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A NOTE ON RANDOM FUZZY RENEWAL PROCESS

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ABSTRACT. Recently, Zhao et.al [European Journal of Operational Research 169 (2006) 189-201] discussed a random fuzzy renewal process based on random fuzzy theory. They considered the rate of the random fuzzy renewal process and presented a random fuzzy elementary renewal theorem. They also established Blackwell's theorem in random fuzzy sense. But all these results are invalid. We give a counter example in this note.

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Zhao et al. [2] discussed a random fuzzy renewal process based on random fuzzy theory. The interarrival times are characterized as nonnegative random fuzzy variables which is a more reasonable consideration in the real world. Under this setting, they discussed the rate of the random fuzzy renewal process and presented a random fuzzy elementary renewal theorem. Furthermore, they investigated the expected value of renewals in an arbitrary interval and also established Blackwell's theorem in random fuzzy sense. But all these results are wrong. We explain the reason in this note.

We refer all notations and definitions to Zhao et al. [2] and briefly introduce some basic definitions.

Let ξ be a fuzzy variable on a possibility space $\Big(\Theta, P(\Theta), Pos\Big)$ (for the concept of the possibility space, see [1]), where Θ is a universe, $P(\Theta)$ is the power set of Θ and Pos is a possibility measure defined on $P(\Theta)$.

Definition 1 (Liu [1]). The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if and only if

$$Pos\Big\{\xi_i \in B_i, i = 1, 2, \cdots, n\Big\} = \min_{1 \le i \le n} Pos\{\xi_i \in B_i\} \tag{1}$$

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for any sets $B_1, B_2, \cdots, B_n$ of $\mathcal{R}$.

Definition 2 (Liu [1]). The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be identically distributed if and only if

$$Pos\{\xi_i \in B\} = Pos\{\xi_j \in B\} \tag{2}$$

for any sets B of $\Re$.

Definition 3 (Liu [1]). A random fuzzy variable is a function from a possibility space $(\Theta, P(\Theta), Pos)$ to a collection of random variables $\mathfrak{F}$.

Definition 4 (Liu [1]). The random fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent and identically distributed (i.i.d.) if and only if

$$\Big(Pr\{\xi_i(\theta) \in B_1\}, Pr\{\xi_i(\theta) \in B_2\}, \cdots, Pr\{\xi_i(\theta) \in B_m\}\Big), i = 1, 2, \cdots, n$$

are i.i.d. fuzzy vectors for any Borel sets $B_1, B_2, \cdots, B_m$ of $\Re$ and any positive integer m.

Definition 5 (Liu [1]). The random fuzzy variables $\xi_i, i \in I$ are said to be i.i.d.if and only if $\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_n}$ are i.i.d. random fuzzy variables for all finite collections $\{i_1, i_2, \cdots, i_n\}$ of I, where I is an index set.

Zhao et al. [2] discussed a random fuzzy renewal process based on random fuzzy theory. Let ξ_n denote the times between the $(n-1)$th and the nth events, known as the inter-arrival times, $n = 1, 2, \cdots$, respectively. Define

$$S_0 = 0, \; S_n = \xi_1 + \xi_2 + \cdots + \xi_n, \; n \geq 1.$$

If the inter-arrival times ξ_n, $n = 1, 2, \cdots$ are fuzzy variables defined on the possibility spaces $\Big(\Theta_i, P(\Theta_i), Pos_i\Big)$, $i = 1, 2, \cdots$, respectively, then the process $\{S_n, n \geq 1\}$ is called a. random fuzzy renewal process on the possibility space $\Big(\Theta_i, P(\Theta_i), Pos_i\Big)$, where $\Big(\Theta, P(\Theta), Pos\Big), i = 1, 2, \cdots$, is the product possibility space of $\Big(\Theta_i, P(\Theta_i), Pos_i\Big)$.

Let $N(t)$ denote the total number of the events that have occurred by time t. Then we have

$$N(t) = \max_{n \geq 0}\{n \mid 0 < S_n \leq t\}.$$

For any fixed $\theta = (\theta_1, \theta_2, \cdots) \in \Theta$, it is clear that $N(t)(\theta)$ is a random variable with the probability distribution

$$P\Big\{N(t)(\theta) = n\Big\} = P\{S_n(\theta) \leq t\} - P\{S_{n+1}(\theta) \leq t\}, n = 0, 1, 2, \cdots,$$

where $S_n(\theta) = \sum_{i=1}^{n} \xi(\theta) = \sum_{i=1}^{n} \xi(\theta_i)$. Thus $N(t)$ is a random fuzzy variable. We call $N(t)$ the random fuzzy renewal variable. Under this setting, the authors provided some useful theorems, for example, random fuzzy elementary renewal theorem, random fuzzy Blackwell's theorem. But there is something wrong in the paper. To show this, we consider the following fact.

Proposition 1. *If* Θ *consists of a single element, say* $\Theta = \Big\{\theta = (\theta, \theta, \theta, \cdots)\Big\}$, *and* $\xi_1(\theta), \xi_2(\theta), \cdots$ *, are identically distributed random variables then* $\{\xi_n\}_{i=1}^{\infty}$ *is i.i.d. random fuzzy variables.*

Proof. Clearly $\{\xi_n\}_{i=1}^{\infty}$ is identically distributed random fuzzy variables. So, we only prove the independence. Let $\{i_1, i_2, \cdots, i_n\} \subset \mathbb{N}$ and $B_{j_k}, D_j,\ k = 1, 2, \cdots, m, j = 1, 2, \cdots, n$ be Borel sets of $\Re$. We show that

$$\Big(Pr\{\xi_{i_j}(\theta) \in D_1\}, Pr\{\xi_{i_j}(\theta) \in D_2\}, \cdots, Pr\{\xi_{i_j}(\theta) \in D_m\}\Big), j = 1, 2, \cdots, n$$

are independent fuzzy vectors. Since $\Theta = \{\theta = (\theta, \theta, \theta, \cdots)\}$ is a singleton set there are only two possible cases: either

$$Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta) \in D_1\}, Pr\{\xi_{i_j}(\theta) \in D_2\}, \cdots, Pr\{\xi_{i_j}(\theta) \in D_m\}\Big) \in (B_{j_1}, B_{j_2}, \cdots, B_{j_m}), j = 1, 2, \cdots, n\Big\} = 1$$

or

$$Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta) \in D_1\}, Pr\{\xi_{i_j}(\theta) \in D_2\}, \cdots, Pr\{\xi_{i_j}(\theta) \in D_m\}\Big) \in (B_{j_1}, B_{j_2}, \cdots, B_{j_m}), j = 1, 2, \cdots, n\Big\} = 0.$$

If

$$Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta) \in D_1\}, Pr\{\xi_{i_j}(\theta) \in D_2\}, \cdots, Pr\{\xi_{i_j}(\theta) \in D_m\}\Big) \in (B_{j_1}, B_{j_2}, \cdots, B_{j_m}), j = 1, 2, \cdots, n\Big\} = 1,$$

then

$$Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta) \in D_1\}, Pr\{\xi_{i_j}(\theta) \in D_2\}, \cdots, Pr\{\xi_{i_j}(\theta) \in D_m\}\Big) \in (B_{j_1}, B_{j_2}, \cdots, B_{j_m})\Big\} = 1$$

for all $j = 1, 2, \cdots, n$. Therefore

$$Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta) \in D_1\}, Pr\{\xi_{i_j}(\theta) \in D_2\}, \cdots, Pr\{\xi_{i_j}(\theta) \in D_m\}\Big) \in (B_{j_1}, B_{j_2}, \cdots, B_{j_m}), j = 1, 2, \cdots, n\Big\}$$

$$= \min_{1\le j\le n} Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta)\in D_1\}, Pr\{\xi_{i_j}(\theta)\in D_2\},\cdots, Pr\{\xi_{i_j}(\theta)\in D_m\}\Big) \in (B_{j_1}, B_{j_2},\cdots, B_{j_m})\Big\}.$$

Similarly, if

$$Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta)\in D_1\}, Pr\{\xi_{i_j}(\theta)\in D_2\},\cdots, Pr\{\xi_{i_j}(\theta)\in D_m\}\Big) \in (B_{j_1}, B_{j_2},\cdots, B_{j_m}), j=1,2,\cdots,n\Big\} = 0,$$

then

$$Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta)\in D_1\}, Pr\{\xi_{i_j}(\theta)\in D_2\},\cdots, Pr\{\xi_{i_j}(\theta)\in D_m\}\Big) \in (B_{j_1}, B_{j_2},\cdots, B_{j_m})\Big\} = 0$$

for all $j = 1, 2, \cdots, n$. Therefore

$$Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta)\in D_1\}, Pr\{\xi_{i_j}(\theta)\in D_2\},\cdots, Pr\{\xi_{i_j}(\theta)\in D_m\}\Big) \in (B_{j_1}, B_{j_2},\cdots, B_{j_m}), j=1,2,\cdots,n\Big\}$$
$$= \min_{1\le j\le n} Pos\Big\{\Big(Pr\{\xi_{i_j}(\theta)\in D_1\}, Pr\{\xi_{i_j}(\theta)\in D_2\},\cdots, Pr\{\xi_{i_j}(\theta)\in D_m\}\Big) \in (B_{j_1}, B_{j_2},\cdots, B_{j_m})\Big\},$$

which completes the proof. □

We now consider the following example which shows that the random fuzzy elementary renewal theorem and the random fuzzy Blackwell's theorem do not hold.

Example 1. Let Θ consist of a single element, say $\Theta = \{\theta = (\theta, \theta, \theta, \cdots)\}$, and $\xi_1(\theta), \xi_2(\theta), \cdots$, are identically distributed random variables with

$$P\Big\{\xi_1(\theta) = \xi_2(\theta) = \xi_3(\theta) = \cdots = 0\Big\} = 2/3,$$
$$P\Big\{\xi_1(\theta) = \xi_2(\theta) = \xi_3(\theta) = \cdots = 1\Big\} = 1/3.$$

Then by Proposition 1, $\{\xi_n\}_{i=1}^{\infty}$ is a sequence of i.i.d. nonnegative random fuzzy variables. It is easy to check that $E[N(t)] = \infty$ for all $t > 0$ and $E[1/\xi_1] = 1/E[\xi_1] = 3$. Therefore Theorem 3 [2], random fuzzy elementary renewal theorem,

$$\lim_{n\to\infty} \frac{E[N(t)]}{t} = E\left[\frac{1}{\xi_1}\right]$$

does not hold and Theorem 5 [2], random fuzzy elementary Blackwell's theorem,

$$\lim_{n\to\infty} E[N(t+a)] - E[N(t)] = E\left[\frac{a}{\xi_1}\right]$$

for any $a > 0$, does not hold true.

Note. Indeed, it was used that for fixed θ, $\{\xi_n(\theta)\}_{i=1}^{\infty}$ are i.i.d. random variables in the proof of most of Theorems in [2]. But as we have shown in Lemma 1, the independence of the process for given θ, is not guaranteed. Therefore, most of the results in this paper are not valid.

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