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INTUITIONISTIC FUZZY INTERIOR IDEALS IN ORDERED SEMIGROUPS

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ABSTRACT. In this paper we define intuitionistic fuzzy interior ideals in ordered semigroups. We prove that in regular(resp. intra-regular and semisimple) ordered semigroups the concepts of intuitionistic fuzzy interior ideals and intuitionistic fuzzy ideals coincide. We prove that an ordered semigroup is intuitionistic fuzzy simple if and only if every intutionistic fuzzy interior ideal is a constant function. We characterize intra-regular ordered semigroups in terms of interior (resp. intuitionistic fuzzy interior) ideals.

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Key words and Phrases : Intuitionistic fuzzy sets, intuitionistic fuzzy interior ideals, regular (resp. intra-regular and semisimple) ordered semigroups, intuitionistic fuzzy left (resp. right) simple ordered semigroup.

1. Introduction

The idea of an intuitionistic fuzzy set was first introduced by Atanassov (see [1,2,3]) as a generalization of the notion of a fuzzy set given by Zadeh [17]. The concept of a fuzzy semigroup was first introduced by Kuroki [15]. Kim and Jun introduced the concepts of intuitionistic fuzzy quasigroups, intuitionistic fuzzy ideals and intuitionistic fuzzy interior ideals in semigroups (cf. [12,13,14]). Kehayopulu and Tsingelis first considered the fuzzy sets in ordered groupoids and ordered semigroups [8]. They discussed fuzzy analogous for several notions that have been proved to be useful in the theory of ordered groupoids/ordered semigroups. In [10], they have shown that the concepts of a fuzzy ideal and a fuzzy interior ideal coincide in case of regular and intra-regular ordered semigroups.

They also shown that an ordered semigroup is simple if and only if it is fuzzy simple.

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In this paper we consider the intuitionistic fuzzification of the notion of interior ideals in ordered semigroups. We prove that in regular, intra-regular and semisimple ordered semigroups the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy interior ideals coincide. Finally, we introduce the concept of an intuitionistic fuzzy simple ordered semigroup and prove that an ordered semigroup is simple if and only if it is intuitionistic fuzzy simple, and we characterize ordered semigroups in terms of interior ideals and in terms of intuitionistic fuzzy interior ideals.

2. Preliminaries

We include here some basic definitions of ordered semigroups that are necessary for the subsequent results and for more details on ordered semigroups we refer to [5,6,7]. By an *ordered semigroup* we mean an *ordered set* S at the same time a semigroup satisfying the following conditions:

$$a \leq b \Longrightarrow xa \leq xb \text{ and } ax \leq bx \text{ for all } x, a, b \in S.$$

If $(S, ., \leq)$ is an ordered semigroup, and A a *subset* of S, we denote by $(A]$ the subset of S defned as follows:

$$(A] := \{t \in S | t \leq a \text{ for some } a \in A\}.$$

For $A, B \subseteq S$, we denote,

$$AB := \{ab | a \in A, b \in B\}.$$

An ordered semigroup $(S, ., \leq)$ is called *regular* if for each $a \in S$ there exists $x \in S$ such that $a \leq axa$.

Equivalent Definitions: (1) $A \subseteq (ASA]$ for each $A \subseteq S$. (2) $a \in (aSa]$ for each $a \in S$ [7]. An ordered semigroup $(S, ., \leq)$ is called *intra-regular* if for each $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. Equivalent Definitions: (1) $A \subseteq (SA^2S]$ for each $A \subseteq S$. (2) $a \in (Sa^2S]$ for each $a \in S$ [7].

An ordered semigroup $(S, ., \leq)$ is called *semisimple* if for each $a \in S$, there exist $x, y, z \in S$ such that $a \leq xayaz$. Equivalent Definition: (1) $A \subseteq (SASAS]$ for each $A \subseteq S$. (2) $a \in (SaSaS]$ for each $a \in S$ [16]. A non-empty subset A of an ordered semigroup $(S, ., \leq)$ is called a *left* (resp. *right*) *ideal* of S if it satisfies:

(1) $SA \subseteq A$ (resp. $AS \subseteq A$),

(2) $a \leq b$ $S \ni b \leq a$ implies $b \in A$ for all $a, b \in S$.

Condition (2) is equivalent to the condition $(A] = A$. Both a left and a right ideal of an ordered semigroups $(S, ., \leq)$ is called an *ideal* of S. A non-empty subset A of an ordered semigroup $(S, ., \leq)$ is called an *interior ideal* of S if it satisfies:

(1) $SAS \subseteq A$;

(2) $a \leq b$ $S \ni b \leq a$ implies $b \in A$ for all $a, b \in S$ [10].

Let $(S, ., \leq)$ be an ordered semigroup. By a *fuzzy subset* f of S, we mean a mapping $f : S \longrightarrow [0, 1]$.

Let $(S, ., \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy interior ideal* of S, if the following assertions are satisfied:
(1) $f(xay) \geq f(a)$ for all $x, a, y \in S$ and
(2) If $x \leq y$, then $f(x) \geq f(y)$ for all $x, y \in S$ [10].

3. Intuitionistic fuzzy interior ideals

As an important generalization of the notion of fuzzy sets in S, Atanassov [1] introduced the concept of an intuitionistic fuzzy set (IFS for short) defined on a non-empty set S as objects having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in S\},$$

where the functions $\mu_A : S \longrightarrow [0,1]$ and $\gamma_A : S \longrightarrow [0,1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\gamma_A(x)$) of each element $x \in S$ to the set A respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in S$.

For any two *IFS*s A and B of an ordered semigroup S we define:
(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in S$,
(2) $A^c = \{\langle x, \gamma_A(x), \mu_A(x)\rangle | x \in S\}$,
(3) $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\gamma_A(x), \gamma_B(x)\}\rangle | x \in S\}$,
(4) $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\gamma_A(x), \gamma_B(x)\}\rangle | x \in S\}$.

In this section, we prove that in regular(resp. intra-regular and semisimple) ordered semigroups the concepts of intuitionistic ideals and the intuitionistic fuzzy interior ideals coincide.

For the sake brivity, we shall use the symbol $A = \langle \mu_A, \gamma_A \rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in S\}$. Let (S,.,) be an ordered semigroup and $A \subseteq S$, the intuitionistic characteristic function

$$\chi_A := \{\langle x, \mu_{\chi_A}, \gamma_{\chi_A}\rangle | x \in S\},$$

where μ_{χ_A} and γ_{χ_A} are fuzzy subsets defined as follows:

$$\mu_{\chi_A} : S \longrightarrow [0,1] | x \longrightarrow \mu_{\chi_A}(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A \end{cases},$$

and

$$\gamma_{\chi_A} : S \longrightarrow [0,1] | x \longrightarrow \gamma_{\chi_A}(x) := \begin{cases} 0 \text{ if } x \in A, \\ 1 \text{ if } x \notin A \end{cases}.$$

Definition 1. *An IFS $A = \langle \mu_A, \gamma_A \rangle$ in an ordered semigroup $(S, ., \leq)$ is called an intuitionistic fuzzy interior ideal of S if it satisfies the following assertions:*

(1) $\mu_A(xaz) \geq \mu_A(a)$ and $\gamma_A(xaz) \leq \gamma_A(a)$ for all $x, a, y \in S$ and
(2) If $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$, and $\gamma_A(x) \leq \gamma_A(y)$ for all $x, y \in S$.

Proposition 2. *Let $(S, ., \leq)$ be an ordered semigroup. If $\{A_i : i \in \Lambda\}$ is a family of intuitionistic fuzzy interior ideals of S. Then $\bigcap_{i \in \Lambda} A_i$, if it is non-empty, is an* *intuitionistic fuzzy interior ideal of S, where* $\bigcap_{i\in\Lambda} A_i = \left\langle \bigcap_{i\in\Lambda} \mu_{A_i}, \bigcup_{i\in\Lambda} \gamma_{A_i} \right\rangle$, *and*

$$\begin{aligned}\bigcap_{i\in\Lambda} \mu_{A_i}(x) &= \bigwedge_{i\in\Lambda} \mu_{A_i}(x) := inf\{\mu_{A_i}(x) : i \in \Lambda, x \in S\},\\ \bigcup_{i\in\Lambda} \gamma_{A_i}(x) &= \bigvee_{i\in\Lambda} \gamma_{A_i}(x) := sup\{\gamma_{A_i}(x) : i \in \Lambda, x \in S\}.\end{aligned}$$

Proof. (1) For $x, y \in S$ if $x \le y$, then we have

$$\begin{aligned}\left(\bigcap_{i\in\Lambda} \mu_{A_i}\right)(x) &= \left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(x) = \bigwedge_{i\in\Lambda}(\mu_{A_i}(x))\\ &\ge \bigwedge_{i\in\Lambda}(\mu_{A_i}(y)) \text{ (since } x \le y \Longrightarrow \mu_{A_i}(x) \ge \mu_{A_i}(y))\\ &= \left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(y) = \left(\bigcap_{i\in\Lambda} \mu_{A_i}\right)(y),\end{aligned}$$

and

$$\begin{aligned}\left(\bigcup_{i\in\Lambda} \gamma_{A_i}\right)(x) &= \left(\bigvee_{i\in\Lambda} \gamma_{A_i}\right)(x) = \bigvee_{i\in\Lambda}(\gamma_{A_i}(x))\\ &\le \bigvee_{i\in\Lambda}(\gamma_{A_i}(y)) \text{ (since } x \le y \Longrightarrow \gamma_{A_i}(x) \le \gamma_{A_i}(y))\\ &= \left(\bigvee_{i\in\Lambda} \gamma_{A_i}\right)(y) = \left(\bigcup_{i\in\Lambda} \mu_{A_i}\right)(y).\end{aligned}$$

Let $a, x, y \in S$. Then

$$\begin{aligned}\left(\bigcap_{i\in\Lambda} \mu_{A_i}\right)(xay) &= \left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(xay) = \bigwedge_{i\in\Lambda}(\mu_{A_i}(xay))\\ &\ge \bigwedge_{i\in\Lambda}(\mu_{A_i}(a)) \text{ (since } \mu_{A_i}(xay) \ge \mu_{A_i}(a))\\ &= \left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(a) = \left(\bigcap_{i\in\Lambda} \mu_{A_i}\right)(a),\end{aligned}$$

and

$$\begin{aligned}\left(\bigcup_{i\in\Lambda} \gamma_{A_i}\right)(xay) &= \left(\bigvee_{i\in\Lambda} \gamma_{A_i}\right)(xay) = \bigvee_{i\in\Lambda}(\gamma_{A_i}(xay))\\ &\le \bigvee_{i\in\Lambda}(\gamma_{A_i}(a)) \text{ (since } \gamma_{A_i}(xay) \le \gamma_{A_i}(a))\end{aligned}$$

$$= \left(\bigvee_{i\in\Lambda} \gamma_{A_i}\right)(a) = \left(\bigcup_{i\in\Lambda} \mu_{A_i}\right)(a).$$

Thus $\bigcap_{i\in\Lambda} A_i$ is an intuitionistic fuzzy interior ideal of S. □

Proposition 3. *Let $(S, ., \leq)$ be an ordered semigroup. Then every intuitionistic fuzzy ideal of S is an intuitionistic fuzzy interior ideal of S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy ideal of an ordered semigroup S. Let $x, a, y \in S$. Then $\mu_A(x(ay)) \geq \mu_A(ay) \geq \mu_A(a)$ and $\gamma_A(x(ay)) \leq \gamma_A((ay)) \leq \gamma_A(a)$. Thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy interior ideal of S. □

Proposition 4. *Let $(S, ., \leq)$ be a regular ordered semigroup. Then every intuitionistic fuzzy interior ideal of S is an intuitionistic fuzzy ideal of S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy interior ideal of a regular ordered semigroup S. Let $a, b \in S$. Then $\mu_A(ab) \geq \mu_A(a)$ and $\gamma_A(ab) \leq \gamma_A(a)$. Indeed: Since S is regular, there exists $x \in S$ such that $a \leq axa$. Then $ab \leq (axa)b = (ax)ab$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy interior ideal of S we have

$$\mu_A(ab) \geq \mu_A((ax)ab) \geq \mu_A(a)$$

and

$$\gamma_A(ab) \leq \gamma_A((ax)ab) \leq \gamma_A(a).$$

Similarly we can show that $\mu_A(ab) \geq \mu_A(b)$ and $\gamma_A(ab) \leq \gamma_A(b)$. Thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal of S. □

Comibining Prop. 3 and 4, we have the following:

Proposition 5. *In regular ordered semigroups the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy interior ideals coincide.*

Proposition 6. *Let $(S, ., \leq)$ be an intra-regular ordered semigroup. Then every intuitionistic fuzzy interior ideal of S is an intuitionistic fuzzy ideal of S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy interior ideal of an intra-regular ordered semigroup S. Let $a, b \in S$. Then $\mu_A(ab) \geq \mu_A(a)$ and $\gamma_A(ab) \leq \gamma_A(a)$. Indeed: Since S is intra-regular, there exists $x, y \in S$ such that $a \leq xa^2y$. Then $ab \leq (xa^2y)b = (xa)a(yb)$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy interior ideal of S we have

$$\mu_A(ab) \geq \mu_A((xa)a(yb)) \geq \mu_A(a)$$

and

$$\gamma_A(ab) \leq \gamma_A((xa)a(yb)) \leq \gamma_A(a).$$

Similarly we can show that $\mu_A(ab) \geq \mu_A(b)$ and $\gamma_A(ab) \leq \gamma_A(b)$. Thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal of S. □

Comibining Prop. 3 and 6, we have the following:

Proposition 7. *In intra-regular ordered semigroups the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy interior ideals coincide.*

Proposition 8. *Let $(S, ., \leq)$ be a semisimple ordered semigroup. Then every intuitionistic fuzzy interior ideal of S is an intuitionistic fuzzy ideal of S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy interior ideal of an intra-regular ordered semigroup S. Let $a, b \in S$. Then $\mu_A(ab) \geq \mu_A(a)$ and $\gamma_A(ab) \leq \gamma_A(a)$. Indeed: Since S is semisimple, there exists $x, y, z \in S$ such that $a \leq xayaz$. Then $ab \leq (xayaz)b = xa(yazb)$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitinistic fuzzy interior ideals of S we have

$$\mu_A(ab) \geq \mu_A(xa(yazb)) \geq \mu_A(a)$$

and

$$\gamma_A(ab) \leq \gamma_A(xa(yazb)) \leq \gamma_A(a).$$

Similarly we can prove that $\mu_A(ab) \geq \mu_A(b)$ and $\gamma_A(ab) \leq \gamma_A(b)$. Thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal of S. □

Combining Prop. 3 and 8, we have the following:

Theorem 9. *In semisimple ordered semigroups the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy interior ideals coincide.*

Theorem 10. *Let $(S, ., \leq)$ be an ordered semigroup, $\emptyset \neq I \subseteq S$. Then I is an interior ideal of S if and only if the intuitionistic characteristic function $\chi_I = \langle \mu_{\chi_I}, \gamma_{\chi_I} \rangle$ of I is an intuitionistic fuzzy interior ideal of S.*

Proof. $\Longrightarrow$. Suppose that I is an interior ideal of S and χ_I the intuitionistic characteristic function of I. Let $a, b \in S$, $a \leq b$ then $\mu_{\chi_I}(a) \geq \mu_{\chi_I}(b)$ and $\gamma_{\chi_I}(a) \leq \gamma_{\chi_I}(b)$. Indeed: If $b \notin I$ then $\mu_{\chi_I}(b) = 0$ and $\gamma_{\chi_I}(b) = 1$. Since $\mu_{\chi_I}(a) \geq 0$ and $\gamma_{\chi_I}(a) \leq 1$, we have $\mu_{\chi_I}(a) \geq \mu_{\chi_I}(b)$ and $\gamma_{\chi_I}(a) \leq \gamma_{\chi_I}(b)$. Let $b \in I$ then $\mu_{\chi_I}(b) = 1$ and $\gamma_{\chi_I}(b) = 0$. Since I is an interior ideal of S and $a \leq b$ we have $a \in I$. Then $\mu_{\chi_I}(a) = 1$ and $\gamma_{\chi_I}(a) = 0$. Again we have $\mu_{\chi_I}(a) \geq \mu_{\chi_I}(b)$ and $\gamma_{\chi_I}(a) \leq \gamma_{\chi_I}(b)$.

Let $x, a, y \in S$. If $a \in I$ then $\mu_{\chi_I}(a) = 1$ and $\gamma_{\chi_I}(a) = 0$. Since I is an interior ideal of S, we have $xay \in SIS \subseteq I$. Then we have $\mu_{\chi_I}(xay) = 1$ and $\gamma_{\chi_I}(xay) = 0$, hence $\mu_{\chi_I}(xay) \geq \mu_{\chi_I}(a)$ and $\gamma_{\chi_I}(xay) \leq \gamma_{\chi_I}(a)$.

$\Longleftarrow$. Assume that χ_I is an intuitionistic fuzzy interior ideal of S. Let $a, b \in S$, $a \leq b$. If $b \in I$, then $\mu_{\chi_I}(b) = 1$ and $\gamma_{\chi_I}(b) = 0$. Since $\mu_{\chi_I}(a) \geq \mu_{\chi_I}(b)$ and $\gamma_{\chi_I}(a) \leq \gamma_{\chi_I}(b)$, we have $\mu_{\chi_I}(a) = 1$ and $\gamma_{\chi_I}(a) = 0$ and so $a \in I$.

Let $x, a, y \in S$. If $a \in I$, then $\mu_{\chi_I}(a) = 1$ and $\gamma_{\chi_I}(a) = 0$. Since $\mu_{\chi_I}(xay) \geq \mu_{\chi_I}(a)$ and $\gamma_{\chi_I}(xay) \leq \gamma_{\chi_I}(a)$, we have $\mu_{\chi_I}(xay) = 1$ and $\gamma_{\chi_I}(xay) = 0$ and so $xay \in I \Longrightarrow SIS \subseteq I$. □

4. Intuitionistic fuzzy simple ordered semigroups

In this section we introduce the concept of intuitionistic fuzzy simple ordered semigroups, we prove that an ordered semigroup is simple if and only if it is intuitionistic fuzzy simple, and we characterize this type of ordered semigroups in terms of intuitionistic fuzzy interior ideals.

An ordered semigroup S is called simple if does not contain proper ideals, that is, for any ideal A of S, we have $A = S$ [10].

Definition 11. *An ordered semigroup S is called intuitionistic fuzzy simple if every intuitionistic fuzzy ideal of S is an intuitionistic fuzzy constant function, that is, for every intuitionistic fuzzy ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, we have $\mu_A(a) = \mu_A(b)$ and $\gamma_A(a) = \gamma_A(b)$ for all $a, b \in S$.*

If $(S, ., \leq)$ is an ordered semigroup and $a \in S$, we denote by I_a the subset of S defines as follows:

$$I_a := \{b \in S : \mu_A(b) \geq \mu_A(a) \text{ and } \gamma_A(b) \leq \gamma_A(a)\}.$$

Proposition 12. *Let $(S, ., \leq)$ be an ordered semigroup and $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy right ideal of S. Then I_a is a right ideal of S for every $a \in S$.*

Proof. Let $(S, ., \leq)$ be an ordered semigroup and $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy right ideal of S. Since $a \in I_a$ for every $a \in S$, we have $I_a \neq \emptyset$. Let $b \in I_a$ and $x \in S$. We have to prove that $bx \in I_a$. Since $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy right ideal of S, we have $\mu_A(bx) \geq \mu_A(b)$ and $\gamma_A(bx) \leq \gamma_A(b)$. Since $b \in I_a$, we have $\mu_A(b) \geq \mu_A(a)$ and $\gamma_A(b) \leq \gamma_A(a)$. Thus $\mu_A(bx) \geq \mu_A(a)$ and $\gamma_A(bx) \leq \gamma_A(a)$, hence $bx \in I_a$.

Let $b \in I_a$ and $S \ni x \leq b$. Then $x \in I_a$. Indeed: Since $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy right ideal of S and $x \leq b$ we have $\mu_A(x) \geq \mu_A(b)$ and $\gamma_A(x) \leq \gamma_A(b)$. Since $b \in I_a$ we have $\mu_A(b) \geq \mu_A(a)$ and $\gamma_A(b) \leq \gamma_A(a)$. Thus $\mu_A(x) \geq \mu_A(a)$ and $\gamma_A(x) \leq \gamma_A(a)$, which implies that $x \in I_a$. □

In a similar way we can prove that:

Proposition 13. *Let $(S, ., \leq)$ be an ordered semigroup and $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy left ideal of S. Then I_a is a left ideal of S for every $a \in S$.*

Combining Propositions 12 and 13, we have the following:

Proposition 14. *Let $(S, ., \leq)$ be an ordered semigroup and $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy ideal of S. Then I_a is an ideal of S for every $a \in S$.*

Lemma 15. *Let $(S, ., \leq)$ be an ordered semigroup, $\emptyset \neq A \subseteq S$. Then A is an ideal of S if and only if the intuitionistic characteristic function $\chi_A = \langle \mu_A, \gamma_A \rangle$ of A is an intuitionistic fuzzy ideal of S.*

Proof. $\Longrightarrow$. Suppose that A is an ideal of S and χ_I the intuitionistic characteristic function of A. Let $a, b \in S$, $a \leq b$ then $\mu_{\chi_A}(a) \geq \mu_{\chi_A}(b)$ and

$\gamma_{\chi_A}(a) \leq \gamma_{\chi_A}(b)$. Indeed: If $b \notin A$ then $\mu_{\chi_A}(b) = 0$ and $\gamma_{\chi_A}(b) = 1$. Since $\mu_{\chi_A}(a) \geq 0$ and $\gamma_{\chi_A}(a) \leq 1$, we have $\mu_{\chi_A}(a) \geq \mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(a) \leq \gamma_{\chi_A}(b)$. Let $b \in A$ then $\mu_{\chi_A}(b) = 1$ and $\gamma_{\chi_A}(b) = 0$. Since A is an ideal of S and $a \leq b$ we have $a \in A$. Then $\mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}(a) = 0$. Again we have $\mu_{\chi_A}(a) \geq \mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(a) \leq \gamma_{\chi_A}(b)$.

Let $x, y \in S$. If $x \in A$ then $\mu_{\chi_A}(x) = 1$ and $\gamma_{\chi_A}(x) = 0$. Since A is a right ideal of S, we have $xy \in AS \subseteq A$. Then we have $\mu_{\chi_A}(xy) = 1$ and $\gamma_{\chi_A}(xy) = 0$, hence $\mu_{\chi_A}(xy) \geq \mu_{\chi_A}(x)$ and $\gamma_{\chi_A}(xy) \leq \gamma_{\chi_I}(x)$.

$\Longleftarrow$. Assume that χ_A is an intuitionistic fuzzy ideal of S. Let $a, b \in S$, $a \leq b$. If $b \in A$, then $\mu_{\chi_A}(b) = 1$ and $\gamma_{\chi_A}(b) = 0$. Since $\mu_{\chi_A}(a) \geq \mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(a) \leq \gamma_{\chi_A}(b)$, we have $\mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}(a) = 0$ and so $a \in A$.

Let $x, y \in S$. If $x \in A$, then $\mu_{\chi_A}(x) = 1$ and $\gamma_{\chi_A}(x) = 0$. Since χ_A is an intuitionistic fuzzy right ideal of S, we have $\mu_{\chi_A}(xy) \geq \mu_{\chi_A}(x)$ and $\gamma_{\chi_A}(xy) \leq \gamma_{\chi_I}(x)$, we have $\mu_{\chi_A}(xy) = 1$ and $\gamma_{\chi_A}(xy) = 0$ and so $xy \in A \Longrightarrow AS \subseteq A$. □

Theorem 16. *An ordered semigroup $(S, ., \leq)$ is simple if and only if it is intuitionistic fuzzy simple.*

Proof. $\Longrightarrow$. Let S be a simple ordered semigroup, $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy ideal of S and $a, b \in S$. Since μ_A and γ_A are fuzzy ideals of S and $a \in S$, by Proposition 14, I_a is an ideal of S. Since S is simple we have $I_a = S$, and we have $b \in I_a$. Thus $\mu_A(b) \geq \mu_A(a)$ and $\gamma_A(b) \leq \gamma_A(a)$. By a similar way we can prove that $\mu_A(a) \geq \mu_A(b)$ and $\gamma_A(a) \leq \gamma_A(b)$. Thus $\mu_A(b) = \mu_A(a)$ and $\gamma_A(b) = \gamma_A(a)$ and so S is intuitionistic fuzzy simple.

$\Longleftarrow$. Suppose S contains proper ideals and let A be an ideal of S such that $A \neq S$. Since A is proper by Lemma 15, $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ is an intuitionistic fuzzy ideal of S. Let $x \in S$. Since $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ is a intuitionistic fuzzy constant ideal of S. We have $\mu_{\chi_A}(x) = \mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(x) = \gamma_{\chi_A}(b)$, for every $b \in S$. Since $A \neq \emptyset$, let $a \in A$. Then $\mu_{\chi_A}(x) = \mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}(x) = \gamma_{\chi_A}(a) = 0$, hence $x \in A$. Thus $S \subset A$, a contradition. Thus $S = A$ and S is simple. □

Lemma 17. *(cf. [6]). An ordered semigroup S is simple if and only if for every $a \in S$, we have $S = (SaS]$.*

Theorem 18. *Let $(S, ., \leq)$ be an ordered semigroup. Then S is simple if and only if every intuitionistic fuzzy interior ideal of S is a constant intuitionistic function.*

Proof. $\Longrightarrow$. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy interior ideal of a simple ordered semigroup S,and $a, b \in S$. Since S is simple and $b \in S$, by Lemma 17, we have $S = (SbS]$. Since $a \in S$, we have $a \in (SbS]$. Then there exist $x, y \in S$ such that $a \leq xby$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionitic fuzzy interior ideal of S, we have $\mu_A(a) \geq \mu_A(xby) \geq \mu_A(b)$ and $\gamma_A(a) \leq \gamma_A(xby) \geq \gamma_A(b)$. Thus we have $\mu_A(a) \geq \mu_A(b)$ and $\gamma_A(a) \leq \gamma_A(b)$. In a similar way we can prove that $\mu_A(a) \leq \mu_A(b)$ and $\gamma_A(a) \geq \gamma_A(b)$., and thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy constant function.

$\Longleftarrow$. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy ideal of S. By Proposition 3, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy interior ideal of S. By hypothesis, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy constant function. Then S is intuitionistic fuzzy simple and, by Theorem 16, S is simple. □

Proposition 19. *Let $(S, ., \leq)$ be an intra-regular ordered semigroups. Then for every interior ideals A and B of S we have,*

$$(A^2] = A \text{ and } (AB] = (BA].$$

Proof. (1) Let S be an intra-regular ordered semigroup and A, B interior ideals of S. Let $a \in A$. Since S is intra-regular, there exist x, $y \in S$ such that

$$\begin{aligned} a &\leq xa^2y = (xa)(ay) \leq x(xa^2y)(xa^2y)y \\ &= ((xxa)a(y))((xa)a(ayy) \in (SAS)(SAS) \subseteq AA = A^2 \\ &\Longrightarrow a \in (A^2] \Longrightarrow A \subseteq (A^2]. \end{aligned}$$

For the reverse inclusion, let $a \in (A^2]$, then $a \leq a_1a_2$ for some $a_1, a_2 \in A$. Then

$$\begin{aligned} a &\leq xa^2y = (xa)(ay) \leq x(a_1a_2)(a_1a_2)y \\ &= (xa_1a_2)a_1(a_2y) \in SAS \subseteq A \\ &\Longrightarrow a \in (A] = A \Longrightarrow (A^2] \subseteq A. \end{aligned}$$

Thus $(A^2] = A$.

(2). Let A, B be interior ideals of S. Then $(AB] = (BA]$. Indeed: By (1) we have

$$\begin{aligned} (AB] &= ((AB]^2] = ((AB](AB]] \\ &= ((AB]^2(AB]^2] = (((AB](AB])((AB](AB])] \\ &\subseteq (((AB)(AB)]((AB)(AB)]] \\ &= (((A)B(AB)]((AB)A(B)]] \subseteq ((SBS](SAS]] \\ &\subseteq ((B](A]] = (BA] \Longrightarrow (AB] \subseteq (BA]. \end{aligned}$$

By symmetry we have $(BA] \subseteq (AB]$.

Show that μ_A and γ_A are fuzzy interior ideals of S, if $A = \langle \mu_A, \gamma_A \rangle$ is intuitionistic fuzzy interior ideal of S. □

Proposition 20. *Let $(S, ., \leq)$ be an intra-regular ordered semigroup and $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy interior ideal of S. Then for every $a \in S$ such that $a^2 \leq a$, we have*

$$\mu_A(a) = \mu_A(a^2),\ \gamma_A(a) = \gamma_A(a^2) \text{ and } \mu_A(ab) = \mu_A(ba),\ \gamma_A(ab) = \gamma_A(ba).$$

Proof. (1) Let S be an intra-regular ordered semigroup, $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy interior ideal of S and $a \in S$. Then $\mu_A(a) = \mu_A(a^2)$, $\gamma_A(a) = \gamma_A(a^2)$. Indeed: Since S is intra-regular and $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$ for some $x, y \in S$. Then

$$\mu_A(a) \geq \mu_A(xa^2y) \geq \mu_A(a^2).$$

Since $a^2 \leq a$ we have $\mu_A(a^2) \geq \mu_A(a)$. Hence $\mu_A(a) = \mu_A(a^2)$, and

$$\gamma_A(a) \leq \gamma_A(xa^2y) \geq \gamma_A(a^2),$$

since $a^2 \leq a$ we have $\gamma_A(a^2) \leq \gamma_A(a)$. Hence $\gamma_A(a) = \gamma_A(a^2)$.

(2) Let $a, b \in S$. Then $\mu_A(ab) = \mu_A(ba)$, $\gamma_A(ab) = \gamma_A(ba)$. Indeed: By (1) we have

$$\mu_A(ab) = \mu_A((ab)^2) = \mu_A((ab)(ab)) = \mu_A(a(ba)b) \geq \mu_A(ba),$$

By symmetry we have $\mu_A(ba) \geq \mu_A(ab)$. Hence $\mu_A(a) = \mu_A(a^2)$ and

$$\gamma_A(ab) = \gamma_A((ab)^2) = \gamma_A((ab)(ab)) = \gamma_A(a(ba)b) \leq \gamma_A(ba),$$

by symmetry we have $\gamma_A(ba) \leq \gamma_A(ab)$. Hence $\gamma_A(ab) = \gamma_A(ba)$. □

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