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A NOTE ON THE REFLECTION SYMMETRIES OF THE GENOCCHI POLYNOMIALS

C. S. RYOO

ABSTRACT. It is the aim of this paper to consider the reflection symmetries of the Genocchi polynomials $G_n^*(x)$. We display the shape of Genocchi polynomials $G_n^*(x)$. Finally, we investigate the roots of the Genocchi polynomials $G_n^*(x)$.

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1. Introduction

Over the years, the computing environment would make more and more rapid progress. Numerical experiments of Bernoulli polynomials and Euler polynomials have been the subject of extensive study in recent year and much progress have been made both mathematically and computationally. Recently, using computer experiments, Agarwal, Kim, and Ryoo [6, 7, 8] described a structure of the complex roots of q-Bernoulli polynomials. Ryoo [6] investigated a structure of the complex roots of Genocchi polynomials. Therefore, using computer, a realistic study for Genocchi polynomials $G_n(x)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the Genocchi polynomials $G_n^*(x)$ in complex plane. The outline of this paper is as follows. We introduce the Genocchi polynomials $G_n(x)$. Genocchi polynomials was introduced in [2]. These Genocchi polynomials are related to ordinary Euler polynomials, (see [1]). The studies in these areas are seems to be interesting and many mathematicians are studying to investigate the relation between Genocchi numbers and polynomials, (see [2, 4]). In Section 2, we describe the beautiful zeros of the Genocchi polynomials $G_n^*(x)$ using a numerical investigation. Finally, we observe the directions for further research. First, we introduce

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the Genocchi numbers and Genocchi polynomials. The Genocchi numbers G_n are defined by the generating function:

$$F(t) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \ (|t| < \pi), \ \text{cf. [1, 4]}$$

where we use the technique method notation by replacing G^n by $G_n (n \geq 0)$ symbolically. In general, it satisfies $G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given $G_n = 2(1-2^{2n})B_{2n} = 2nE_{2n-1}$, where B_n are Bernoulli numbers and E_n are Euler numbers. For $x \in \mathbb{R}$ (= the field of real numbers), we consider the Genocchi polynomials as follows:

$$F(x,t) = F(t)e^{xt} = \frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \ \text{see [4, 5, 6]}.$$

Note that $G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}$. In the special case $x = 0$, we define $G_n(0) = G_n$. Since

$$\begin{aligned} F(1-x,-t) &= \frac{-2t}{e^{-t}+1}e^{(1-x)(-t)} = \sum_{n=0}^{\infty} G_n(1-x)\frac{(-t)^n}{n!} \\ &= \frac{-2t}{e^t+1}e^{xt} = -F(x,t) = -\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \end{aligned}$$

we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{N}$, we have*

$$G_n(1-x) = (-1)^{n-1} G_n(x).$$

We consider the reflection symmetries of the Genocchi polynomials. Set

$$G_n^*(x) = G_n(1-x).$$

We also have the following corollary.

Corollary 2. *If $G_n^*(x) = 0$, then $G_n(x) = 0$.*

Because

$$\frac{\partial F}{\partial x}(x,t) = tF(x,t) = \sum_{n=0}^{\infty} \frac{dG_n}{dx}(x)\frac{t^n}{n!},$$

it follows the important relation

$$\frac{dG_n^*}{dx}(x) = -nG_{n-1}^*(x).$$

We have the integral formula as follows:

$$\int_a^b G_{n-1}^*(x)dx = -\frac{1}{n}(G_n^*(b) - G_n^*(a)).$$

Then, it is easy to deduce that $G_n^*(x)$ are polynomials of degree $n-1$. Here is the list of the first Genocchi's polynomials $G_n^*(x)$.

$G_1^*(x) = 1, \quad G_2^*(x) = -2x+1, \quad G_3^*(x) = 3x^2 - 3x, \quad G_4^*(x) = -4x^3 + 6x^2 - 1$

$G_5^*(x) = 5x^4 - 10x^3 + 5x, \quad G_6^*(x) = -6x^5 + 15x^4 - 15x^2 + 3,$

$G_7^*(x) = 7x^6 - 21x^5 + 35x^3 - 21x,$

$G_8^*(x) = -8x^7 + 28x^6 - 70x^4 + 84x^2 - 17,$

$G_9^*(x) = 9x^8 - 36x^7 + 126x^5 - 242x^3 + 153x,$

$G_{10}^*(x) = -10x^9 + 45x^8 - 210x^6 + 630x^4 - 765x^2 + 155,$

$G_{11}^*(x) = 11x^{10} - 55x^9 + 330x^7 - 1386x^5 + 2805x^3 - 1705x,$

$G_{12}^*(x) = -12x^{11} - 66x^{10} - 495x^8 + 2772x^6 - 8415x^4 + 10230x^2 - 2073, \cdots$

2. Zeros of the Genocchi polynomials $G_n^*(x)$

The purpose of this section is to investigate the roots of the Genocchi polynomials $G_n^*(x)$ for values of the index n by using computer. We display the shapes of the Genocchi polynomials $G_n^*(x)$. For $n = 1, \cdots, 10$, we can draw a plot of the Genocchi polynomials $G_n^*(x)$, respectively. This shows the ten plots combined into one. For $n = 1, \cdots, 10$, we display the shapes of $G_n^*(x), n = 1, \cdots, 10, -10 \le x \le 10$. (Figures 1, 2).

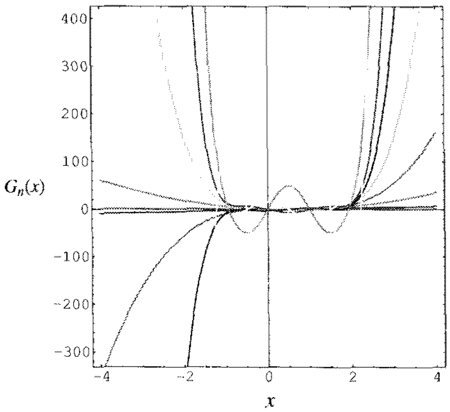

FIGURE 1. Curves of $G_n(x)$

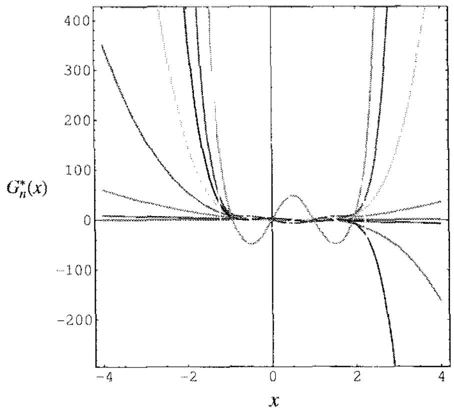

FIGURE 2. Curves of $G_n^*(x)$

We plot the zeros of $G_n(x)$ and $G_n^*(x)$ for $n = 40$ and $x \in \mathbb{C}$. (Figures 3, 4).

Comparing this paper with paper [6], there is no difference between these two results about the roots (see [6]). By numerical experiments, we can verify the

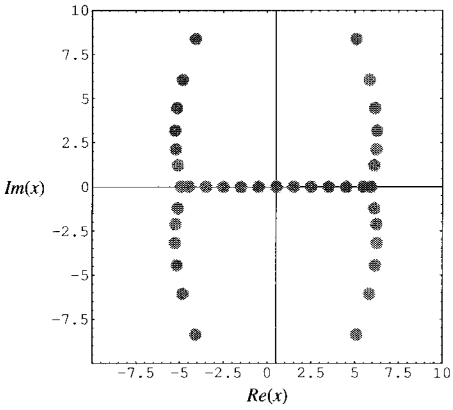

FIGURE 3. Zeros of $G_n(x)$

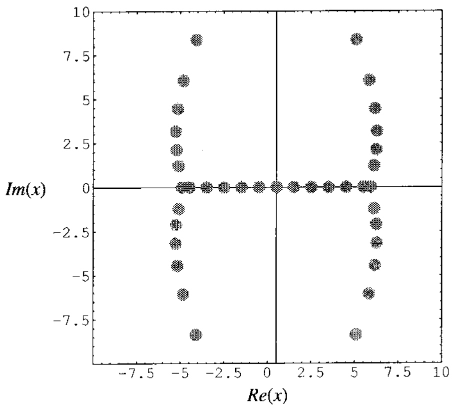

FIGURE 4. Zeros of $G_n^*(x)$

Corollary 2. We investigate the beautiful zeros of the $G_n^*(x), n = 15, 20, 25, 30$ by using a computer. (Figure 5).

Next, we calculated an approximate solution satisfying $G_n^*(x), x \in \mathbb{C}$. The results are given in Table 1.

Table 1. Approximate solutions of $G_n^*(x) = 0, x \in \mathbb{C}$

degree n	x
7	0.0000000, $-0.8606180 - 0.3182471i$, $-0.8606180 + 0.3182471i$ 1.0000000 $1.860618 - 0.318247i$, $1.860618 + 0.318247i$
8	$-1.0356003 - 0.4804187i$, $-1.0356003 + 0.4804187i$, -0.4977314 0.5000000 1.497731, $2.035600 - 0.480419i$, $2.035600 + 0.480419i$
9	$-1.172878 - 0.670300i$, $-1.172878 + 0.670300i$, -1.21973, 0.0000 1.000000, 1.9323, $2.17288 - 0.67030i$ $2.17288 + 0.67030i$
10	$-1.31362 - 0.87637i$, $-1.31362 + 0.87637i$, -0.932328, -0.500080 0.500000, 1.5001, 2.220, $2.3136 - 0.8764i$ $2.3136 + 0.8764i$

In Figures 4, 5 and Table1, $G_n^*(x), x \in \mathbb{C}$, has $Im(z) = 0$ reflection symmetry. This translates to the following open problem: Prove that $G_n^*(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry. We show the stacks of zeros of $G_n^*(x), 1 \leq n \leq 40$ (Figure 6).

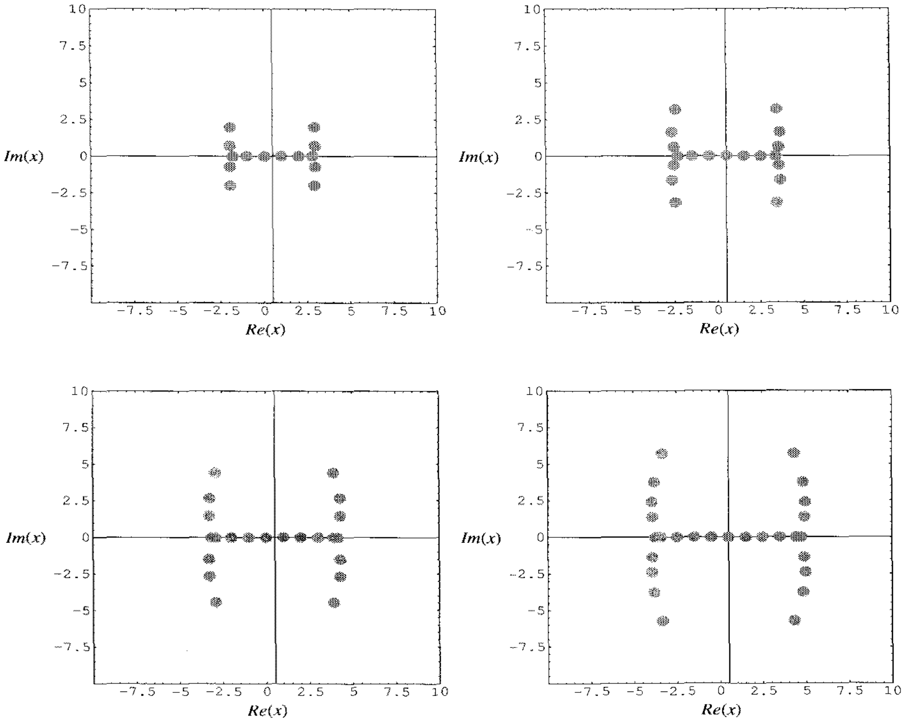

FIGURE 5. Zeros of $G_n^*(x)$ for $n = 15, 20, 25, 30$

We calculated an approximate solution satisfying $G_n^*(x), x \in \mathbb{R}$. The results are given in Table 2.

Our numerical results for approximate solutions of real zeros of $G_n^*(x)$ are displayed (Table 3).

We observe a remarkably regular structure of the complex roots of Genocchi polynomials $G_n^*(x)$. We hope to verify a remarkably regular structure of the complex roots of Genocchi polynomials $G_n^*(x)$ (Table 3).

3. Directions for further research

In this section we discuss the further research. We shall consider the more general open problems. In general,how many roots does $G_n^*(x)$ have ? Prove that $G_n^*(x)$ has $n-1$ distinct solutions. Find the numbers of complex zeros $C_{G_n^*(x)}$ of $G_n^*(x), Im(x) \neq 0$. Prove or give a counterexample: *Conjecture*: Since $n-1$ is the degree of the polynomial $G_n^*(x)$, the number of real zeros $R_{G_n^*(x)}$

lying on the real plane $Im(x) = 0$ is then $R_{G_n^*(x)} = n-1-C_{G_n^*(x)}$, where $C_{G_n^*(x)}$ denotes complex zeros.

See Table 3 for tabulated values of $R_{G_n^*(x)}$ and $C_{G_n^*(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs. Prove or disprove: $G_n^*(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry. For $n = 1, \cdots, 10$, we can draw a plot of the Genocchi polynomials, respectively. This shows the ten curves combined into one (Figures 1, 2).

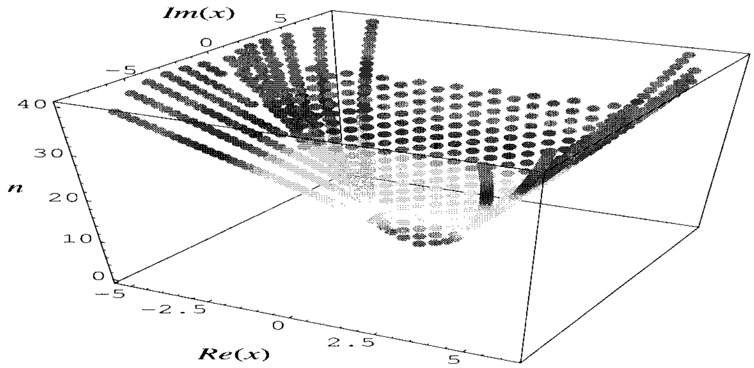

FIGURE 6. Stacks of zeros of $G_n^*(x), 1 \leq n \leq 40$

Table 2. Approximate solutions of $G_n^*(x) = 0, x \in \mathbb{R}$

degree n	x
2	0.500000000
3	0.000000000, 1.000000000
4	−0.366025404, 0.500000000, 1.366025404
5	−0.61803399, 0.0000000, 1.00000000, 1.61803399
6	−0.6180, −0.6180, 0.50000000, 1.618, 1.618
7	0.0000000, 1.0000000
8	−0.4977314, 0.5000000, 1.497731
9	−0.932328, 0.0000000, 1.000000, 1.9323
10	−1.21973, −0.500080, 0.500000, 1.5001, 2.220
11	−1.3652, −1.0150, 0.0000000, 1.0000, 2.015, 2.37

These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the Genocchi numbers, polynomials, and roots of the Genocchi polynomials polynomials. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the Genocchi polynomials to appear in mathematics and physics. For related topics the interested reader is referred to [6], [7], [8].

Table 3. Numbers of real and complex zeros of $G_n^*(x)$

degree n	real zeros	complex zeros
2	1	0
3	2	0
4	3	0
5	4	0
6	6	0
⋮	⋮	⋮
10	5	4
11	6	4
12	3	8
13	4	8
14	5	8
15	6	8

References

1. T. Kim, *A note on q-Volkenborn integration*, Proc.Jangjeon Math. Soc., **8**(2005), 13-17.
2. L. C. Jang, T. Kim, D.H. Lee, D.W. Park, *An application of polylogarithms in the analogs of Genocchi numbers*, Notes Number Theory Discrete Math. **7**(2001), 35-69.
3. T. Kim, *On p-adic q-L-functions and sums of powers*, Discrete Math., **252**(2002), 179-187.
4. T. Kim, S. H. Rim, *On Changhee-Barnes' q-Euler numbers and polynomials*, Adv. Stud. Contemp. Math. **9**(2004), 81-86.
5. T. Kim, *On Euler-Barnes multiple zeta functions*, Russ. J. Math. Phys **10**(2003), 261-267.
6. C. S. Ryoo, *A numerical investigation on the zeros of the Genocchi polynomials*, J. Appl. Math. Comput. , **22** (2006),125-132.

7. C. S. Ryoo , *Calculating zeros of the twisted Genocchi polynomials,* Advan. Stud. Contemp. Math., **17**(2008) , 147-159.
8. C. S. Ryoo, T. Kim, R. P. Agarwal, *Exploring the multiple Changhee q-Bernoulli polynomials,* Inter. J. Comput. Math., v.82(4)(2005), 483-493.

Cheon Seoung Ryoo received his Ph.D. degree from Kyushu University in 1998. His research interests focus on the numerical analysis and scientific computing.

Department of Mathematics, Hannam University, Daejeon 306-791, Korea
e-mail: ryoocs@hnu.kr