J. Appl. Math. & Informatics Vol. **27**(2009), No. 5 - 6, pp. 1435 - 1446
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BLOW UP OF SOLUTIONS TO A SEMILINEAR PARABOLIC SYSTEM WITH NONLOCAL SOURCE AND NONLOCAL BOUNDARY

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ABSTRACT. In this paper we investigate the blow up properties of the positive solutions to a semilinear parabolic system with coupled nonlocal sources $u_t = \Delta u + k_1 \int_\Omega u^\alpha(y,t)v^p(y,t)dy$, $v_t = \Delta v + k_2 \int_\Omega u^q(y,t)v^\beta(y,t)dy$ with nonlocal Dirichlet boundary conditions. We establish the conditions for global and non-global solutions respectively and obtain its blow up set..

AMS Mathematics Subject Classification: 35J25; 35J65
Key words and phrases: Parabolic system; nonlocal sources; nonlocal boundary conditions; blow-up

1. Introduction

In this paper, we consider the following semilinear parabolic system with nonlinear nonlocal sources subject to nonlocal Dirichlet boundary conditions

$$\begin{cases} u_t = \Delta u + k_1 \int_\Omega u^\alpha(y,t)v^p(y,t)dy, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + k_2 \int_\Omega u^q(y,t)v^\beta(y,t)dy, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \int_\Omega f(x,y)u(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ v(x,t) = \int_\Omega g(x,y)v(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where k_1, k_2 are positive constants, and $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. The two equations in (1.1) are completely coupled via the nonlocal nonlinear sources with positive constants α, β, p, q, while the functions $f(x,y), g(x,y)$ in the boundary conditions are continuous, nonnegative on $\partial\Omega \times \overline{\Omega}$, and $\int_\Omega f(x,y)dy$, $\int_\Omega g(x,y)dy > 0$ on $\partial\Omega$. The initial data $u_0(x), v_0(x) \in$

Received April 9, 2008. Revised May 2, 2008. Accepted May 27, 2008. *Corresponding author.
Project Supported by the National Natural Science Foundation of China(Grant No.10571022). Project Supported by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No.04KJB110062; No.06KJB110056).

$C^{2+\nu}(\overline{\Omega})$ with $0<\nu<1$, $u_0, v_0 \geq 0, \not\equiv 0$, and satisfy the compatibility conditions.

The global solutions and blow-up problems for the single parabolic equation with nonlocal nonlinearities had been studied extensively, see [9-14][16],[17],[22-25] and the references therein. The local existence of classical solution of problem (1.1) can be obtained by a simple modification of arguments given in [12], and the uniqueness results can be obtained by similar arguments as [2]. Particularly, in the paper [25], the authors established the critical Fujita exponent for the Cauchy problem

$$\begin{cases} u_t = \Delta u^m + (\int_{R^N} K(y)u^q(y,t)dy)^{\frac{p-1}{q}} u^{r+1}, x \in R^N, t>0, \\ u(x,0) = u_0(x), \qquad x \in R^N, \end{cases}$$

where parameters $m, p > 1, q > 0, r \geq 0$, the initial data $u_0(x)$ is a bounded nonnegative function, and the kernel function K is nonnegative and measurable. In the paper [7], Li, Huang and Xie studied the following problem

$$\begin{cases} u_t = \Delta u + \int_\Omega u^m(x,t)v^n(x,t)dx, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + \int_\Omega u^p(x,t)v^q(x,t)dx, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

and they get the following result.

Theorem A. *The system (1.1) with null Dirichlet boundary conditions admits a unique global solution for any nonnegative initial data* $u_0, v_0 \geq 0, \not\equiv 0$, *if*

$$\alpha < 1, \beta < 1, \textit{and} \quad pq < (1-\alpha)(1-\beta).$$

There have been many articles which deal with properties of solutions to partial differential equations with local boundary conditions(see [4],[5],[6],[8],[15] and [18]). However, there are some important phenomena formulated into parabolic equations which are coupled with nonlocal boundary conditions in mathematical modelling such as thermoelasticity theory (see [1],[2],[3],[20] and [21]). In this case, the solution $(u(x,t), v(x,t))$ describes entropy per volume of the material.

The problem of nonlocal boundary conditions for linear scalar parabolic equations of the form

$$\begin{cases} u_t - A(u) = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \int_\Omega \varphi(x,y)u(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

with uniform elliptic

$$A = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x)\frac{\partial}{\partial x_j} + c(x),$$

and $c(x) \leq 0$ was studied by Friedman [4]. It was proved that the unique solution of (1.2) tends to 0 monotonically and exponentially as $t \to +\infty$ provided

$$\int_\Omega |\varphi(x,y)|dy \leq \rho < 1, \quad x \in \partial\Omega.$$

In [2], Deng obtained the uniqueness and existence of local solutions to the semilinear scalar parabolic equations subject to nonlocal boundary conditions

$$\begin{cases} u_t - \Delta u = g(x,u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \int_\Omega \varphi(x,y)u(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x). & x \in \Omega, \end{cases} \tag{1.3}$$

In addition, the exponential decay for the global solutions of (1.3) was proved under the assumption

$$\int_\Omega \varphi^2(x,y)dy \leq \frac{1}{|\Omega|}, \quad x \in \partial\Omega.$$

As for more general discussions on the dynamics of parabolic problem with nonlocal boundary conditions, one can see, e.g. [5] by Pao, where the following problem

$$\begin{cases} u_t - L(u) = f(x,u), & (x,t) \in \Omega \times (0,T), \\ Bu(x,t) = \int_\Omega K(x,y)u(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

was considered with

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{j=1}^{n} b_j(x)u_{x_j}, \qquad Bu = \alpha_0\frac{\partial u}{\partial \nu} + u.$$

The scalar problems with both nonlocal sources and nonlocal boundary conditions have been studied as well. For example, the problem of the form

$$\begin{cases} u_t - \Delta u = \int_\Omega g(u)dy, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \int_\Omega \varphi(x,y)u(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

was studied by Lin and Liu [6], where and in the sequel denote $\int_\Omega g(u)dy = \int_\Omega g(u(y,t))dy$.

In [8], S.N.Zheng and L.H.Kong investigate the following problem

$$\begin{cases} u_t = \Delta u + u^m(y,t)\int_\Omega v^n(y,t)dy, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + v^q(y,t)\int_\Omega u^p(y,t)dy, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \int_\Omega \varphi(x,y)u(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ v(x,t) = \int_\Omega \psi(x,y)v(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.6}$$

They investigated the roles of weight functions in nonlocal boundary conditions to the blow up of the solutions. Interestedly, they found that the solutions would blow up with any positive initial data under some conditions.

In this paper we discuss the reaction-diffusion system (1.1). We assume that $u_0(x), v_0(x)$ satisfy

(H) $\Delta u_0(x)+\int_\Omega u_0^\alpha(x)v_0^p(x)dx \geq 0, \Delta v_0(x)+\int_\Omega u_0^q(x)v_0^\beta(x)dx \geq 0, \quad x\in\Omega.$

The present work is partially motivated by [7],[8],[12] and [17]. The main purpose of this paper is to discuss global existence and global nonexistence (finite time blowing up) of the solution. We use some ideas developed by Souplet [16] and Wang [14] and extend them to nonlocal parabolic systems. Also we will obtain the similar results with [8] and obtain the blow up set of (1.1).

This paper is organized as follows. Section 2 deals with the maximum principle and comparison principle used for the model. Theorems 1 and 2 on global solutions and the blow-up conditions for large initial data (Theorem 3) and any positive initial data (Theorems 4 and 5) respectively will be proved in Section 3. Finally, we discuss in the last section the blow up set of (1.1) and we obtain the blow up set of (1.1) is whole domain to contrast the local problems.

2. Comparison principle

Let $Q_T=\Omega\times(0,T), S_T=\partial\Omega\times(0,T)$ and $\overline{Q}_T=\overline{\Omega}\times(0,T)$. We begin with the definition of subsolution and supersolution of (1.1).

Definition 2.1. A pair of functions $\underline{u},\underline{v}\in C^{2,1}(Q_T)\bigcap C(\overline{Q}_T)$ is called a subsolution of (1.1) if

$$\begin{cases} \underline{u}_t \leq \Delta\underline{u}+\int_\Omega \underline{u}^\alpha(y,t)\underline{v}^p(y,t)dy, & (x,t)\in Q_T,\\ \underline{v}_t \leq \Delta\underline{v}+\int_\Omega \underline{u}^q(y,t)\underline{v}^\beta(y,t)dy, & (x,t)\in Q_T,\\ \underline{u}(x,t)\leq \int_\Omega f(x,y)\underline{u}(y,t)dy, & (x,t)\in S_T,\\ \underline{v}(x,t)\leq \int_\Omega g(x,y)\underline{v}(y,t)dy, & (x,t)\in S_T,\\ \underline{u}(x,0)\leq u_0(x), \underline{v}(x,0)\leq v_0(x), & x\in\Omega. \end{cases} \tag{2.1}$$

A supersolution is defined in similar way with each inequality reversed.

Using an argument as that [8] we have the maximum principle and comparison principle.

Lemma 2.1. *Suppose that $w(x,t), z(x,t)\in C^{2,1}(Q_T)\bigcap C(\overline{Q}_T)$ satisfies*

$$\begin{cases} w_t-\Delta w\geq \int_\Omega(b_1(y,t)w(y,t)+c_1(y,t)z(y,t))dy, & (x,t)\in\Omega\times(0,T),\\ z_t-\Delta z\geq \int_\Omega(b_2(y,t)w(y,t)+c_2(y,t)z(y,t))dy, & (x,t)\in\Omega\times(0,T),\\ w(x,t)\leq \int_\Omega f(x,y)w(y,t)dy, & (x,t)\in\partial\Omega\times(0,T),\\ z(x,t)\leq \int_\Omega g(x,y)z(y,t)dy, & (x,t)\in\partial\Omega\times(0,T),\\ w(x,0)>0, z(x,0)>0, & x\in\Omega, \end{cases} \tag{2.2}$$

where $b_i, c_i\geq 0, i=1,2,$ on $\overline{Q}_T$ and $f(x,y), g(x,y)\geq 0$ on $\partial\Omega\times\Omega$, $\int_\Omega f(x,y)dy>0, \int_\Omega g(x,y)dy>0$ on $\partial\Omega$, then $w>0, z>0$ on $\overline{Q}_T$.

Lemma 2.2. *Suppose that $w(x,t), z(x,t)\in C^{2,1}(Q_T)\bigcap C(\overline{Q}_T)$ satisfies*

$$\begin{cases} w_t - \Delta w \geq \int_\Omega (b_1(y,t)w(y,t) + c_1(y,t)z(y,t))dy, & (x,t) \in \Omega \times (0,T), \\ z_t - \Delta z \geq \int_\Omega (b_2(y,t)w(y,t) + c_2(y,t)z(y,t))dy, & (x,t) \in \Omega \times (0,T), \\ w(x,t) \leq \int_\Omega f(x,y)w(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,t) \leq \int_\Omega g(x,y)z(y,t)dy, & (x,t) \in \partial\Omega \times (0,T), \\ w(x,0) \geq 0, z(x,0) \geq 0, & x \in \Omega, \end{cases} \tag{2.3}$$

where $b_i, c_i, i = 1, 2$ are nonnegative and bounded functions in Q_T, $f(x,y), g(x,y) \geq 0$ on $\partial\Omega \times \Omega$, $\int_\Omega f(x,y)dy > 0$, $\int_\Omega g(x,y)dy > 0$ on $\partial\Omega$, then $w \geq 0, z \geq 0$ on $\overline{Q}_T$.

Remark 2.1. The nonnegativity of $f(x,y), g(x,y)$ plays an important role, see Remark 2.2 of [18]. The following counter example shows that the nonnegativity of $b_i, c_i (i = 1, 2)$ are also necessary. Let

$$\Omega = (-1, 1), f(x,y) = g(x,y) = \frac{1}{2},$$

$$w(x,t) = z(x,t) = x^2 - t, b_1 = b_2 = -5, c_1 = c_2 = -4.$$

It is clear that

$$\begin{cases} w_t - w_{xx} = -3 > \int_{-1}^{1} (b_1(y,t)w(y,t) + c_1(y,t)z(y,t))dy, & |x| < 1, 0 < t < \frac{1}{6}, \\ z_t - z_{xx} = -3 > \int_{-1}^{1} (b_2(y,t)w(y,t) + c_2(y,t)z(y,t))dy, & |x| < 1, 0 < t < \frac{1}{6}, \\ w(x,t) = 1 - t > \int_{-1}^{1} f(x,y)w(y,t)dy, & |x| = 1, 0 < t < \frac{1}{6}, \\ z(x,t) = 1 - t > \int_{-1}^{1} g(x,y)z(y,t)dy, & |x| = 1, 0 < t < \frac{1}{6}, \\ w(x,0) \geq 0, z(x,0) \geq 0, & |x| < 1. \end{cases} \tag{2.4}$$

But $w(0,t) = -t < 0, z(0,t) = -t < 0$ for $0 < t < \frac{1}{6}$. If the nonlocal reactions

$$\int_\Omega (b_1(y,t)w(y,t) + c_1(y,t)z(y,t))dy \quad \text{and} \quad \int_\Omega (b_2(y,t)w(y,t) + c_2(y,t)z(y,t))dy$$

is replaced by the local reactions

$$b_1(x,t)w(x,t) + c_1(x,t)z(x,t) \quad \text{and} \quad b_2(x,t)w(x,t) + c_2(x,t)z(x,t),$$

the nonnegativity of $b_i, c_i (i = 1, 2)$ is not necessary, see Theorem 2.1 in [18].

Based on the above Lemmas, we obtain the following comparison principle of (1.1).

Lemma 2.3. *Let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be sub- and supersolutions of problem (1.1) on $\overline{Q}_T$ respectively. Then $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v})$ on $\overline{Q}_T$.*

3. Global existence and blow up in finite time

We give the following Lemma without proof for our global result of (1.1).

Lemma 3.1. [See Lemma 3.1 in [8]] *Let $\varphi(x,y)$ and $w_0(x)$ be continuous, nonnegative functions on $\partial\Omega \times \overline{\Omega}$ and $\overline{\Omega}$, respectively, and the nonnegative constants*

$\theta_1,\dots,\theta_4$ satisfy $0<\theta_1+\theta_2\le 1, 0<\theta_3+\theta_4\le 1$, and k_1,k_2 are positive constants. Then the solutions of the nonlocal problem

$$\begin{cases} w_t-\Delta w = kw^{\theta_1}\int_\Omega w^{\theta_2}dy + k_2w^{\theta_3}\int_\Omega w^{\theta_4}dy, x\in\Omega, t>0, \\ w(x,t)=\int_\Omega \varphi(x,y)w(y,t)dy, & x\in\partial\Omega, t>0, \\ w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{3.1}$$

are global.

By a modification of the method given in [8], we obtain the following results.

Theorem 3.1. *Suppose that the condition (1.6) holds, then all solutions of problem (1.1) exist globally.*

Proof. Using the condition (1.6) and $p>0, q>0$, there exist $m,l>1$ such that

$$\frac{1-\alpha}{p}\ge\frac{l}{m},\quad \frac{1-\beta}{q}\ge\frac{m}{l}. \tag{3.2}$$

Denote $k=\frac{1}{m}+\frac{1}{l}$. Let $\Phi(x,y)\ge\max\{f(x,y),g(x,y)\}$ be a continuous functions defined for $(x,y)\in\partial\Omega\times\overline{\Omega}$, with

$$a(x)=(\int_\Omega\Phi(x,y)dy)^{\frac{1-m}{m}}, b(x)=(\int_\Omega\Phi(x,y)dy)^{\frac{1-l}{l}},\quad x\in\partial\Omega.$$

Suppose z solves

$$\begin{cases} z_t-\Delta z = kk_1z^{1-m}\int_\Omega z^{m\alpha+lp}dy + kk_2z^{1-l}\int_\Omega z^{mq+l\beta}dy, x\in\Omega, t>0, \\ z(x,t)=(a(x)+b(x))\int_\Omega \varphi(x,y)z(y,t)dy, & x\in\partial\Omega, t>0, \\ z(x,0)=1+u_0^{\frac{1}{m}}(x)+v_0^{\frac{1}{l}}(x), & x\in\Omega, \end{cases} \tag{3.3}$$

Notice that (3.2) implies $0<1-m+m\alpha+lp\le 1, 0<1-l+mq+l\beta\le 1$. In view of Lemma 3.1, we know that z is global. Moreover, $z>1$ in $\overline{\Omega}\times[0,\infty)$ by the maximum principle. Set $(\overline{u},\overline{v})=(z^m,z^l)$. A simple computations shows

$$\begin{aligned}\overline{u}_t = mz^{m-1}z_t &\ge mz^{m-1}(\Delta z+kk_1z^{1-m}\textstyle\int_\Omega z^{m\alpha+lp}dy) \\ &= mz^{m-1}\Delta z + kk_1m\textstyle\int_\Omega z^{m\alpha+lp}dy,\end{aligned}$$

$$\Delta\overline{u}=mz^{m-1}\Delta z+m(m-1)z^{m-2}|\nabla z|^2\le mz^{m-1}\Delta z,$$

and thus

$$\begin{aligned}\overline{u}_t-\Delta\overline{u} &\ge kk_1m\textstyle\int_\Omega z^{m\alpha+lp}dy \ge k_1\int_\Omega z^{m\alpha+lp}dy \\ &= k_1\textstyle\int_\Omega \overline{u}^\alpha(y,t)\overline{v}^p(y,t)dy.\end{aligned}$$

When $(x,t)\in\partial\Omega\times(0,\infty)$, in view of Hölder's inequality,

$$\begin{aligned}\overline{u} &\ge (a(x))^m\{\textstyle\int_\Omega\Phi(x,y)z(y,t)dy\}^m \\ &= \{\textstyle\int_\Omega\Phi(x,y)dy\}^{1-m}\{\int_\Omega\Phi(x,y)z(y,t)dy\}^m \\ &\ge \{\textstyle\int_\Omega f(x,y)dy\}^{1-m}\{\int_\Omega f(x,y)z(y,t)dy\}^m \\ &\ge \textstyle\int_\Omega f(x,y)z^m(y,t)dy = \int_\Omega f(x,y)\overline{u}(y,t)dy.\end{aligned}$$

Similarly, we have also for v that

$$\overline{v}_t-\Delta\overline{v}\ge k_2\int_\Omega z^{mq+l\beta}dy = k_2\int_\Omega \overline{u}^q(y,t)\overline{v}^\beta(y,t)dy,\quad x\in\Omega,\quad t>0,$$

$$\overline{v} \geq \int_\Omega g(x,y)\overline{v}(y,t)dy, \quad x \in \partial\Omega, \quad t > 0.$$

Noticing $u_0(x) \leq \overline{u}(x,0), v_0(x) \leq \overline{v}(x,0)$ in Ω, we have by Lemma 2.3 that $(\overline{u}, \overline{v})$ is a global supersolution of (1.1). □

Theorem 3.2. *Suppose that the condition (1.6) fails with at least one of the following*

$$(a) \ \ \alpha > 1, \quad (b) \ \ \beta > 1, \quad (c) \ \ pq > (1-\alpha)(1-\beta).$$

If $\int_\Omega f(x,y)dy, \int_\Omega g(x,y)dy < 1$ *for all* $x \in \partial\Omega$*, then the solution of (1.1) are global for small initial data.*

Proof. (i) Denote

$$\max\{\max_{\partial\Omega}\int_\Omega f(\cdot,y)dy, \max_{\partial\Omega}\int_\Omega g(\cdot,y)dy\} = \delta_0 \in (0,1). \tag{3.4}$$

Let w be the unique solution of the elliptic problem:

$$-\Delta w = 1 \quad \text{in } \Omega; w = C_0 \quad \text{on } \partial\Omega.$$

Then $C_0 \leq w \leq C_0 + M$ for some $M > 0$ independent of C_0. Let C_0 be large that

$$\frac{1+C_0}{1+C_0+M} \geq \delta_0. \tag{3.5}$$

Due to $\alpha > 1, \beta > 0$, it is easy to verify that for fixed positive constants C_0, M and b, there exists $a > 0$ small such that

$$a \geq k_1 a^\alpha b^p (1+C_0+M)^{\alpha+p}|\Omega|, \quad b \geq k_2 a^\beta b^q (1+C_0+M)^{\beta+q}|\Omega|. \tag{3.6}$$

Set $\overline{u}(x,t) = a(1+w(x)), \overline{v}(x,t) = b(1+w(x))$. We know

$$\begin{aligned} \overline{u}_t - \Delta\overline{u} &= -a\Delta w(x) = a \\ &\geq k_1 a^\alpha b^p (1+C_0+M)^{\alpha+p}|\Omega| \\ &\geq k_1 \int_\Omega a^\alpha (1+w(y))^\alpha b^p (1+w(y))^p dy \\ &= k_1 \int_\Omega \overline{u}^\alpha(y,t)\overline{v}^p(y,t)dy. \end{aligned}$$

in $\Omega \times (0,\infty)$. Similarly, we have $\overline{v}_t - \Delta\overline{v} \geq k_2 \int_\Omega \overline{u}^q(y,t)\overline{v}^\beta(y,t)dy$, in $\Omega \times (0,\infty)$. Moreover, we have on the boundary that

$$\begin{aligned} \overline{u} &= a(1+C_0) \geq a\delta_0(1+C_0+M) \\ &\geq \int_\Omega a\delta_0(1+C_0+M)f(x,y)dy \\ &\geq \int_\Omega f(x,y)\overline{u}(y,t)dy. \end{aligned}$$

And similarly, $\overline{v} \geq \int_\Omega g(x,y)\overline{v}(y,t)dy$, in $\partial\Omega \times (0,\infty)$. By Lemma 2.3, $(\overline{u}, \overline{v})$ is a global super solution of (1.1) provided the initial data small that $u_0(x) \leq a(1+w(x)), v_0(x) \leq b(1+w(x))$ for $x \in \Omega$.

(ii) Due to $\beta > 1, \alpha > 0$, we can prove it by exchanging the roles of u and v in the case (i).

(iii) For $\alpha \leq 1, \beta \leq 1$ and $pq > (1-\alpha)(1-\beta)$, the conclusion is obviously true by the proof of Theorem 2 in [6]. □

Now we prove the blow up conclusions with or without large initial data.

Theorem 3.3. *Suppose (1.5) fails with at least one of the following*

$$(a)\ \alpha > 1;\ \ (b)\ \beta > 1;\ \ (c)\ pq > (1-\alpha)(1-\beta),$$

then the solutions of (1.1) blow up in finite time for large initial data.

Proof. Let λ be the first eigenvalue of the eigenvalue problem

$$-\Delta\phi = \lambda\phi \quad \text{in } \Omega; \qquad \phi = 0 \quad \text{on } \partial\Omega,$$

and ϕ be the corresponding eigenfunction. We choose $\varphi(x)$ such that $\varphi(x) > 0$ in Ω and $\max_{x\in\Omega} = 1$.

(i) Assume that $\alpha > 1$. Since $q > 0$, we can choose $m > 1, l > 1$ such that $mq + l\beta - l > 0$. Set $\gamma = \min\{1 - m + m\alpha + pl, 1 - l + mq + l\beta\}$, then $\gamma > 1$.

Let $s(t)$ be the unique solution of the ODE problem

$$\begin{cases} s'(t) = -\lambda s(t) + \min\{\frac{k_1}{m}\int_\Omega \varphi^{m\alpha+pl}dx, \frac{k_2}{l}\int_\Omega \varphi^{mq+l\beta}dx\}s^\gamma(t), \\ s(0) = s_0 > 0, \end{cases}$$

then $s(t)$ blows up in finite time $T(s_0)$ for sufficiently large data s_0.

Set $\underline{u}(x,t) = s^m(t)\varphi^m(x), \underline{v}(x,t) = s^l(t)\varphi^l(x), (x,t) \in \overline{\Omega} \times [0, T(s_0))$. We assert that $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1).

A simple computation yields

$$\begin{aligned} &\Delta\underline{u} + k_1\int_\Omega \underline{u}^\alpha(y,t)\underline{v}^p(y,t)dy \\ &= s^m(t)[m\varphi^{m-1}\Delta\varphi + m(m-1)\varphi^{m-2}|\nabla\varphi|^2] + k_1 s^{m\alpha+pl}(t)\int_\Omega \varphi^{m\alpha+pl}dy \\ &\geq ms^{m-1}(t)\varphi^m[-\lambda s(t) + (\tfrac{k_1}{m}\int_\Omega \varphi^{m\alpha+pl}dx)s^{1-m+m\alpha+pl}(t)] \\ &\geq ms^{m-1}(t)s'(t)\varphi^m \\ &= \underline{u}_t, \end{aligned}$$

in $\overline{\Omega} \times [0, T(s_0))$, and similarly,

$$\Delta\underline{v} + k_2\int_\Omega \underline{u}^q(y,t)\underline{v}^\beta(y,t)dy \geq \underline{v}_t \text{ in } \overline{\Omega} \times [0, T(s_0)).$$

On the other hand, we have clearly for $(x,t) \in \partial\Omega \times (0, T(s_0)$ that

$$\underline{u}(x,t) = 0 \leq \int_\Omega f(x,y)\underline{u}(y,t)dy, \quad \underline{v}(x,t) = 0 \leq \int_\Omega g(x,y)\underline{v}(y,t)dy.$$

By Lemma 2.3, $(\underline{u}, \underline{v})$ is a blow up subsolution of (1.1) provided the initial data large that $u_0(x) \geq \underline{u}(x,0) = s^m(0)\varphi^m(x), v_0(x) \geq \underline{v}(x,0) = s^l(0)\varphi^l(x)$ for $x \in \Omega$.

(ii) For the case of $\beta > 1$, the proof is similarly.

(iii) If $0 < \alpha \leq 1, 0 < \beta \leq 1$ and $pq > (1-\alpha)(1-\beta)$, noticing $p > 0, q > 0$, there exist two positive constants $m, l > 2$ such that

$$\frac{1-\alpha}{p} < \frac{l}{m}, \quad \frac{1-\beta}{q} < \frac{m}{l}. \tag{3.7}$$

Let $\gamma = \min\{1 - m + m\alpha + pl, 1 - l + mq + l\beta\}$, then $\gamma > 1$. Set $\underline{u}(x,t) = s^m(t)\varphi^m(x), \underline{v}(x,t) = s^l(t)\varphi^l(x)$, where $s(t), \varphi(x)$ is defined in the case (i). The left arguments are as same as those of the case (i). □

Theorem 3.4. *Assume $\alpha > 1$(or $\beta > 1$). If $\int_\Omega f(x,y)dy \geq 1$(or $\int_\Omega g(x,y)dy \geq 1$) for all $x \in \partial\Omega$. Then the solutions of (1.1) blow up in finite time under any positive initial data.*

Proof. In view of $u_0, v_0 > 0$ in Ω, $\int_\Omega f(x,y)dy, \int_\Omega g(x,y)dy > 0$ on $\partial\Omega$,and

$$u_0(x) = \int_\Omega f(x,y)u_0(y)dy, v_0(x) = \int_\Omega g(x,y)v_0(y)dy, \quad x \in \partial\Omega,$$

by the compatibility conditions, we have $u_0, v_0 > 0$ on $\partial\Omega$. Denote by η the positive constant such that $u_0, v_0 > \eta$ on $\overline{\Omega}$.

The assumption (H) implies $u_t, v_t \geq 0$ by the comparison principle and in turn $u, v \geq \eta$ on $\overline{\Omega} \times [0, T)$. Furthermore,$u$ satisfies

$$\begin{cases} u_t \geq \Delta u + k_1\eta^p \int_\Omega u^\alpha(y,t)dy, (x,t) \in Q_T, \\ u = \int_\Omega f(x,y)u(y,t)dy, \qquad (x,t) \in S_T, \\ u(x,0) = u_0(x), \qquad x \in \Omega. \end{cases}$$

Let $\underline{u}(x,t) = s(t)$ be the unique solution of the ODE problem

$$\begin{cases} s'_t = k_1\eta^p|\Omega|s^\alpha(t), (x,t) \in Q_T, \\ s(0) = \frac{\eta}{2}, \qquad x \in \Omega. \end{cases}$$

Then $\underline{u}$ blows up in a finite time since $\alpha > 1$. Clearly,

$$\underline{u}_t = \Delta\underline{u} + k_1\eta^p|\Omega|\underline{u}^\alpha = \Delta\underline{u} + k_1\eta^p \int_\Omega u^\alpha(y,t)dy, \ \ \underline{u}(x,0) \leq u_0(x).$$

Furthermore, the assumption $\int_\Omega f(x,y)dy \geq 1$ implies

$$\underline{u}(x,t) \leq \underline{u}(x,t) \int_\Omega f(x,y)dy = \int_\Omega f(x,y)\underline{u}(y,t)dy, \quad (x,t) \in \partial\Omega \times (0,T).$$

By Lemma 2.3, $u \geq \underline{u}$ as long as both u and $\underline{u}$ exist, and thus u blows up in finite time for any positive initial data u_0.

For the case of $\beta > 1$ and $\int_\Omega g(x,y)dy \geq 1$, the proof is similar. □

Theorem 3.5. *Assume $pq > (1-\alpha)(1-\beta)$. If $\int_\Omega f(x,y)dy, \int_\Omega g(x,y)dy \geq 1$ for all $x \in \partial\Omega$, then the solutions of (1.1) blow up in finite time under any positive initial data.*

Proof. The proof is similar with the Theorem 5 of [6], so we omit it. □

4. Blow up Set

We will fix the blow up set to (1.1) if the solutions of it blow up in finite time. To do this, we introduce a definition as preliminaries:

Definition 4.1. A point $x_0 \in \overline{\Omega}$ is called a blow up point of (u, v) if there exist a sequence $\{(x_n, t_n)\}, x_n \in \Omega, t_n < T^*, (x_n, t_n) \to (x_0, T^*)$ as $n \to \infty$ such that

$$\lim_{n\to\infty} (u(x_n, t_n) + v(x_n, t_n)) = \infty.$$

The blow up set is the set of all blow up points.

Set $h_1(t) = k_1 \int_\Omega u^\alpha(y,t)v^p(y,t)dy, h_2(t) = k_2 \int_\Omega u^q(y,t)v^\beta(y,t)dy$, we have the following Theorem.

Theorem 4.1. *Suppose that the solution of (1.1) blows up in finite time, then the blow up set is the whole domain.*

Proof.(This proof uses some ideas of [17]) Let $G(x,\xi;t,\tau)$ be the Green's function associated with the operator $L=\frac{\partial}{\partial t}-\Delta$ along with null Dirichlet boundary condition in $\Omega\times(0,T^*)$. Then for any $t<T^*$, the solution (u,v) can be written as

$$u(x,y)=\int_\Omega G(x,\xi;t,0)u_0(\xi)d\xi+\int_0^t\int_\Omega G(x,\xi;t,\tau)h_1(\tau)d\xi d\tau \\ -\int_0^t\int_{\partial\Omega}\frac{\partial G}{\partial\nu}(x,\xi;t,\tau)\int_\Omega f(\xi,y)u(y,\tau)dyd\xi d\tau,(x,t)\in Q_T, \quad (4.4)$$
$$v(x,y)=\int_\Omega G(x,\xi;t,0)v_0(\xi)d\xi+\int_0^t\int_\Omega G(x,\xi;t,\tau)h_2(\tau)d\xi d\tau \\ -\int_0^t\int_{\partial\Omega}\frac{\partial G}{\partial\nu}(x,\xi;t,\tau)\int_\Omega g(\xi,y)v(y,\tau)dyd\xi d\tau,(x,t)\in Q_T. \quad (4.6)$$

For the Green's function $G(x,\xi;t,\tau)$, we have $G(x,\xi;t,\tau)=0$ and $\frac{\partial G}{\partial\nu}(x,\xi;t,\tau)\le 0$ on $\partial\Omega$,and we also have the estimates (see [19])

$$0\le G(x,\xi;t,\tau)\le C_1(t-\tau)^{-\frac{N}{2}}\exp\{-\theta_1\frac{|x-\xi|^2}{t-\tau}\},\ \ C_1>0,\theta_1>0, \quad (4.3)$$

and

$$0\le|D_\xi G(x,\xi;t,\tau)|\le C_2(t-\tau)^{-\frac{N+1}{2}}\exp\{-\theta_2\frac{|x-\xi|^2}{t-\tau}\},\ \ C_2>0,\theta_2>0. \quad (4.4)$$

Therefore,

$$\int_\Omega G(x,\xi;t,\tau)d\xi\le C_3, \quad (4.5)$$

and

$$\int_{\partial\Omega}\frac{\partial G}{\partial\nu}(x,\xi;t,\tau)d\xi\le C_4, \quad (4.6)$$

where C_3,C_4 is independent of t. Let x' be a blow up point of (u,v),then there exists a sequence $\{(x_n,t_n)\},x_n\in\Omega,t_n<T^*,(x_n,t_n)\to(x',T^*)$,as $n\to\infty$ such that

$$\lim_{n\to\infty}(u(x_n,t_n)+v(x_n,t_n))=\infty.$$

By (4.1)-(4.6), we obtain

$$\lim_{n\to\infty}\int_0^{t_n}(h_1(\tau)+h_2(\tau))d\tau=\infty. \quad (4.7)$$

For any given $x_0\in\overline{\Omega}$, to prove x_0 is a blow up point, it needs only to prove that for any $\epsilon>0$,$u(x,t)$ is unbounded in $(B(x_0,\epsilon)\bigcap\Omega)\times(0,T)$, where $B(x_0,\epsilon)$ is a ball of R^n centered at x_0 with the radius ϵ. Denote $\Omega_0=B(x_0,\epsilon)\bigcap\Omega$. On the contrary, we suppose that there exists an $M>0$ such that $u(x,t)\le M$ in $\Omega_0\times(0,T)$. Let $G_0(x,\xi;t,\tau)$ be the Green's function in Ω_0 associated with the operator $L=\frac{\partial}{\partial t}-\Delta$ along with the null Dirichlet boundary condition. Then we have

$$u(x,y)=\int_{\Omega_0}G_0(x,\xi;t,0)u_0(\xi)d\xi+\int_0^t\int_{\Omega_0}G_0(x,\xi;t,\tau)h_1(\tau)d\xi d\tau \\ -\int_0^t\int_{\partial\Omega_0}\frac{\partial G_0}{\partial\nu}(x,\xi;t,\tau)u(\xi,\tau)dSd\tau,\ (x,t)\in\Omega_0\times(0.T), \quad (4.8)$$
$$v(x,y)=\int_{\Omega_0}G_0(x,\xi;t,0)v_0(\xi)d\xi+\int_0^t\int_{\Omega_0}G_0(x,\xi;t,\tau)h_2(\tau)d\xi d\tau$$

$$-\int_0^t \int_{\partial\Omega_0} \frac{\partial G_0}{\partial \nu}(x,\xi;t,\tau)v(\xi,\tau)dSd\tau,\ (x,t)\in\Omega_0\times(0.T). \qquad (4.9)$$

Since $G_0(x,\xi;t,\tau)\geq 0, u_0(x)\geq 0, v_0(x)\geq 0$, we have

$$u(x,y)\geq \int_0^t\int_{\Omega_0} G_0(x,\xi;t,\tau)h_1(\tau)d\xi d\tau$$
$$-\int_0^t\int_{\partial\Omega_0}\frac{\partial G_0}{\partial\nu}(x,\xi;t,\tau)u(\xi,\tau)dSd\tau,\ \ (x,t)\in\Omega_0\times(0.T),$$
$$v(x,y)\geq \int_0^t\int_{\Omega_0} G_0(x,\xi;t,\tau)h_2(\tau)d\xi d\tau$$
$$-\int_0^t\int_{\partial\Omega_0}\frac{\partial G_0}{\partial\nu}(x,\xi;t,\tau)v(\xi,\tau)dSd\tau,\ \ (x,t)\in\Omega_0\times(0.T).$$

Since $G_0>0$ in Ω_0 and $G_0=0$ on $\partial\Omega_0$, we have $\frac{\partial G_0}{\partial\nu}\leq 0$ on $\partial\Omega_0$. Thus

$$-\int_0^t\int_{\partial\Omega_0}\frac{\partial G_0}{\partial\nu}(x,\xi;t,\tau)u(\xi,\tau)dSd\tau\geq 0,$$

$$-\int_0^t\int_{\partial\Omega_0}\frac{\partial G_0}{\partial\nu}(x,\xi;t,\tau)v(\xi,\tau)dSd\tau\geq 0.$$

These estimates give

$$u(x,t)+v(x,t)$$
$$\geq \int_0^t(h_1(\tau)+h_2(\tau))\int_{\Omega_0}G_0(x,\xi;t,\tau)d\xi d\tau,(x,t)\in\Omega_0\times(t,T^*). \qquad (4.10)$$

Choose a compact subset $\Omega'\subset\subset\Omega_0$, by use of the strong maximum principle, we have that there exists $\delta=\delta(\Omega')$ such that

$$\int_{\Omega_0}G_0(x,\xi;t,\tau)d\xi\geq\delta(\Omega')\ \text{ for all }\ (x,t)\in\Omega'\times[t,T^*),\ \ \tau>0. \qquad (4.11)$$

From (4.10) and (4.11) it follows that for all $(x,t)\in\Omega'\times[0,T^*)$,

$$u(x,t)+v(x,t)\geq\delta(\Omega')\int_0^t(h_1(\tau)+h_2(\tau))d\tau.$$

Therefore, we have $\lim_{t\to T^*}(u(x,t)+v(x,t))=\infty$. It is in contradiction with the assumption of $u(x,t),v(x,t)\leq M$ in $\Omega_0\times(0,T]$. Therefore x_0 is a blow up point. By the arbitrariness of $x_0\in\overline{\Omega}$ we obtain that the blow up set is $\overline{\Omega}$. The proof is completed. □

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