=1}^{s}\kappa_{4j}X_{a_j:n}\\
&+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\kappa_{3j}X_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{1j}X_{a_j:n}+\gamma_{1j}X_{a_{j-1}:n})\\
&-\Bigg[(a_1-1)\kappa_{31}-(n-a_s)\alpha_2+s-\sum_{j=1}^{s}\kappa_{4j}+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\kappa_{3j}\\
&+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{1j}+\gamma_{1j})\Bigg]\hat{\theta},\\
C_2 =& (a_1-1)\delta_{31}(X_{a_1:n}-\hat{\theta})^2-(n-a_s)\beta_2(X_{a_s:n}-\hat{\theta})^2\\
&-\sum_{j=1}^{s}\delta_{4j}(X_{a_j:n}-\hat{\theta})^2+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\delta_{3j}(X_{a_j:n}-\hat{\theta})^2.
\end{aligned}
$$

Third, we can approximate these functions by

$$\frac{f(z_{a_j:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}\simeq \alpha_{2j}+\beta_{2j}Z_{a_j:n}+\gamma_{2j}Z_{a_{j-1}:n} \tag{4.15}$$

and

$$\frac{f(z_{a_{j-1}:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}\simeq \alpha_{3j}+\beta_{3j}Z_{a_j:n}+\gamma_{3j}Z_{a_{j-1}:n}, \tag{4.16}$$

where

$$
\begin{aligned}
&\alpha_{2j}=\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[(1+K_j)f(\xi_{a_j})-\xi_{a_j}f'(\xi_{a_j})\right],\ \beta_{2j}=\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[f'(\xi_{a_j})-\frac{f^2(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\right],\\
&\gamma_{2j}=\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{[p_{a_j}-p_{a_{j-1}}]^2},\ \alpha_{3j}=\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[(1+K_j)f(\xi_{a_{j-1}})-\xi_{a_{j-1}}f'(\xi_{a_{j-1}})\right],\\
&\beta_{3j}=-\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{[p_{a_j}-p_{a_{j-1}}]^2}=-\gamma_{2j},\ \gamma_{3j}=\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[f'(\xi_{a_{j-1}})+\frac{f^2(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\right].
\end{aligned}
$$

By substituting the equations (4.6), (4.7), (4.8), (4.15), and (4.16) into the equation (4.5), we can derive the other estimator of σ as follows;

$$\hat{\sigma_3}=\frac{-B_3+\sqrt{B_3^2-4A_3C_3}}{2A_3}, \tag{4.17}$$

where

$$
\begin{aligned}
A_3 =& s+(a_1-1)\kappa_{11}-(n-a_s)\alpha_1-\sum_{j=1}^{s}\kappa_{2j}+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\kappa_{1j},\\
B_3 =& (a_1-1)\delta_{11}X_{a_1:n}-(n-a_s)\beta_1X_{a_s:n}+\sum_{j=1}^{s}X_{a_j:n}-\sum_{j=1}^{s}\delta_{2j}X_{a_j:n}\\
&+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\delta_{1j}X_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{2j}X_{a_j:n}-\alpha_{3j}X_{a_{j-1}:n})\\
&-\Bigg[(a_1-1)\delta_{11}-(n-a_s)\beta_1+s-\sum_{j=1}^{s}\delta_{2j}+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\delta_{1j}\\
&+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{2j}-\alpha_{3j})\Bigg]\hat{\theta},\\
C_3 =& \sum_{j=1}^{s}(a_j-a_{j-1}-1)\Big\{\beta_{2j}(X_{a_j:n}-\hat{\theta})^2+2\gamma_{2j}(X_{a_j:n}-\hat{\theta})(X_{a_{j-1}:n}-\hat{\theta})\\
&-\gamma_{3j}(X_{a_{j-1}:n}-\hat{\theta})^2\ \Big\}.
\end{aligned}
$$

By substituting the equations (4.11), (4.12), (4.13), (4.15), and (4.16) into the equation (4.5), we can derive the other estimator of σ as follows;

$$
\hat{\sigma_4}=\frac{-B_4+\sqrt{B_4^2-4sC_4}}{2s}, \tag{4.18}
$$

where

$$
\begin{aligned}
B_4 =& (a_1-1)\kappa_{31}X_{a_1:n}-(n-a_s)\alpha_2X_{a_s:n}+\sum_{j=1}^{s}X_{a_j:n}-\sum_{j=1}^{s}\kappa_{4j}X_{a_j:n}\\
&+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\kappa_{3j}X_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{2j}X_{a_j:n}-\alpha_{3j}X_{a_{j-1}:n})\\
&-\Bigg[(a_1-1)\kappa_{31}-(n-a_s)\alpha_2+s-\sum_{j=1}^{s}\kappa_{4j}+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\kappa_{3j}\\
&+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}-\alpha_{2j})\Bigg]\hat{\theta},\\
C_4 =& C_2+C_3.
\end{aligned}
$$

From the equation (4.3), on differentiating the log-likelihood equation for θ is obtained as

$$\begin{aligned}\frac{\partial lnL}{\partial\theta} =& -\frac{1}{\sigma}\left[(a_1-1)\frac{f(z_{a_1:n})}{F(z_{a_1:n})}-(n-a_s)\frac{f(z_{a_s:n})}{1-F(z_{a_s:n})}+s-\sum_{j=1}^{s}e^{z_{a_j:n}}\right.\\ &\left.+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\frac{f(z_{a_j:n})}{F(z_{a_j:n})}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(z_{a_j:n})-f(z_{a_{j-1}:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}\right]\\ &=0.\end{aligned} \tag{4.19}$$

The equation (4.19) does not admit an explicit solution for θ. But we can expand the function $(f(z_{a_j:n})-f(z_{a_{j-1}:n}))/(F(z_{a_j:n})-F(z_{a_{j-1}:n}))$ as follows;

$$\frac{f(z_{a_j:n})-f(z_{a_{j-1}:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}\simeq\alpha_{4j}+\beta_{4j}Z_{a_j:n}+\gamma_{4j}Z_{a_{j-1}:n}, \tag{4.20}$$

where $\alpha_{4j}=\alpha_{2j}-\alpha_{3j}$, $\beta_{4j}=\beta_{2j}-\beta_{3j}$, $\gamma_{4j}=\gamma_{2j}-\gamma_{3j}$.

By substituting the equations (4.11), (4.12), (4.13), and (4.20) into the equation (4.19), we can derive an estimator of θ as follows;

$$\hat{\theta}=\frac{A_1B_0-A_0B_1}{A_0C_1-A_1C_0}, \tag{4.21}$$

where

$$\begin{aligned}A_0 =&(a_1-1)\kappa_{31}-(n-a_s)\alpha_2+s-\sum_{j=1}^{s}\kappa_{4j}+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\kappa_{3j}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\alpha_{4j},\\ B_0 =&(a_1-1)\delta_{31}X_{a_1:n}-(n-a_s)\beta_2X_{a_s:n}-\sum_{j=1}^{s}\delta_{4j}X_{a_j:n}+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\delta_{3j}X_{a_j:n}\\ &+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{4j}X_{a_j:n}+\gamma_{4j}X_{a_{j-1}:n}),\\ C_0 =&(a_1-1)\delta_{31}-(n-a_s)\beta_2-\sum_{j=1}^{s}\delta_{4j}+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{s}\delta_{3j}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{4j}+\gamma_{4j}).\end{aligned}$$

From the above formula, the mean squared errors of the estimators are simulated by Monte Carlo method for sample size n =20 and 50, and various choices of censoring. These values are given in Table 4.1. From Table 4.1, the estimators $\hat{\sigma}_2$ and $\hat{\sigma}_4$ are generally more efficient than the estimators $\hat{\sigma}_1$ and $\hat{\sigma}_3$ in the sense of the MSE. The estimator $\hat{\sigma}_1$ and $\hat{\theta}$ are the linear function of available order statistics. So the estimator $\hat{\sigma}_1$ is a more simple estimator than the others. The MSEs of the proposed estimators decrease as λ increases. As expected, the MSEs of all estimators decrease as sample size n increases. From Table 4.2, the MSE of the MLE $\tilde{\sigma}$ decrease as λ increases. The MLE $\tilde{\sigma}$ is more efficient than the estimators $\hat{\sigma}_1$, $\hat{\sigma}_2$, $\hat{\sigma}_3$ an $\hat{\sigma}_4$ in the sense of the MSE when $\lambda=1.5,\ 3.0$.But the estimators $\hat{\sigma}_1$, $\hat{\sigma}_2$, $\hat{\sigma}_3$ and $\hat{\sigma}_4$ are more efficient than the MLE $\tilde{\sigma}$ in the sense of the MSE when $\lambda=0.5$.

Table 4.1 The relative mean squared errors for the estimators of the location parameter θ and scale parameter σ ($\lambda = 0.5$)

n	k	a_i	MSE $\hat{\theta}$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\sigma}_3$	$\hat{\sigma}_4$
20	0	1∼20	0.111274	0.037224	0.035546	0.037224	0.035546
	2	1∼18	0.128542	0.044130	0.043801	0.044130	0.043801
		3∼20	0.111178	0.040714	0.038709	0.040714	0.038709
		2∼19	0.118780	0.042508	0.041675	0.042508	0.041675
	4	2∼17	0.144669	0.050849	0.050516	0.050849	0.050516
		4∼19	0.118602	0.046868	0.045832	0.046868	0.045832
		3∼18	0.128565	0.049054	0.048595	0.049054	0.048595
		2∼4, 7∼14, 16∼20	0.111147	0.039013	0.037211	0.040173	0.040602
	5	3∼17	0.144837	0.053929	0.053553	0.053929	0.053553
		4∼18	0.128554	0.051835	0.051267	0.051835	0.051267
		2∼6 10∼19	0.118757	0.042836	0.042065	0.043793	0.044200
	6	4∼17	0.145125	0.057329	0.056862	0.057329	0.056862
		1 2 6∼9 12∼15 17∼20	0.111248	0.037767	0.036315	0.042147	0.043843
50	0	1∼50	0.040727	0.014471	0.014079	0.014471	0.014079
	2	1∼48	0.042719	0.015776	0.015647	0.015776	0.015647
		3∼50	0.040736	0.015022	0.014601	0.015022	0.014601
		2∼49	0.041610	0.015426	0.015221	0.015426	0.015221
	4	2∼47	0.044030	0.016761	0.016654	0.016761	0.016654
		4∼49	0.041601	0.015943	0.015719	0.015943	0.015719
		3∼48	0.042720	0.016401	0.016258	0.016401	0.016258
		2∼4 7∼14 16∼50	0.040748	0.014760	0.014363	0.015196	0.015244
	5	3∼47	0.044027	0.017106	0.016992	0.017106	0.016992
		4∼48	0.042715	0.016634	0.016486	0.016634	0.016486
		2∼6 10∼19 21∼50	0.040744	0.014775	0.014379	0.015155	0.015211
	6	4∼47	0.044022	0.017364	0.017245	0.017364	0.017245
		1 2 6∼9 12∼15 17∼50	0.040761	0.014536	0.014185	0.015673	0.016061

Table 4.1 Continued ($\lambda = 1.5$)

n	k	a_i	MSE $\hat{\theta}$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\sigma}_3$	$\hat{\sigma}_4$
20	0	1∼20	0.042674	0.031047	0.029599	0.031047	0.029599
	2	1∼18	0.043243	0.036648	0.036055	0.036648	0.036055
		3∼20	0.043531	0.034434	0.032638	0.034434	0.032638
		2∼19	0.043208	0.035617	0.034573	0.035617	0.034573
	4	2∼17	0.044172	0.042576	0.041846	0.042576	0.041846
		4∼19	0.044266	0.040074	0.038826	0.040074	0.038826
		3∼18	0.043935	0.041277	0.040402	0.041277	0.040402
		2∼4 7∼14 16∼20	0.043128	0.032704	0.031151	0.032436	0.032204
	5	3∼17	0.044544	0.045534	0.044712	0.045534	0.044712
		4∼18	0.044434	0.044172	0.043188	0.044172	0.043188
		2∼6 10∼19	0.043508	0.035767	0.034829	0.035697	0.035497
	6	4∼17	0.044952	0.049043	0.048121	0.049043	0.048121
		1 2 6∼9 12∼15 17∼20	0.043245	0.031527	0.030400	0.032796	0.033601
50	0	1∼50	0.016897	0.012140	0.011836	0.012140	0.011836
	2	1∼48	0.016936	0.013142	0.012995	0.013142	0.012995
		3∼50	0.017028	0.012635	0.012304	0.012635	0.012304
		2∼49	0.016977	0.012898	0.012696	0.012898	0.012696
	4	2∼47	0.017027	0.013919	0.013777	0.013919	0.013777
		4∼49	0.017082	0.013390	0.013168	0.013390	0.013168
		3∼48	0.017062	0.013686	0.013518	0.013686	0.013518
		2∼4 7∼14 16∼50	0.016984	0.012394	0.012085	0.012484	0.012440
	5	3∼47	0.017087	0.014227	0.014077	0.014227	0.014077
		4∼48	0.017103	0.013917	0.013742	0.013917	0.013742
		2∼6 10∼19 21∼50	0.016987	0.012406	0.012098	0.012473	0.012429
	6	4∼47	0.017126	0.014480	0.014321	0.014480	0.014321
		1 2 6∼9 12∼15 17∼50	0.016965	0.012217	0.011954	0.012638	0.012822

Table 4.1 Continued ($\lambda = 3.0$)

			MSE				
n	k	a_i	$\hat{\theta}$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\sigma}_3$	$\hat{\sigma}_4$
20	0	1∼20	0.032661	0.029029	0.027696	0.029029	0.027696
	2	1∼18	0.032874	0.033915	0.033104	0.033915	0.033104
		3∼20	0.035156	0.032703	0.031285	0.032703	0.031285
		2∼19	0.033893	0.033429	0.032439	0.033429	0.032439
	4	2∼17	0.034091	0.039712	0.038889	0.039712	0.038889
		4∼19	0.037092	0.038286	0.037272	0.038286	0.037272
		3∼18	0.035585	0.038902	0.038066	0.038902	0.038066
		2∼4 7∼14 16∼20	0.033879	0.030783	0.029514	0.030008	0.029442
	5	3∼17	0.035700	0.042869	0.042058	0.042869	0.042058
		4∼18	0.037416	0.042033	0.041177	0.042033	0.041177
		2∼6 10∼19	0.034175	0.033485	0.032614	0.033073	0.032584
	6	4∼17	0.037617	0.046635	0.045815	0.046635	0.045815
		1 2 6∼9 12∼15 17∼20	0.033335	0.029360	0.028370	0.029009	0.029091
50	0	1∼50	0.013108	0.011336	0.011076	0.011336	0.011076
	2	1∼48	0.013177	0.012189	0.012032	0.012189	0.012032
		3∼50	0.013457	0.011880	0.011619	0.011880	0.011619
		2∼49	0.013321	0.012039	0.011853	0.012039	0.011853
	4	2∼47	0.013378	0.012915	0.012768	0.012915	0.012768
		4∼49	0.013648	0.012597	0.012410	0.012597	0.012410
		3∼48	0.013536	0.012787	0.012630	0.012787	0.012630
		2∼4 7∼14 16∼50	0.013303	0.011608	0.011360	0.011539	0.011427
	5	3∼47	0.013568	0.013254	0.013108	0.013254	0.013108
		4∼48	0.013683	0.013056	0.012898	0.013056	0.012898
		2∼6 10∼19 21∼50	0.013309	0.011617	0.011368	0.011543	0.011430
	6	4∼47	0.013719	0.013545	0.013399	0.013545	0.013399
		1 2 6∼9 12∼15 17∼50	0.013205	0.011409	0.011192	0.011423	0.011440

Table 4.2 The relative mean squared errors for the MLEs of the location parameter θ and scale parameter σ

	$\lambda = 0.5$		$\lambda = 1.5$		$\lambda = 3.0$	
n	$\tilde{\theta}$	$\tilde{\sigma}$	$\tilde{\theta}$	$\tilde{\sigma}$	$\tilde{\theta}$	$\tilde{\sigma}$
20	0.039995	0.440530	0.039995	0.022182	0.039995	0.014959
50	0.039995	0.410290	0.039995	0.007049	0.039995	0.005313

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