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A NOTE ON LINEAR COMBINATIONS OF AN IDEMPOTENT MATRIX AND A TRIPOTENT MATRIX

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ABSTRACT. Let A_1 and A_2 be nonzero complex idempotent and tripotent matrix, respectively. Denote a linear combination of the two matrices by $A = c_1A_1 + c_2A_2$, where c_1, c_2 are nonzero complex scalars. In this paper, under an assumption of $A_1A_2 = A_2A_1$, we characterize all situations in which the linear combination is tripotent. A statistical interpretation of this tripotent problem is also pointed out. Moreover, In [2], Baksalary characterized all situations in which the above linear combination is idempotent by using the property of decomposition of a tripotent matrix, i.e. if A_2 is tripotent, then $A_2 = B_1 - B_2$, where $B_i^2 = B_i$, $i = 1, 2$ and $B_1B_2 = B_2B_1 = 0$. While in this paper, by utilizing a method different from the one used by Baksalary in [2], we prove the theorem 1 in [2] again.

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1. Introduction

Let C and C_{nn} be the sets of complex numbers and $n \times n$ complex matrices, respectively. It is assumed throughout this paper that c_1, $c_2 \in C$ are both nonzero and $A_1 \in C_{nn}$ is idempotent matrix while $A_2 \in C_{nn}$ is tripotent matrix, i.e. $A_1^2 = A_1$ and $A_2^3 = A_2$. Denote a linear combination of the two matrices by

$$A = c_1A_1 + c_2A_2, \tag{1}$$

where c_1, c_2 are nonzero complex scalars.

Because the idempotency or tripotency of a linear combination of some idempotent or tripotent matrices has its statistical applications, the problems of characterizing some and even all situations where a linear combination of some idempotent or tripotent matrices is an idempotent or a tripotent matrix are studied extensively,(see [1]-[4]). In [1], Baksalary studied the idempotency of

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linear combinations of two idempotent matrices, in [2], Baksalary gave the sufficient and necessary conditions of the idempotency of linear combination of an idempotent matrix and a tripotent matrix by using the property of decomposition of a tripotent matrix, i.e. if A_2 is tripotent, then $A_2 = B_1 - B_2$, where $B_i^2 = B_i$, $i = 1,\ 2$ and $B_1B_2 = B_2B_1 = 0$, in [3], Baksalary characterized all situations in which the linear combination of commuting tripotent matrices is tripotent. Inspired by the above work, this paper deals with the problem of characterizing all situations in which the linear combination (1) with an assumption $A_1A_2 = A_2A_1$ is a tripotent matrix in section 2. While in section 3, by utilizing a method different from the one used by Baksalary in [2], we gave a new proof of the theorem 1 in [2].

It should be emphasized that an essential motivation of this paper originates from statistics, a fact that if A_1 and A_2 are $n\times n$ real symmetric matrices and X is a $n \times 1$ real random vector having the multivatiate normal distribution $N_n(0, I)$, where I stands for the identity matrix of order n, then necessary and sufficient conditions for the quadratic forms $X^{'}A_1X$ and $X^{'}A_2X$ to be distributed as a Chi-square variable and as a difference of two independent Chi-square variables are the idempotency property $A_1^2 = A_1$ and the tripotency property $A_2^3 = A_2$, respectively, see [5] and [6].

2. Main result

As already pointed out, the main result of this section provides a complete solution to the problem of characterizing all situations in which a linear combination (1) with an assumption $A_1A_2 = A_2A_1$ is a tripotent matrix.

Lemma 1. [7] *Suppose A is an idempotent matrix in C_{nn}, then there exists a nonsingular matrix $P \in C_{nn}$, such that*

$$A = P(I_s \oplus 0)P^{-1}, \quad \textit{where } s = \text{rankA}.$$

Lemma 2. [7] *Suppose A is a tripotent matrix in C_{nn}, then there exists a nonsingular matrix $P \in C_{nn}$, such that*

$$A = P(I_p \oplus -I_q \oplus 0)P^{-1}, \quad \textit{where } p + q = \text{rankA}.$$

Theorem 1. *Let A_1 and A_2 be nonzero complex idempotent and tripotent matrix, respectively, and satisfying condition $A_1A_2 = A_2A_1$. Let A be a linear combination of the form (1), with nonzero $c_1,\ c_2 \in C$. Then the following list comprises characterization of all cases in which A is a tripotent matrix:*

(i) $A_1A_2 = -\varepsilon A_1A_2^2$, holds along with $c_1 = 1,\ c_2 = \varepsilon$ or $c_1 = -1,\ c_2 = -\varepsilon$, where $\varepsilon = \pm 1$;

(ii) $A_1A_2 = -\varepsilon A_1$, holds along with $c_1 = 2,\ c_2 = \varepsilon$ or $c_1 = -2,\ c_2 = -\varepsilon$, where $\varepsilon = \pm 1$;

(iii) $A_1A_2 = A_2$, holds along with $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ or $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$ or $c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$ or $c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$.

Proof. The sufficiency is easy to prove by direct calculation. Now, we only prove the necessity. From Lemma 2, there exists a nonsingular matrix $P \in C_{nn}$ such that $A_2 = P(I_p \oplus -I_q \oplus 0)P^{-1}$, where $p+q = \text{rankA}_2$ and the note $'\oplus'$ is a direct sum. Since $A_1A_2 = A_2A_1$ and $A_1^2 = A_1$, A_1 can be represented as $A_1 = P(X_1 \oplus X_2 \oplus X_3)P^{-1}$ with $X_1^2 = X_1 \in C_{pp}$, $X_2^2 = X_2 \in C_{qq}$ and $X_3^2 = X_3 \in C_{tt}$ where $t = n-p-q$. So the tripotency of the linear combination (1) is equivalent to finding the conjunction of the tripotency of $c_1X_1 + c_2I_p$, $c_1X_2 - c_2I_q$ and c_1X_3.

In the consecutive steps, we establish necessary and sufficient conditions ensuring that this conjunction is satisfied, expressed in terms of scalars c_1, c_2 and matrices X_1, X_2, X_3, and then reexpress these conditions in terms of scalars c_1, c_2 and matrices A_1, A_2.

The first step is to characterize all situations in which the linear combination $c_1X_1+c_2I_p$ is tripotent. Similar to the above, by Lemma 1, $X_1 = Q(I_s \oplus 0)Q^{-1}$ where $Q \in C_{pp}$ is a nonsingular matrix and $s = \text{rankX}_1$.

Case 1. Suppose $s = 0$, then $X_1 = 0$, from the tripotency of $c_1X_1 + c_2I_p$, we have $c_2 = \pm 1$ and $c_1 \in C\backslash\{0\}$;

Case 2. Suppose $s = p$, then $X_1 = I_p$, it is easy to get $c_1 + c_2 = 0$ or $c_1 + c_2 = \pm 1$;

Case 3. Suppose $0 < s < p$, in order to ensure the tripotency of $c_1X_1+c_2I_p$, the scalars should satisfy $c_1 = -1,\ c_2 = 1$ or $c_1 = -2,\ c_2 = 1$ or $c_1 = 1,\ c_2 = -1$ or $c_1 = -2,\ c_2 = -1$.

Concluding the above three cases, we have the sufficienct and necessary conditions for the tripotency of $c_1X_1 + c_2I_p$ as follows:

$$X_1 = 0,\ \text{holds along with } c_2 = \pm 1 \text{ and } c_1 \in C\backslash\{0\}; \tag{2}$$

$$X_1 = I_p,\ \text{holds along with } c_1 + c_2 = 0 \text{ or } c_1 + c_2 = \pm 1; \tag{3}$$

$$X_1 = Q(I_s \oplus 0)Q^{-1},\ \text{holds along with } c_1 \in \{-\varepsilon, -2\varepsilon\},\ c_2 = \varepsilon \tag{4}$$

where $0 < s = rankX_1 < p$, and $\varepsilon = \pm 1$.

Similar to the first step, we get the sufficienct and necessary conditions for the tripotency of $c_1X_2 - c_2I_q$ as follows:

$$X_2 = 0,\ \text{holds along with } c_2 = \pm 1 \text{ and } c_1 \in C\backslash\{0\}; \tag{5}$$

$$X_2 = I_q,\ \text{holds along with } c_1 - c_2 = 0 \text{ or } c_1 - c_2 = \pm 1; \tag{6}$$

$$X_2 = Q(I_k \oplus 0)Q^{-1},\ \text{holds along with } c_1 \in \{\varepsilon,\ 2\varepsilon\},\ c_2 = \varepsilon \tag{7}$$

where $0 < k = \text{rankX}_2 < \text{p}$, and $\varepsilon = \pm 1$.

The third step is characterizing all situations in which the tripotency of c_1X_3, there are two situations:

$$X_3 = 0, \text{holds along with } c_1 \in C\backslash\{0\},\ c_2 \in C\backslash\{0\}; \tag{8}$$

$$X_3 \neq 0, \text{holds along with } c_1 = \pm 1,\ c_2 \in C\backslash\{0\}. \tag{9}$$

The present step is to establish necessary and sufficient conditions ensuring the simutaneous tripotency of $c_1X_1 + c_2I_p$, $c_1X_2 - c_2I_q$ and c_1X_3. This aim is accomplished by combining each of (2)-(4) with (5)-(7), then combining each of the corresponding results with (8) and (9).

Observing that each combination of ((2), (5), (8)), ((3), (6), (9)), ((3),(7)), ((4),(6)) and ((4), (7)) is a contradiction.

Combining (2) with a version of (6) having $c_1 - c_2 = 0$ and (8) , yields $c_1 = c_2 = \pm 1$ and $A_1A_2 = -A_1A_2^2$. The same characterization follows by combining (2) with a version of (7) having $c_1 = c_2 = \pm 1$ and (8) or (9). Combining (3) under $c_1 + c_2 = 0$ with (5) and (8), entails $c_1 = -c_2 = \pm 1$ and $A_1A_2 = A_1A_2^2$. The same characterization follows by combining (4) under $c_1 = -c_2 = \pm 1$ with (5) and (8) or (9). Hence, consequently taking $\varepsilon = \pm 1$, we obtain the characterization (i). Combining (2)with (5) and (9), or combining (2) with (6) and (9) also get the situation (i).

Combining again (2) with a version of (6) having $c_1 - c_2 = \pm 1$ and (8), yields $c_1 = 2$, $c_2 = 1$ or $c_1 = -2$, $c_2 = -1$ and $A_1A_2 = -A_1$. The same characterization follows by combining (2) with (7) under $c_1 = 2$, $c_2 = 1$ or $c_1 = -2$, $c_2 = -1$ and (8). Combining (3) under $c_1 + c_2 = \pm 1$ with (5) and (8) , entails $c_1 = -2$, $c_2 = 1$ or $c_1 = 2$, $c_2 = -1$ and $A_1A_2 = A_1$. The same characterization follows by combining (4) under $c_1 = -2$, $c_2 = 1$ or $c_1 = 2$, $c_2 = -1$ with (5) and (8). Hence, consequently taking $\varepsilon = \pm 1$, we obtain the characterization (ii).

Combining (3) with (6) and (8) can get the characterization (iii). The proof is completed. □

Theorem 1 shows that a linear combination of two quadratic forms in normal variables, one distributed as a Chi-square variable and another distributed as a difference of two independent Chi-square variable can also be distributed as a difference.

3. Additional result

In this section, we characterize all situations in which a linear combination (1) is idempotent with a new method different from the method used in [2].

Theorem 2. *Let A_1 and A_2 be nonzero complex idempotent and tripotent matrix, respectively. Let A be a linear combination of the form (1), with nonzero c_1, $c_2 \in C$. Then the following list comprises characterization of all cases in which A is an idempotent matrix:*

(i) $A_1A_2 + A_2A_1 + c_2A_2^2 - A_2 = 0$, holds along with $c_1 = 1$, $c_2 \in C\backslash\{0\}$, and when $c_2 \neq \pm 1$, $p = q$, where $p + q = rankA_2$;

(ii) $A_1A_2 + A_2A_1 = A_1 + \frac{1}{2}(A_2^2 + A^2)$, holds along with $c_1 = 2$, $c_2 = -1$;

(iii) $A_1A_2 + A_2A_1 = \frac{1}{2}(A_1 + 2A_2 - A_2^2)$, holds along withc$_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$;

(iv) $A_1A_2 + A_2A_1 = -A_1 + \frac{1}{2}(A_2 - A_2^2)$, holds along with $c_1 = 2$, $c_2 = 1$;

(v) $A_1A_2 + A_2A_1 = A_2^2 + 2A_2 - A_1$, holds along with $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$.

Proof. The sufficiency is obvious. Now, we only prove the necessity. From Lemma 2, there exists a nonsingular matrix $P \in C_{nn}$ such that $A_2 = P(I_p \oplus -I_q \oplus 0)P^{-1}$, where $p + q = \mathrm{rankA}_2$ and the note $'\oplus'$ is a direct sum.

Let

$$A_1 = P \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} P^{-1} \tag{10}$$

where $B_{11} \in C_{pp}$, $B_{12} \in C_{pq}$, $B_{13} \in C_{pt}$, $B_{21} \in C_{qp}$, $B_{22} \in C_{qq}$, $B_{23} \in C_{qt}$, $B_{31} \in C_{tp}$, $B_{32} \in C_{tq}$, $B_{33} \in C_{tt}$. From $A_1^2 = A_1$, we know the following relations:

$$B_{ii}^2 + B_{ij}B_{ji} + B_{ik}B_{ki} = B_{ii} \tag{11}$$

where $i, j, k \in \{1, 2, 3\}$, and $i \neq j \neq k$,

$$B_{ii}B_{ij} + B_{ij}B_{jj} + B_{ik}B_{kj} = B_{ij} \tag{12}$$

where $i, j, k \in \{1, 2, 3\}$, and $i \neq j \neq k$.

Taking the form A_1, A_2 into (1), and from $A^2 = A$ and equations(11), (12), we have

$$c_1(c_1 + 2c_2 - 1)B_{11} = c_2(1 - c_2)I_p, \tag{13}$$

$$c_1(c_1 - 2c_2 - 1)B_{22} = -c_2(1 + c_2)I_q, \tag{14}$$

$$c_1(c_1 - 1)B_{12} = 0,\ c_1(c_1 - 1)B_{21} = 0,\ c_1(c_1 - 1)B_{33} = 0, \tag{15}$$

$$c_1(c_1 + c_2 - 1)B_{13} = 0,\ c_1(c_1 + c_2 - 1)B_{31} = 0, \tag{16}$$

$$c_1(c_1 - c_2 - 1)B_{23} = 0,\ c_1(c_1 - c_2 - 1)B_{32} = 0. \tag{17}$$

Case 1. Suppose $c_1 + 2c_2 - 1 = 0$, and from (13), we have $c_2(1 - c_2) = 0$, then $c_2 = 1$, $c_1 = -1$. Combining $c_2 = 1$, $c_1 = -1$ with equations (14)-(17), it entails

$$B_{12} = 0,\ B_{13} = 0,\ B_{21} = 0,\ B_{31} = 0,\ B_{23} = 0,\ B_{32} = 0,\ B_{33} = 0,\ B_{22} = -\frac{1}{2}I_q.$$

The result contradicts with the equation (11) where $i = 2$. And the same contradiction by $c_1 - 2c_2 - 1 = 0$.

Case 2. Suppose $c_1 = 1$, because of $c_2 \neq 0$, then $c_1 + c_2 - 1 \neq 0$, $c_1 - c_2 - 1 \neq 0$, $c_1 + 2c_2 - 1 \neq 0$. So

$$B_{13} = 0,\ B_{31} = 0,\ B_{32} = 0,\ B_{23} = 0,\ B_{22} = \frac{1 + c_2}{2}I_q,\ B_{11} = \frac{1 - c_2}{2}I_p.$$

Hence, take the values of these B_{ij}, $i, j \in \{1, 2, 3\}$ into equation (11), we have

$$B_{12}B_{21} = \frac{1 - c_2^2}{4}I_p,\ B_{21}B_{12} = \frac{1 - c_2^2}{4}I_q, \tag{18}$$

$$B_{33}^2 = B_{33},$$

then yield the relation between A_1 and A_2, i.e. $A_1A_2 + A_2A_1 + c_2A_2^2 - A_2 = 0$; when $c_2 \neq \pm 1$, and from (18), we can know $p = q$. So we complete the characterization of (i).

Case 3. Suppose $c_1 + c_2 = 1$, then $c_1 - c_2 - 1 \neq 0,\ c_1 + 2c_2 - 1 \neq 0,\ c_1 - 1 \neq 0,\ c_1 - 2c_2 - 1 \neq 0$, so

$$B_{12} = 0,\ B_{23} = 0,\ B_{32} = 0,\ B_{33} = 0,\ B_{21} = 0,\ B_{11} = I_p,\ B_{22} = \frac{2 - c_1}{3c_1} I_q.$$

Taking these values into equation(11), we have

$$(\frac{2 - c_1}{3c_1})^2 I_q = \frac{2 - c_1}{3c_1} I_q, \tag{19}$$

$$B_{31}B_{13} = 0,\ B_{13}B_{31} = 0,$$

from equation (19), we get $c_1 = 2,\ c_2 = -1$ or $c_1 = \frac{1}{2},\ c_2 = \frac{1}{2}$.

When $c_1 = 2,\ c_2 = -1$, entails $A_1A_2 + A_2A_1 = A_1 + \frac{1}{2}(A_2^2 + A_2)$, we finish the characterization of (ii).

When $c_1 = \frac{1}{2},\ c_2 = \frac{1}{2}$, yields $A_1A_2 + A_2A_1 = \frac{1}{2}(A_1 + 2A_2 - A_2^2)$, it is the form (iii).

Case 4. Suppose $c_1 - c_2 = 1$, then $c_1 + c_2 - 1 \neq 0,\ c_1 - 2c_2 - 1 \neq 0,\ c_1 - 1 \neq 0,\ c_1 + 2c_2 - 1 \neq 0$, so

$$B_{12} = 0,\ B_{13} = 0,\ B_{31} = 0,\ B_{33} = 0,\ B_{21} = 0,\ B_{22} = I_q,\ B_{11} = \frac{2 - c_1}{3c_1} I_p.$$

Similar to the above case, we can prove the situations (iv) and (v).

Case 5. Suppose $c_1 - 1 \neq 0,\ c_1 + c_2 - 1 \neq 0,\ c_1 - c_2 - 1 \neq 0$, then we have

$$B_{ij} = 0, (i \neq j,\ i, j \in \{1, 2, 3\}),\ B_{33} = 0,$$

$$B_{11} = \frac{c_2(1 - c_2)}{c_1(c_1 + 2c_2 - 1)} I_p,\ B_{22} = -\frac{c_2(1 + c_2)}{c_1(c_1 - 2c_2 - 1)} I_q,$$

from equation (11), yields

$$(\frac{c_2(1 - c_2)}{c_1(c_1 + 2c_2 - 1)})^2 I_p = \frac{c_2(1 - c_2)}{c_1(c_1 + 2c_2 - 1)} I_p, \tag{20}$$

$$(-\frac{c_2(1 + c_2)}{c_1(c_1 - 2c_2 - 1)})^2 I_q = -\frac{c_2(1 + c_2)}{c_1(c_1 - 2c_2 - 1)} I_q. \tag{21}$$

When $c_2 = 1$, from equation (21), we get $c_1 = 1$ or $c_1 = 2$, which contradict with what we have supposed. The same $c_2 = -1$ is also a contradiction.

When $c_2 \neq \pm 1$, from equations (20) and (21), by direct calculation, we have $c_1 = \frac{1}{2},\ c_2 = -\frac{1}{2}$ or $c_1 = \frac{1}{2},\ c_2 = \frac{1}{2}$, which also contradict with what we have supposed. So we finish the proof. □

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