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A BIO-ECONOMIC MODEL OF TWO-PREY ONE-PREDATOR SYSTEM

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ABSTRACT. We propose a model based on Lotka-Volterra dynamics with two competing spices which are affected not only by harvesting but also by the presence of a predator, the third species. Hyperbolic and linear response functions are considered. We derive the conditions for global stability of the system using Lyapunov function. The optimal harvest policy is studied and the solution is derived in the interior equilibrium case using Pontryagin's maximal principle. Finally, some numerical examples are discussed. The nature of variations in the two prey species and one predator species is studied extensively through graphical illustrations.

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1. Introduction

Bioeconomic models integrate economic and biological influences with the goal of assisting natural resource managers in determining appropriate levels of stocks and catches. The biological resources which have socio-economic values are of great importance for such modeling. Fishery is one of the such biological renewable resources. The rise in interest in managing fisheries is due to a globally perceived decrease in ocean and sea productivity as a result of over fishing and mismanagement of the fishing resources. An increase in fishing power due to advancement of technologies, high growth rate of world population, and a lack of knowledge about the characteristics of exploited species, are all causes of decrement in different species of fishes.

In recent years many works on optimal management of renewable resources are done. An excellent introduction to optimal management of renewable resources is given by Clark [4]. Policies related to bioeconomic exploitation of renewable resources are discussed by Clark [4,5]. Harvesting of fish population in an open access fishery model is studied in detail by Clark [4]. In this article we consider an

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open access fishery model with two-prey and one-predator. In recent past, some works on two-prey and one-predator are done. Kar and Chaudhuri [8] discussed a bioeconomic model of two-prey and one-predator fishery with linear functional response. Models on combined harvesting of two species prey-predator fishery have been discussed by Kar et al. [9], Samanta et al.[16], Chaudhuri and Saha [3], Mesterton - Gibbons [12], Ragozin and Brown [15], Hanneson [6]. Bioeconomic exploitation of both the species in a Lotka - Volterra prey-predtor system was discussed by Chaudhuri and Saha Ray[2]. Krishna et al [11] discussed the conservation of an exploited ecosystem with optimal taxation on harvesting. A capital theoretic model of two species fishery with taxation as a control instrument was developed by Pradhan and Chaudhuri [14].

Our present paper consists of two preys which are continuously being harvested in time by harvesting agencies and one predator which survives on the two prey species. We assume that the predator is not affected directly by the harvesting agencies. The harvesting activities reduce the predator population indirectly by reducing the availability of the two prey species to the predator. Thus more the harvesting effort, the predator population is less. Due to obligation to the society and necessity of preserving all three species we wish to evolve a optimal harvesting policy. A good example of such system is minke whales and two of its main fish prey items : juvenile herring and capelin. It is to be noted that IWC (International Whaling Commission) has imposed a complete ban on killing whales. As capelin and herring declined due to fishing, their availability as food for whales declined, resulting in a reduction of minlke's consumption rate and consequently their production and biomass. Our model is based on the following assumption :

(i) One prey is much higher in abundance and more vulnerable compared to other prey.
(ii) Interspecific competition among the prey species is exploitative.
(iii) Handling time for one prey is negligible(minke whale and capeline), whereas the predator needs sufficient handling time for other prey (mink whale and juveline herring) and these are incorporated using two different functional responses of Holling types I and II [7].

The bioeconomic model presented in this work has the following organisation. The problem is formulated in Section 2. Boundedness of solution is discussed in Section 3. Finding the equilibrium points and the analysis of local stability at these points are given in Section 4. Global stability analysis at the interior equililbrium is carried out in Section 5. The optimal harvest policy is discussed in Section 6 and some numerical examples are given in Section 7.

2. Formulation of the problem

We assume two competiting fish species which compete with each other for the use of a common resource and both of them are subjected to continuous independent harvesting. There is a predator feeding on both of them and is assumed that the predator population is not harvested.

Since we are not making a case study in respect of a specific prey-predator community, we have opted for logistic growth function for the two prey species. It is assumed that the feeding rate of predator species increases linearly with first prey density and non-linearly (more specifically with hyperbolic functional response) with second prey species. The governing equations of the system are written as

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - a_1xy - b_1xz - q_1E_1x, \tag{1}$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) - a_2xy - \frac{b_2yz}{m+y} - q_2E_2y, \tag{2}$$

$$\frac{dz}{dt} = \alpha_1b_1xz + \frac{\alpha_2b_2yz}{m+y} - cz, \tag{3}$$

where $r, s, K, L, a_1, b_1, a_2, b_2, \alpha_1, \alpha_2, c, q_1, q_2$ and m are positive constants. More clearly, r and s are biotic potentials and K, L are environmental carrying capacities of the two prey species; a_1, a_2 are co-efficients of interspecific competition between two prey species; b_1, b_2 are predation co-efficients; $\alpha_1\ (< 1)$ and $\alpha_2\ (< 1)$ are conversion factors; E_1, E_2 are harvesting efforts and q_1, q_2 are catchability co-efficients of the two prey population x and y respectively; m is the half-saturation constant; c is the natural mortality rate of the predator population z. The catch-rate functions q_1E_1x and q_2E_2y are based on CPUE (catch-per-unit-effort) hypothesis. We consider that

p_1 = constant price per unit of biomass of first species;

p_2 = constant price per unit of biomass of second species;

and c_1, c_2 are the constant costs of fishing for the two prey species per unit effort. Our objective is to study the optimal harvest policy i.e. $E_i(t) \in [0, E_i^{\max}]$ where $i = 1, 2$ and $t \in [0, \infty)$ to maximize the profits to the harvesting agencies and to keep the predator population at an optimal level. The objective functional representing the present value J of a continuous time stream of revenues, is given by

$$J(E_1, E_2) = \int_0^\infty e^{-\delta t}[p_1q_1E_1x + p_2q_2E_2y - c_1E_1 - c_2E_2]dt \tag{4}$$

where δ is instantaneous annual rate of discount. In the Section 6, we will find an optimal harvest policy solving the following optimization problem :

$$\left.\begin{array}{ll} \text{Maximize} & J(E_1, E_2) \\ \text{subject to} & (1) - (3) \\ \text{and} & 0 \le E_i(t) \le E_i^{\max} \\ \text{for} & i = 1, 2 \end{array}\right\} \tag{5}$$

Before doing this, it is important to study the nature of the equilibrium solutions associated with the system. This is made in the Section 4. We now examine the boundedness of the solution of the system (1)-(3) in the next section.

3. Boundedness of the system

If the variables x, y, z and the parameters involved in the equations (1)-(3) are positive then the right hand sides of these equations are smooth functions of x, y, z and the parameters. So local existence, uniqueness and continuation properties hold in the positive octant R_+^3. We now establish the following lemma for boundedness.

Lemma 1. *All solutions of (1) - (3) which initiate in* R_+^3 *are uniformly bounded if* $E_2 > \frac{s}{q_2}$.

Proof. We introduce a function

$$\omega = \alpha_1 x + \alpha_2 y + z \tag{6}$$

The time derivative of (6) along the solutions of (1) - (3) is given by

$$\frac{d\omega}{dt} = \alpha_1 r x\left(1 - \frac{x}{K}\right) - a_1\alpha_1 xy - b_1\alpha_1 xz - \alpha_1 q_1 E_1 x + \alpha_2 sy\left(1 - \frac{y}{L}\right) - a_2\alpha_2 xy$$
$$-\frac{b_2\alpha_2 zy}{m+y} - q_2E_2\alpha_2 y + \alpha_1 b_1 zx + \frac{\alpha_2 b_2 yz}{m+y} - cz$$

$\therefore$ For each $\eta > 0$,

$$\frac{d\omega}{dt} + \eta\omega \le \alpha_1 x\left[r\left(1 - \frac{x}{K}\right) + \eta - q_1E_1\right] + \alpha_2[\eta - (q_2E_2 - s)]y + (\eta - c)z \tag{7}$$

Now using $[K(r + \eta - q_1E_1) - 2rx]^2 \ge 0$, we can derive that

$$x\left[r\left(1 - \frac{x}{K}\right) + \eta - q_1E_1\right] \le \frac{K}{4r}(r + \eta - q_1E_1)^2.$$

Hence (7) can be written as,

$$\frac{d\omega}{dt} + \eta\omega \le \alpha_1\frac{K}{4r}(r + \eta - q_1E_1)^2 + [\eta - \alpha_2(q_2E_2 - s)y + (\eta - c)z.$$

Since $E_2 > \frac{s}{q_2}$, if we choose $\eta < \min\{(q_2E_2 - s), c\}$, then right hand side of the above inequation is bounded i.e. $\exists\, l > 0$ such that $\frac{d\omega}{dt} + \eta\omega < l$. Applying a theory of differential equation (Birkhoff and Rota[1]), we get

$$0 < \omega(x, y, z) < \frac{l}{\eta}(1 - e^{-\eta t}) + \omega(x(0), y(0), z(0))e^{-\eta t}$$

and for $t \to \infty$, $0 < \omega(x, y, z) < \frac{l}{\eta}$. Hence all the solutions of (1) - (3) that initiate at $(x(0), y(0), z(0))$ lie in R_+^3 and are confined in the region

$$\Gamma = \left\{(x, y, z) \in R_+^3; \alpha_1 x + \alpha_2 y + z = \frac{l}{\eta} + h \text{ for any } h > 0\right\},$$

for all $t \ge T$ where T depends upon the initial values $(x(0), y(0), z(0))$. Thus the set Γ is an invariant set and contains the Ω - limit set of all the paths of the system(1) that initiate in the positive octant R_+^3. □

4. Equilibria and stability analysis

The equilibrium points of the system are the solution of the steady state equations

$$x\left[r\left(1 - \frac{x}{K}\right) - a_1 y - b_1 z - q_1E_1\right] = 0, \tag{8}$$

$$y\left[s\left(1-\frac{y}{L}\right)-a_2x-\frac{b_2z}{m+y}-q_2E_2\right]=0, \tag{9}$$

$$z\left[\alpha_1b_1x+\alpha_2\frac{b_2y}{m+y}-c\right]=0. \tag{10}$$

Since we assume that the predator species has a positive mortality rate c, the possible nonnegative equilibria of the system are P_0, P_1, P_2, P_3, P_4, P_5, and P_6 which are given below :

$P_0 : (0,0,0)$

$$P_1 : (0, e_{12}, 0) \text{ where } e_{12}=L\left(1-\frac{q_2E_2}{s}\right) \tag{11}$$

$$\text{with } s > q_2E_2. \tag{12}$$

$$P_2 : (0, e_{22}, e_{23}) \text{ where, } e_{22}=\frac{cm}{\alpha_2b_2-c} \tag{13}$$

$$e_{23}=\frac{m\alpha_2}{\alpha_2b_2-c}\left[s\left(1-\frac{cm}{L(\alpha_2b_2-c)}-q_2E_2\right)\right], \tag{14}$$

with the conditions

$$\left.\alpha_2b_2>c, 1-\frac{cm}{L(\alpha_2b_2-c)}>\frac{q_2E_2}{s}\right\}. \tag{15}$$

$P_3 : (e_{31}, 0, 0)$ where

$$e_{31}=K\left(1-\frac{q_1E_1}{r}\right) \tag{16}$$

$$\text{with } r > q_1E_1. \tag{17}$$

$P_4 : (e_{41}, e_{42}, 0)$ where the coordinates of P_4 are given by

$$e_{41}=\frac{\left[\frac{s}{L}(r-q_1E_1)-a_1(s-q_2E_2)\right]}{\left(\frac{r}{K}\frac{s}{L}-a_1a_2\right)} \tag{18}$$

$$e_{42}=\frac{\left[\frac{r}{K}(s-q_2E_2)-a_2(r-q_1E_1)\right]}{\left(\frac{r}{K}\frac{s}{L}-a_1a_2\right)}. \tag{19}$$

The conditions of existence of the equilibrium point P_4 are

$$\left.\begin{array}{c}\frac{La_1}{s}(s-q_2E_2)<(r-q_1E_1)<\frac{r}{Ka_2}(s-q_2E_2)\\ \text{and} \quad \frac{La_1}{s}<\frac{r}{Ka_2}\end{array}\right\} \tag{20}$$

or,

$$\left.\begin{array}{c}\frac{r}{Ka_2}(s-q_2E_2)<(r-q_1E_1)<\frac{La_1}{s}(s-q_2E_2)\\ \text{and} \quad \frac{r}{Ka_2}<\frac{La_1}{s}\end{array}\right\} \tag{21}$$

$$P_5 : (e_{51}, 0, e_{53}) \text{ where } e_{51}=\frac{c}{\alpha_1b_1}, \tag{22}$$

$$e_{53}=\frac{1}{b_1}\left[r\left(1-\frac{c}{K\alpha_1b_1}\right)-q_1E_1\right], \tag{23}$$

with the condition $(r - q_1E_1) > \dfrac{rc}{K\alpha_1 b_1}$. (24)

The interior equilibrium point $P_6\,(x^*, y^*, z^*)$ is the solution of the system of equations,

$$r\left(1-\frac{x}{K}\right) - a_1 y - b_1 z - q_1 E_1 = 0, \quad (25)$$

$$s\left(1-\frac{y}{L}\right) - a_2 x - \frac{b_2 z}{m+y} - q_2 E_2 = 0, \quad (26)$$

$$\alpha_1 b_1 x + \alpha_2 b_2 \frac{y}{m+y} - c = 0. \quad (27)$$

From (27), we get

$$x = \frac{c_1}{\alpha_1 b_1} - \frac{\alpha_2 b_2}{\alpha_1 b_1}\frac{y}{m+y}, \quad (28)$$

and from (26), we get

$$z = \frac{1}{b_2}\left[\frac{s}{L}(L-y)(m+y) - \left(\frac{a_2 c}{\alpha_1 b_1} + q_2 E_2\right)(m+y) + \frac{a_2\alpha_2}{\alpha_1 b_1}\,y. \quad (29)$$

Using these expression of x and z we get from (25) that y satisfies the cubic

$$Ay^3 + By^2 + Cy + D = 0 \quad (30)$$

where

$$A = \frac{b_1 s}{b_2 L} > 0, \quad (31)$$

$$B = \frac{b_1}{b_2}\left(\frac{a_2 c}{\alpha_1 b_1} + q_2 E_2 - s + \frac{2sm}{L}\right) - \left(a_1 + \frac{a_2\alpha_2}{\alpha_1}\right), \quad (32)$$

$$C = r\left(1 - \frac{c}{K\alpha_1 b_1} + \frac{\alpha_2 b_2}{K\alpha_1 b_1}\right) + \frac{b_1 m}{b_2}\left(\frac{2a_2 c}{\alpha_1 b_1} + 2q_2E_2 - 2s + \frac{sm}{L}\right)$$
$$-\left(a_1 m + \frac{a_2\alpha_2}{\alpha_1}m + q_1 E_1\right), \quad (33)$$

and

$$D = m\left[\left(r - \frac{rc}{K\alpha_1 b_1} - q_1 E_1\right) + \frac{m a_2 c}{\alpha_1 b_2} - \frac{m b_1}{b_2}(s - q_2 E_2)\right]. \quad (34)$$

In the cubic (30), $A = \dfrac{b_1 s}{b_2 L} > 0$ and therefore by Descarte's rule of signs this cubic has atleast one positive root if $D < 0$, i.e. if

$$r - q_1 E_1 - \frac{rc}{K_1\alpha_1 b_1} + \frac{m a_2 c}{\alpha_1 b_2} < \frac{m b_1}{b_2}(s - q_2 E_2) \quad (35)$$

Let $y = y^*$ be a positive root of the cubic (30) and the corresponding values of x and z obtained from (28) and (29) respectively are $x = x^*$ and $z = z^*$. Thus we have the interior equilibrium point $P_6(x^*, y^*, z^*)$ provided (35) holds and $x^* > 0,\ y^* > 0$.

To analyze the local stability at the equilibrium of the system, we consider the variational matrix

$$V(x,y,z) = \begin{bmatrix} v_{11} & -a_1 x & -b_1 x \\ -a_2 y & v_{22} & \frac{-b_2 y}{m+y} \\ \alpha_1 b_1 z & \frac{\alpha_2 b_2 m z}{(m+y)^2} & v_{33} \end{bmatrix} \quad (36)$$

where

$$\left.\begin{aligned} v_{11} &= \left[r\left(1-\frac{x}{K}\right)-a_1y-b_1z-q_1E_1\right]-\frac{rx}{K} \\ v_{22} &= \left[s\left(1-\frac{y}{L}\right)-a_2x-\frac{b_2z}{m+y}-q_2E_2\right]+\frac{b_2yz}{(m+y)^2}-\frac{sy}{L} \\ v_{33} &= \alpha_1b_1x+\frac{\alpha_2b_2y}{m+y}-c \end{aligned}\right\} \tag{37}$$

The stability of the equilibrium points will be determined by the nature of the eigenvalues of the Jaccobian matrix evaluated at the corresponding equilibrium points. At $P_0\,(0,0,0)$, $V(P_0)$ has three eigenvalues given by

$$\lambda_0^1 = r-q_1E_1,\ \lambda_0^2 = s-q_2E_2, \quad \text{and} \ \lambda_0^3 = -c.$$

Thus origin is a stable node if $E_1 > \dfrac{r}{q_1}$ and $E_2 > \dfrac{s}{q_2}$, i.e. if the harvesting efforts are more than the corresponding biotechnical productivities (BTP) of the two prey species. At $P_1\,(0,e_{12},0)$, the variational matrix $V(P_1)$ has eigenvalues given by

$$\lambda_1^1 = r-a_1e_{12}-q_1E_1,\ \lambda_1^2 = -\frac{se_{12}}{L},\ \text{and}\ \lambda_1^3 = \frac{\alpha_2b_2e_{13}}{m+e_{12}}-c.$$

Therefore, the equilibrium P_1 is stable if $(r-q_1E_1) < a_1e_{12}$ and $\dfrac{\alpha_2b_2e_{12}}{m+e_{12}} < c$.

At $P_2\,(0,e_{22},e_{23})$, one of the eigenvalues of the variational matrix $V(P_2)$ is given by $\lambda_2^1 = r-a_1e_{22}-b_1e_{23}-q_1E_1$ which is positive or negative according as $(r-q_1E_1) >$ or $< (a_1e_{22}+b_1e_{23})$.

The other two eigenvalues of $V(P_2)$ are the roots of $\lambda^2-u\lambda-v\omega=0$ whose sum $= u = \dfrac{b_2\,e_{22}\,e_{23}}{(m+e_{22})^2}-\dfrac{se_{22}}{L}$, and product $= -v\omega = \dfrac{\alpha_2\,b_2^2\,me_{22}\,e_{23}}{(m+e_{23})^3} > 0$.

Hence by Routh - Hurwitz condition [10] the equilibrium at P_2 is asymptotically stable if $(r-q_1E_1) < (a_1e_{22}+b_1e_{23})$ and $\dfrac{b_2\,e_{23}}{(m+e_{22})^2} < \dfrac{s}{L}$. At $P_3\,(e_{31},0,0)$, the variational matrix $V(P_3)$, has three eigenvalues given by $\lambda_3^1 = \dfrac{-re_{31}}{K}$, $\lambda_3^2 = s-a_2e_{31}-q_2E_2$, $\lambda_3^3 = \alpha_1b_1e_{31}-c$.

This shows that the equilibrium at P_3 is asymptotically stable if $(s-q_2E_2) < a_2\,e_{31}$ and $\alpha_1b_1e_{31} < c$.

The eigenvalues of the variational matrix at $P_4\,(e_{41},e_{42},0)$, are given by $\lambda_4^3 = \alpha_1\,b_1\,e_{41}+\dfrac{\alpha_2\,b_2\,e_{42}}{(m+e_{42})}$, and λ_4^1, λ_4^2 are the roots of the following quadratic equation in λ $\lambda^2+\left(\dfrac{re_{41}}{K}+\dfrac{se_{42}}{L}\right)\lambda+\left(\dfrac{rs}{KL}-a_1\,a_2\right)e_{41}\,e_{42}=0$, where $\left(\frac{re_{41}}{K}+\frac{se_{42}}{L}\right) > 0$.

Thus by Routh-Hurwitz condition this quadratic equation has negative real roots or complex conjugates with negative real part iff $rs > a_1a_2KL$.

Thus together with the condition of existence of equilibrium at P_4 we have that the equilibrium at P_4 is a stable node or a stable focus if $rs > a_1 a_2 KL$, $\frac{La_1}{s}(s - q_2E_2) < (r - q_1E_1) < \frac{r}{Ka_2}(s - q_2E_2)$, and $\left(\alpha_1 b_1 e_{41} + \frac{\alpha_2 b_2 e_{42}}{m + e_{42}}\right) < c$.

At $P_5(e_{51}, 0, e_{53})$, the eigenvalues $\lambda_5^i\,(i = 1,2,3)$ of $V(P_5)$ are the roots of the following cubic equation in λ, $\lambda^3 - Trace\,|V(P_5)|\,\lambda^2 + Trace\,|adj\,V(P_5)|\,\lambda - |V(P_5)| = 0$. By Routh-Hurwitz's condition this cubic equation has roots with negative real parts if and only if $Trace\,|V(P_5)| < 0$, $|V(P_5)| < 0$, and $Trace\,|V(P_5)|\,Trace\,|adj\,V(P_5)| < |V(P_5)|$.

If we denote $\left(s - a_2 e_{51} - \frac{b_2 e_{53}}{m} - q_2 E_2\right)$ by M, then Routh - Hurwitz's conditions become $M - \frac{re_{51}}{K} < 0$, $\alpha_1 b_1^2 e_{51} e_{53} M < 0$ and

$$\left(M - \frac{re_{51}}{K}\right)\left(\alpha_1 b_1^2 e_{51} e_{53} - \frac{re_{51}M}{K}\right) < (\alpha_1 b_1^2 e_{51} e_{53})M,$$

which are all satisfied if $M < 0$. Thus the equilibrium at P_5 is stable (when it exists) if $M < 0$, i.e., if $(s - q_2 E_2) < a_2 e_{51} + \frac{b_2 e_{53}}{m}$.

The eigenvalues $\lambda_6^i\,(i = 1,2,3)$ of $V(P_6)$ are the roots of the following characteristic cubic equation in λ, $\lambda^3 - Trace\,|V(P_6)|\,\lambda^2 + Trace\,|adj\,V(P_6)|\,\lambda - |V(P_6)| = 0$.

By Routh - Hurwitz's conditions, this cubic equation has roots with negative real parts if and only if

$Trace\,|V(P_6)| < 0, |V(P_6)| < 0$, and $Trace\,|V(P_6)|\,Trace\,|adj\,V(P_6)| < |V(P_6)|$.

Let $\xi = |V(P_6)| =$

$$x^*\,y^*\,z^*\left[b_1\left(\frac{a_2\alpha_2 b_2 m}{(m+y^*)^2} - \frac{s\alpha_1 b_1}{L} + \frac{\alpha_1 b_1 b_2 z^*}{(m+y^*)^2}\right) - \frac{b_2}{m+y^*}\left(\frac{r\alpha_2 b_2 m}{K(m+y^*)^2} - a_1\alpha_1 b_1\right)\right],$$

$\eta = Trace\,|V(P_6)| = \frac{b_2 y^* z^*}{(m+y^*)^2} - \frac{sy^*}{L} - \frac{rx^*}{K}$, and

$\zeta = Trace\,|adj\,V(P_6)| = \alpha_1 b_1^2 x^* z^* - \frac{\alpha_2 b_2^2 m y^* z^*}{(m+y^*)^3} - \frac{r}{K}x^* y^*\left[\frac{b_2 z^*}{(m+y^*)^2} - \frac{s}{L} + \frac{K}{r} a_1 a_2\right]$.

The conditions that make the interior equilibrium $P_6(x^*, y^*, z^*)$, if it exists, asymptotically stable can be expressed as $\eta < 0, \xi < 0$ and $\zeta\eta < \xi$.

5. Global stability

In this section, we consider the global stability of the given system by constructing a suitable Lyapunov function. We define a Lyapunov function $v(x,y,z) = (x - x^*) - x^* \log(\frac{x}{x^*}) + (y - y^*) - y^* log(\frac{y}{y^*}) + (z - z^*) - z^* log(\frac{z}{z^*})$.

The time derivative of $v(x, y, z)$ is given by

$$\frac{dv}{dt} = (x - x^*)\left[r\left(1 - \frac{x}{K}\right) - a_1 y - b_1 z - q_1 E_1\right] + (y - y^*)$$

$$\left[s\left(1-\frac{y}{L}\right)-a_2x-\frac{b_2z}{m+y}-q_2E_2\right]+(z-z^*)\left[\alpha_1b_1x+\frac{\alpha_2b_2y}{m+y}-c\right]$$
$$=-\frac{r}{k}(x-x^*)^2-\left[\frac{s}{L}-\frac{b_2z^*}{(m+y)(m+y^*)}\right](y-y^*)^2$$
$$-(a_1+a_2)(x-x^*)(y-y^*)-b_1(1-\alpha_1)(x-x^*)(z-z^*)$$
$$-(y-y^*)(z-z^*)\left[\frac{b_2y^*-\alpha_2b_2m+mb_2}{(m+y)(m+y^*)}\right]$$
$$=-X^TAX \quad \text{where, } X^T=[x-x^*,y-y^*,z-z^*]$$

and
$$A=\begin{bmatrix}\frac{r}{K} & \frac{a_1+a_2}{2} & \frac{1}{2}b_1(1-\alpha_1)\\ \frac{a_1+a_2}{2} & \frac{s}{L}-\frac{b_2z^*}{(m+y^*)(m+y)} & \frac{b_2(m+y^*-m\alpha_2)}{2(m+y)(m+y^*)}\\ \frac{1}{2}b_1(1-\alpha_1) & \frac{b_2}{2}\frac{m+y^*-\alpha_2m}{(m+y^*)(m+y)} & 0\end{bmatrix}.$$

Let $u_1=\dfrac{a_1+a_2}{2}, u_2=\dfrac{1}{2}b_1(1-\alpha_1), u_3=\dfrac{b_2z^*}{m+y^*}, u_4=\dfrac{b_2}{2}\dfrac{m+y^*-m\alpha_2}{m+y^*}$ (38)

The point $P_6(x^*,y^*,z^*)$ is globally asymptotically stable if $\frac{dv}{dt}<0$ i.e. if A is positive definite. Now A is positive definite if

$$\frac{r}{K}\left(\frac{s}{L}-\frac{u_3}{m+y}\right)>u_1^2,$$

or, $$(m+y)>\frac{b_2z^*}{(m+y^*)\left[\frac{s}{L}-\frac{K}{4r}(a_1+a_2)\right]} \tag{39}$$

and, $$-\frac{r}{K}\frac{u_4^2}{(m+y)^2}-u_1\left(-\frac{u_2u_4}{m+y}\right)+u_2\left\{\frac{u_1u_4}{m+4}-\frac{u_2s}{L}+\frac{u_2u_3}{m+y}\right\}>0.$$

or, $$\frac{s}{L}(m+y)^2-u_4\left(2\frac{u_1}{u_2}+\frac{u_3}{u_4}\right)(m+y)+\frac{r}{K}\frac{u_4^2}{u_2^2}<0.$$

This implies that(m+y) lies between

$$\frac{u_4\left(\frac{2u_1}{u_2}+\frac{u_3}{u_4}\right)\pm\sqrt{u_4^2\left(\frac{2u_1}{u_2}+\frac{u_3}{u_4}\right)^2-\frac{4sru_4^2}{LKu_2^2}}}{2\frac{s}{L}}$$

That is $U<(m+y)<W$ (40)

where

$$U=\frac{b_2L}{2b_1s(m+y^*)(1-\alpha_1)}\Bigg[(a_1+a_2)(m+y^*-m\alpha_2)+b_1z^*(1-\alpha_1)$$
$$-\sqrt{\left\{[(a_1+a_2)(m+y^*-m\alpha_2)+b_1z^*(1-\alpha_1)]^2-\frac{4sr}{LK}(m+y^*-m\alpha_2)^2\right\}}\Bigg] \tag{41}$$

and,

$$W=\frac{b_2L}{2b_1s(m+y^*)(1-\alpha_1)}\Bigg[(a_1+a_2)(m+y^*-m\alpha_2)+b_1z^*(1-\alpha_1)$$
$$-\sqrt{\left\{[(a_1+a_2)(m+y^*-m\alpha_2)+b_1z^*(1-\alpha_1)]^2-\frac{4sr}{LK}(m+y^*-m\alpha_2)^2\right\}}\Bigg] \tag{42}$$

Thus $P_6(x^*,y^*,z^*)$ is globally asymptotically stable in the region which satisfies (39) and (40). The feasible region is therefore given by $U<(m+y)<W$ where

$\frac{b_2z^*}{(m+y^*)[\frac{s}{L}-\frac{K}{4r}(a_1+a_2)^2]} < U$ and the region of stability is $\frac{b_2z^*}{(m+y^*)[\frac{s}{L}-\frac{K}{4r}(a_1+a_2)^2]} <$ $(m+y) < W$ where $\frac{b_2z^*}{(m+y^*)[\frac{s}{L}-\frac{K}{4r}(a_1+a_2)^2]}$ lies between U and W. Moreover, this feasible region of global stability is obtained if

$$[(a_1+a_2)(m+y^*-m\alpha_2)+b_1z^*(1-\alpha)]^2 > \frac{4sr}{LK}(m+y^*-m\alpha_2)^2$$

and
$$\frac{b_2z^*}{(m+y^*)\left[\frac{s}{L}-\frac{K}{4r}(a_1+a_2)^2\right]} < W.$$

6. Optimal harvest policy

Once the process of harvesting the resource is started, the problem of management of the fisheries can be viewed in terms of rent maximization. We have already discussed the dynamics of the system governed by the equations (1) - (3) in the Section 2 and now we are in a position to apply maximum principle [13] to solve the optimization problem (5) and obtain optimal harvesting policy i.e. the values of $E_1(t)$ and $E_2(t)$ such that $J(E_1, E_2)$ is maximized. Importance of discount rate cannot be under-estimated in addressing environmental and resource issues. In fact, the optimal stock size for a given fishery will vary depending on the discount rate. We restate our problem to maximize the objective functional

$$J(E_1,E_2)=\int_0^{\infty} e^{-\delta t}[p_1q_1E_1x+p_2q_2E_2y-c_1E_1-c_2E_2]dt \tag{43}$$

subject to the state equations (1) -(3) and the control constraints

$$0 \le E_i(t) \le E_i^{\max}; \quad i=1,2. \tag{44}$$

The Hamiltonian for this optimal control problem

$$H=e^{-\delta t}[p_1q_1E_1x+p_2q_2E_2y-c_1E_1-c_2E_2+\lambda_1\left[rx\left(1-\frac{x}{K}\right)-a_1xy-b_1xz-q_1E_1x\right]$$
$$+\lambda_2\left[sy\left(1-\frac{y}{L}\right)-a_2xy-b_2\frac{yz}{m+y}-q_2E_2y\right]+\lambda_3\left[\alpha_1b_1xz+\alpha_2\frac{b_2yz}{m+y}-cz\right] \tag{45}$$

where $\lambda_i=\lambda_i(t), i=1,2,3$ are adjoint variables.
We have

$$\frac{\partial H}{\partial E_1}=e^{-\delta t}(p_1q_1x-c_1)-\lambda_1q_1x=\sigma_1(t)(say), \tag{46}$$

$$\frac{\partial H}{\partial E_2}=e^{-\delta t}(p_2q_2y-c_2)-\lambda_2q_2y=\sigma_2(t)(say). \tag{47}$$

The optimal control $E_i(t), (i=1,2)$ must clearly satisfy the conditions

$$E_i(t)=\begin{cases} E_i^{\max} & \text{when } \sigma_i(t)>0, \\ 0 & \text{when } \sigma_i(t)<0. \end{cases}$$

Since $\sigma_i(t)$ causes $E_i(t)$ to switch between the levels 0 and $E^{\max}, \sigma_i(t)(i=1,2)$ are called switching function. Depending on the sign of the switching function $\sigma_i(t)$, the optimal control $E_i(t)$ is a bang-bang control switching from one extreme level to other one .

Once $\sigma_i(t)(i = 1, 2)$ vanishes, the Hamiltonian function H becomes independent of the control variable $E_i(t)(i = 1, 2)$ and its optimal value cannot be determined by the above procedure. It is then called a singular control $E_i^*(t), 0 < E_i^*(t) < E_i^{\max}$. Hence the optimal harvest policy is

$$E_i(t) = \begin{cases} E_i^{\max} & \text{when} \quad \sigma_i(t) > 0, \\ 0 & \text{when} \quad \sigma_i(t) < 0, \\ E_i^*(t) & \text{when} \quad \sigma_i = 0 \end{cases}$$

for $i = 1, 2$.

For singular control, $\sigma_1(t) = 0$ and $\sigma_2(t) = 0$. Then from (46) and (47) we get,

$$\lambda_1 = e^{-\delta t}\left(p_1 - \frac{c_1}{q_1 x}\right), \tag{48}$$

$$\lambda_2 = e^{-\delta t}\left(p_2 - \frac{c_2}{q_2 y}\right). \tag{49}$$

Now the adjoint equations are

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x}, \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y}, \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial z}. \tag{50}$$

Using (45) and the third equation of (50) we get,

$$\frac{d\lambda_3}{dt} = \lambda_1 b_1 x + \lambda_2 \frac{b_2 y}{m + y} - \lambda_3\left(\alpha_1 b_1 x + \frac{\alpha_2 b_2 y}{m + y} - c\right)$$

and at equilibrium it becomes $\frac{d\lambda_3}{dt} = \lambda_1 b_1 x + \lambda_2 \frac{b_2 y}{m + y}$. By (48) and (49) we get

$$\frac{d\lambda_3}{dt} = e^{-\delta t}\left[b_1\left(p_1 - \frac{c_1}{q_1 x}\right)x + b_2\left(p_2 - \frac{c_2}{q_2 y}\right)\frac{y}{m + y}\right].$$

$$\therefore \lambda_3 = -\frac{e^{-\delta t}}{\delta}\left[b_1\left(p_1 - \frac{c_1}{q_1 x}\right)x + b_2\left(p_2 - \frac{c_2}{q_2 y}\right)\frac{y}{m + y}\right]. \tag{51}$$

We here consider that the constant of integration vanishes so that the shadow prices $\lambda_i e^{\delta t}(i = 1, 2, 3)$ of three species are bounded.

Now, using (48) and (49) we get from the first two adjoint equations in (50),

$$\delta e^{-\delta t}\left(p_1 - \frac{c_1}{q_1 x}\right) = \left[e^{-\delta t} p_1 q_1 E_1 - \lambda_1 \frac{rx}{K} - \lambda_2 a_2 y + \lambda_3 \alpha_1 b_1 z\right] \tag{52}$$

and, $\delta e^{-\delta t}\left(p_2 - \frac{c_2}{q_2 y}\right) = e^{-\delta t} p_2 q_2 E_2 - \lambda_1 a_1 x$

$$+ \lambda_2\left[-\frac{sy}{L} + \frac{b_2 y z}{(m + y)^2}\right] + \lambda_3 \frac{\alpha_2 b_2 m z}{(m + y)^2} \tag{53}$$

Using the equilibrium conditions and the values of λ_1, λ_2 and λ_3 in (52), (53) we get,

$$p_1 q_1 E_1 = \delta\left(p_1 - \frac{c_1}{q_1 x}\right) + \frac{rx}{K}\left(p_1 - \frac{c_1}{q_1 x}\right) + a_2 y\left(p_2 - \frac{c_2}{q_2 y}\right)$$
$$+ \frac{\alpha_1 b_1 z}{\delta}\left[b_1\left(p_1 - \frac{c_1}{q_1 x}\right)x + b_2\left(p_2 - \frac{c_2}{q_2 y}\right)\frac{y}{m + y}\right],$$

and,

$$p_2q_2E_2 = \delta\left(p_2 - \frac{c_2}{q_2y}\right) + a_1x\left(p_1 - \frac{c_1}{q_1x}\right) - \left(p_2 - \frac{c_2}{q_2y}\right)\left[-\frac{sy}{L} + \frac{b_2yz}{(m+y)^2}\right]$$

$$+\frac{\alpha_2b_2mz}{\delta(m+y)^2}\left[b_1x\left(p_1 - \frac{c_1}{q_1x}\right) + b_2\frac{y}{m+y}\left(p_2 - \frac{c_2}{q_2y}\right)\right].$$

These equations give the optimal harvesting efforts as

$$E_1 = \frac{\left[\left(p_1 - \frac{c_1}{q_1x}\right)\left(\delta + \frac{rx}{K} + \frac{\alpha_1b_1^2zx}{\delta}\right) + \left(p_2 - \frac{c_2}{q_2y}\right)\left(a_2y + \frac{\alpha_1b_1b_2yz}{\delta(m+y)}\right)\right]}{p_1q_1} \quad (54)$$

and,

$$E_2 = \frac{\left[\left(p_1 - \frac{c_1}{q_1x}\right)\left(a_1x + \frac{\alpha_2b_1b_2mxz}{\delta(m+y)^2}\right) + \left(p_2 - \frac{c_2}{q_2y}\right)\left(\delta + \frac{sy}{L} - \frac{b_2yz}{(m+y)^2} + \frac{\alpha_2b_2^2myz}{\delta(m+y)^3}\right)\right]}{p_2q_2} \quad (55)$$

Hence solving the steady state equations together with (54) and (55) we get an optimal solution $(x_\delta, y_\delta, z_\delta,)$ and the optimal harvesting efforts E_1 and E_2 .

7. Numerical examples

Example 1. Let us consider a set of values of parameters as follows in appropriate units: $r = 2.09, K = 100, a_1 = 0.01, b_1 = 0.05, q_1 = 0.04, E_1 = 15, s = 1.5, L = 100,\ a_2 = 0.001, b_2 = 0.02, m = 15, q_2 = 0.05, E_2 = 20, \alpha_1 = 0.8, \alpha_2 = 0.5, c = 1$.

Then the equilibrium points (which exist) and the eigenvalues of variational matrices at the corresponding points with their are given in the table below:

Table 1

Equilibrium	Eigen values	Nature
$P_o(0,0,0)$	1.34,0.50, -1.0	Unstable
$P_1(0,33.33,0)$	-0.50,1.76,0.99	Unstable
$P_3(71.29,0,0)$	-1.4900,-0.92,1.85	Unstable
$P_4(57.17,29.52,0)$	-1.51,-0.13,1.3	Unstable
$P_5(25,0,19.35)$	-0.26+0.95i, -0.26-0.95i, -0.026	Stable
$P_6(24.83,31.3,13.16)$	-0.2+0.68i,-0.2-0.68i,-0.59	Stable

We consider another example to find optimal equilibrium and optimal harvests.

Example 2. Let $r = 2.09, K = 100, a_1 = 0.01, b_1 = 0.05, \alpha_2 = 0.5, q_1 = 0.04, s = 1.5, L = 100, a_2 = 0.001, c_1 = 1, b_2 = 0.02, m = 15, q_2 = 0.05, \alpha_1 = 0.8, p_1 = 8, p_2 = 25, c_1 = 7.5, c_2 = 9, \delta = 0.04$ in appropriate units.

For the above set of values of the parameters the optimal equilibrium is $(x_\delta, y_\delta, z_\delta,) = (24.82, 38.88, 6.75)$ and the optimal harvesting efforts are given by $E_1 = 21.12$ and $E_2 = 17.79$.

8. Concluding remarks

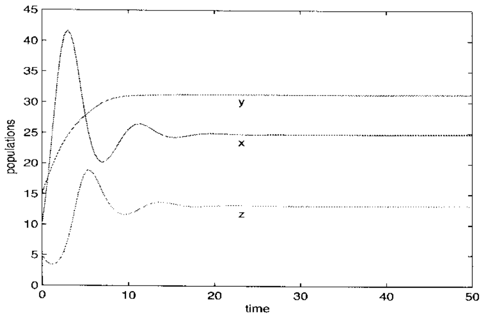

Fig 1.Solution curves

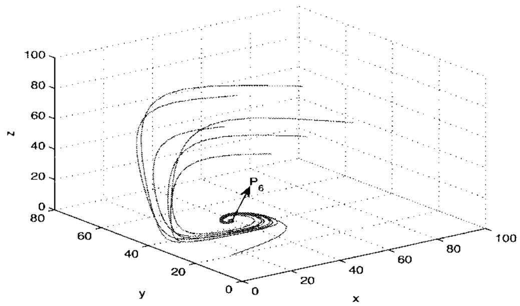

Fig 2. Phase space trajectories corresponding to the singular optimal efforts $E_1 = 21.12$ and $E_2 = 17.79$, beginning with different initial levels. Trajectories clearly indicate that the optimal equilibrium (24.82, 38.88, 6.75) is globally asymptotically stable.

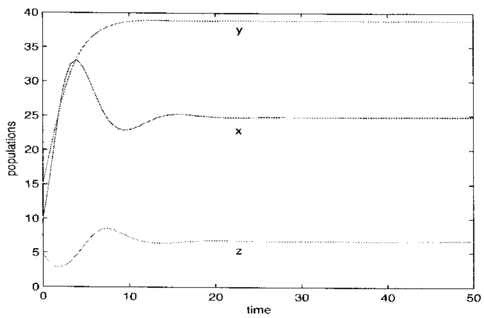

Fig 3. Solution curves corresponding to singular optimal efforts $E_1 = 21.12$ and $E_2 = 17.79$.

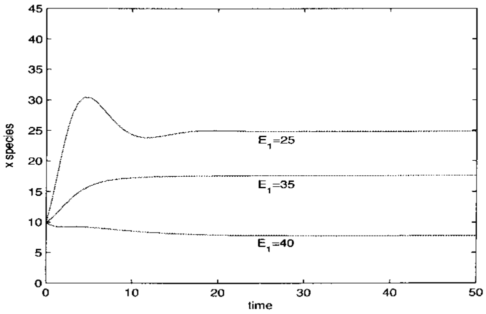

Fig 4. Variation of x for different values of E_1 with fixed $E_2 = 20$

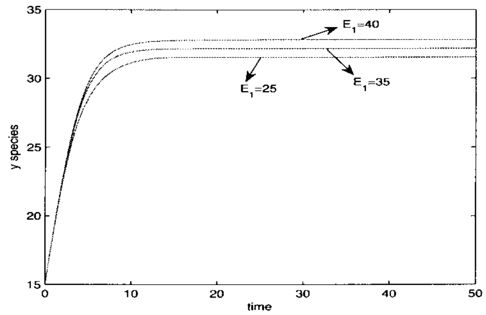

Fig 5. Variation of y for different values of E_1 with fixed $E_2 = 20$.

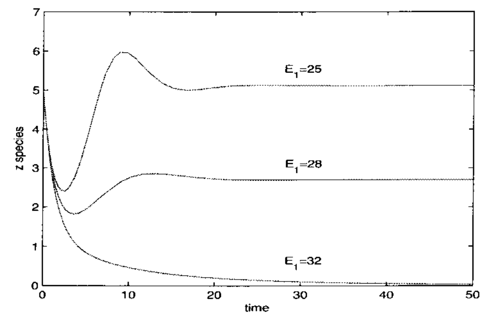

Fig 6. Variation of z for different values of E_1 with fixed $E_2 = 20$.

In this paper, we have discussed the effects of harvesting in a two-species competitive system in the presence of a predator. We have first studied the existence and stability (local as well as global) of the possible steady states. In this bioeconomic model, we then discussed the optimal harvest policy. The present value

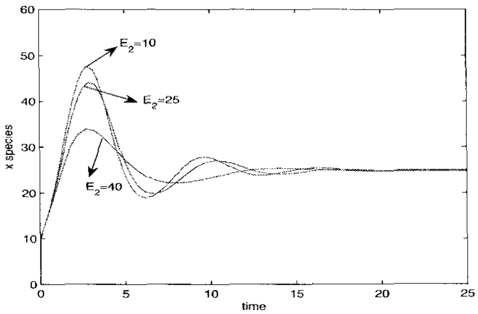

Fig 7. Variation of x for different values of E_2 with fixed $E_1 = 15$.

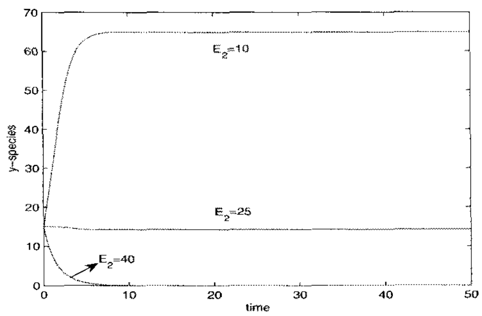

Fig 8. Variation of y for different values of E_2 with fixed $E_1 = 15$.

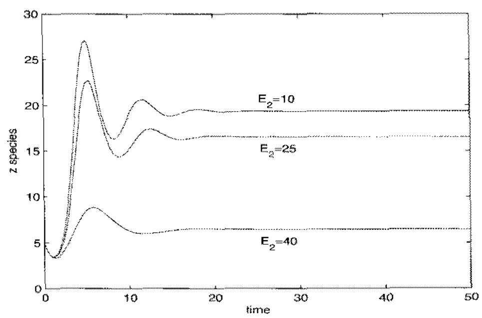

Fig 9. Variation of z for different values of E_2 with fixed $E_1 = 15$.

of a continuous time-stream of revenues is maximized by Pontryagin's maximum principle. The case of an optimal equilibrium solution is studied.

We show several numerical examples of the prey-predator population system with increasing harvesting efforts. Predator population always decreases with increased harvesting efforts. Responses of the two types of the fishing efforts of preys are complicated. This is due to inter-specific competition between the species. Predator can survive even on the extinction of one prey species. Responses of predator to preys fishing efforts are different. This is due to different functional responses. Optimal equilibrium and optimal harvests are found for examples. To get the numerical results and graphs, we used Lingo and MatLab.

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