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HYBRID-TYPE SET–VALUED VARIATIONAL–LIKE INEQUALITIES IN REFLEXIVE BANACH SPACES

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ABSTRACT. In this paper, we introduce a relaxed hybrid-type η-f-α-pseudomonotonicity. By using the KKM-technique, we establish some existence results for set-valued variational-like inequalities with η-f-α-pseudomonotone mappings in reflexive Banach spaces.

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1. Introduction and preliminaries

A monotone concept of a defined mapping is very important together with the continuity and the convexity in nonlinear analysis including variational inequality problems, complementarity problems, optimization problems, programming problems, equilibrium problems, game theory and so on.

Many authors have proposed some important generalizations of monotonicity such as pseudomonotonicity, relaxed monotonicity, η-α-monotonicity, η-f-monotonicity and quasi-monotonicity, see [1-3, 5-6, 9-14] and the references therein. In [14] Verma studied a class of variational inequalities with relaxed monotone operators. Recently, Fang and Huang [5] defined a relaxed η-α-monotone concept and Bai et al. [1] defined a relaxed η-α-pseudomonotone concept for single-valued mappings. For set-valued mappings, Kang *et al.* [10] defined a relaxed η-f-α-pseudomonotone concept, which generalizes monotone concept for single-valued mappins in Fang *et al.* and Bai *et al.*.

Inspired and motivated by [1, 5, 10], in this paper we introduce a more generalized new concept of relaxed η-f-α-pseudomonotone mappings with respect

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to hybrid-type mappings. By using the KKM-technique, we establish some existence results of solutions to set-valued variational-like inequalities with η-f-α-pseudomonotone mappings in reflexive Banach spaces. Our results are generalizations of many existing works of [1, 3, 5, 7, 8, 10].

Let E be a real reflexive Banach space with the dual space E^* and $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. Let K be a nonempty subset of E and 2^E denote the family of all nonempty subsets of E.

Definition 1.1. A set-valued mapping $T : K \to 2^{E^*}$ is said to be relaxed η-f-α-pseudomonotone with respect to the second argument of a hybrid-type mapping $N : K \times E^* \to E^*$ if there exist a mapping $\eta : K \times K \to E$ and functions $f : K \times K \to \mathbb{R}$, $\alpha : E \to \mathbb{R}$ with $\alpha(tz) = k(t)\alpha(z)$ for $z \in E$, where $k : (0, \infty) \to (0, \infty)$ is a function with $\lim_{t \to 0} \frac{k(t)}{t} = 0$, such that for any $x, y \in K$,

$$\Big\langle N(x,u),\ \eta(y,x) \Big\rangle + f(y,x) \ \geq \ 0 \text{ for all } u \in T(x)$$

implies

$$\Big\langle N(x,v),\ \eta(y,x) \Big\rangle + f(y,x) \ \geq \ \alpha(y-x) \text{ for all } v \in T(y).$$

Remark 1.1. If $N(x,u) = u$ and $N(x,v) = v$ in Definition 1.1, then we have the following pseudomonotone concept defined in [10];
For all $x, y \in K$

$$\Big\langle u,\ \eta(y,x) \Big\rangle + f(y,x) \ \geq \ 0 \text{ for all } u \in T(x)$$

implies

$$\Big\langle v,\ \eta(y,x) \Big\rangle + f(y,x) \ \geq \ \alpha(y-x) \text{ for all } v \in T(y).$$

Remark 1.2. If T is a single-valued mapping and $k(t) = t^p$ for $p > 1$, then we have the following relaxed η-α-monotone concept (i) defined in [5] and the following relaxed η-α-pseudomonotone concept (ii) defined in [1];

(i) For any $x,\ y \in K$

$$\Big\langle T(x) - T(y),\ \eta(x,y) \Big\rangle \ \geq \ \alpha(x-y),$$

(ii) For any $x,\ y \in K$,

$$\Big\langle T(y),\ \eta(x,y) \Big\rangle \ \geq \ 0 \text{ implies } \langle T(x),\ \eta(x,y) \rangle \ \geq \ \alpha(x-y).$$

2. Existence results

In this paper, we consider the following two hybrid-type set-valued variational-like inequalities in reflexive Banach spaces;

⟨Stampacchia-type⟩

(i) Find $x \in K$ satisfying

$$\Big\langle N(x,u), \eta(y,x) \Big\rangle + f(y,x) \geq 0 \text{ for } y \in K \text{ and } u \in T(x). \tag{1.1}$$

⟨Minty-type⟩

(ii) Find $x \in K$ satisfying

$$\Big\langle N(x,v), \eta(y,x) \Big\rangle + f(y,x) \geq \alpha(y-x) \text{ for } y \in K \text{ and } v \in T(y). \tag{1.2}$$

Definition 2.1. Let K be a nonempty convex subset of E. Let $T : K \to 2^{E^*}$ and $\eta : K \times K \to E$ be mappings. T is said to be *η-hemicontinuous* if for any $x, y \in K$, a mapping $g : [0,1] \to 2^{\mathbb{R}}$ defined by

$$g(t) \;=\; \bigcup_{u_t \in T(x_t)} \langle u_t, \eta(y,x) \rangle, \text{ where } x_t = x + t(y-x)$$

is upper semicontinuous at 0^+.

Definition 2.2. Let $T : K \to 2^{E^*}$, $N : K \times E^* \to E^*$, $\eta : K \times K \to E$ be mappings and $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ be a proper function. T is said to be η-coercive with respect to f if there exists an $x_0 \in K$ such that for all $u \in T(x)$ and $u_0 \in T(x_0)$.

$$\frac{\langle N(x,u) - N(x_0,u_0),\ \eta(x,x_0) \rangle + f(x,x_0)}{\|\eta(x_0,x)\|} \to \infty,$$

whenever $\|x\| \to \infty$.

Definition 2.3. [4] A mapping $F : K \to 2^E$ is said to be a KKM-mapping if for any finite subset $\{x_1, x_2, \ldots, x_n\} \subset K$, $\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$, where $co\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

Fan-KKM Theorem. [4] *Let K be a nonempty subset of a Hausdorff topological vector space E and let $F : K \to 2^E$ be a KKM-mapping. If $F(x)$ is closed in E for every $x \in K$ and compact for some $x_0 \in K$, then*

$$\bigcap_{x \in K} F(x) \;\neq\; \emptyset.$$

We discuss existence of solutions to (1.1) in a reflexive Banach space.

Theorem 2.1. *Let K be a nonempty closed convex subset of a real reflexive Banach space E. Let $T : K \to 2^{E^*}$ be an η-hemicontinuous and relaxed η-f-α-pseudomonotone set-valued mapping with respect to the second argument of a mapping $N : K \times E^* \to E^*$. Assume that*

(i) $\eta(x,y) + \eta(y,x) = 0$ *and* $f(x,y) + f(y,x) = 0$ *for all* $x, y \in K$;
(ii) $x \mapsto \eta(x,\cdot)$ *and* $x \mapsto f(x,\cdot)$ *are convex.*

Then $x \in K$ *is a solution of* (1.1) *if and only if it is a solution of* (1.2).

Proof. If x is a solution of (1.1), by Definition 1.1 it is a solution of (1.2).

Conversely, suppose that $x \in K$ is a solution of (1.2) and $y \in K$ is any point. Let

$$x_t = ty + (1-t)x, \ t \in [0,1], \text{ then } x_t \in K.$$

It follows from (1.2) that, for $u_t \in T(x_t)$,

$$\begin{aligned}\Big\langle N(x,u_t),\ \eta(x_t,x)\Big\rangle + f(x_t,x) &\geq \alpha(x_t - x)\\ &= \alpha(t(y-x))\\ &= k(t)\alpha(y-x). \end{aligned} \tag{2.1}$$

By conditions (i) and (ii), we have

$$\begin{aligned}&\Big\langle N(x,u_t), \eta(x_t,x)\Big\rangle + f(x_t,x)\\ &= \Big\langle N(x,u_t),\ \eta(ty+(1-t)x,x)\Big\rangle + f(ty+(1-t)x,x)\\ &\leq t\Big\langle N(x,u_t),\ \eta(y,x)\Big\rangle + (1-t)\langle N(x,u_t),\ \eta(x,x)\rangle + tf(y,x) + (1-t)f(x,x)\\ &= t\Big[\langle N(x,u_t),\ \eta(y,x)\rangle + f(y,x)\Big]. \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2)

$$\langle N(x,u_t),\ \eta(y,x)\rangle + f(y,x) \ \geq \ \frac{k(t)}{t}\alpha(y-x), \tag{2.3}$$

for all $y \in K$, $u_t \in T(x_t)$.

Since T is η-hemicontinuous, letting $t \to 0$ in (2.3), we get

$$\langle N(x,u),\ \eta(y,x)\rangle + f(y,x) \ \geq \ 0, \text{ for all } y \in K \text{ and } u \in T(x).$$

Therefore $x \in K$ is a solution of (1.1). □

Theorem 2.2. *Let K be a bounded closed convex subset of a reflexive Banach space E. Let $T : K \to 2^{E^*}$ be an η-hemicontinuous and relaxed η-f-α-pseudomonotone mapping with respect to the second argument of a mapping $N : K \times E^* \to E^*$. Assume that*

(i) $\eta(x,y) + \eta(y,x) = 0$ *and* $f(x,y) + f(y,x) = 0$ *for all* $x, y \in K$;
(ii) $x \mapsto \eta(x,\cdot)$ *and* $x \mapsto f(x,\cdot)$ *are convex and lower semicontinuous;*

(iii) $\alpha : E \to \mathbb{R}$ *is weakly lower semicontinuous, i.e., for any net* $\{x_\beta\}$ *in* E *converging weakly to* x_0,

$$\alpha(x_0) \ \le \ \underline{\lim}_{\beta}\alpha(x_\beta).$$

Then there exists at least one solution for (1.1).

Proof. For any $y \in K$, define two set-valued mappings $F, G : K \to 2^E$ as follows

$$F(y) = \Big\{x \in K : \ \langle N(x,u), \ \eta(y,x)\rangle + f(y,x) \ \ge \ 0, u \in T(x)\Big\},$$
$$G(y) = \Big\{x \in K : \ \langle N(x,v), \ \eta(y,x)\rangle + f(y,x) \ \ge \ \alpha(y-x), v \in T(y)\Big\},$$

We claim that F is a KKM mapping. If F is not a KKM mapping, then there exists $\{y_1, y_2, \dots, y_n\} \subset K$ such that

$$\mathrm{co}\{y_1, y_2, \dots, y_n\} \not\subset \bigcup_{i=1}^{n} F(y_i),$$

i.e., there exists a $y_0 \in \mathrm{co}\{y_1, y_2, \dots, y_n\}$, so, $y_0 = \sum_{i=1}^{n} t_i y_i$, where $t_i \ge 0$, $i = 1, \dots, n$, $\sum_{i=1}^{n} t_i = 1$, but $y_0 \notin \bigcup_{i=1}^{n} F(y_i)$.

By the definition of F, we have

$$\Big\langle N(y_0,v), \ \eta(y_i,y_0)\Big\rangle + f(y_i,y_0) < 0, \text{ for some } v \in T(y_0),$$

for $i = 1, 2, \dots, n$. It follows from conditions (i)–(ii) that

$$\begin{aligned} 0 &= \Big\langle N(y_0,v), \ \eta(y_0,y_0)\Big\rangle + f(y_0,y_0) \\ &= \Big\langle N(y_0,v), \ \eta\Big(\sum_{i=1}^{n} t_i y_i, y_0\Big)\Big\rangle + f\Big(\sum_{i=1}^{n} t_i y_i, y_0\Big) \\ &\le \sum_{i=1}^{n} t_i \langle N(y_0,v), \ \eta(y_i,y_0)\rangle + \sum_{i=1}^{n} t_i f(y_i,y_0) \\ &= \sum_{i=1}^{n} t_i \Big[\langle N(y_0,v), \ \eta(y_i,y_0)\rangle + f(y_i,y_0)\Big] \\ &< 0, \end{aligned}$$

for some $v \in T(y_0)$, which is a contradiction. This implies that F is a KKM mapping. Now we prove that

$$F(y) \subset G(y), \text{ for all } y \in K.$$

For any given $y \in K$, letting $x \in F(y)$, we have

$$\langle N(x,u), \ \eta(y,x)\rangle + f(y,x) \ \ge \ 0 \text{ for } u \in T(x).$$

Since T is relaxed η-f-α-pseudomonotone,

$$\Big\langle N(x,v),\ \eta(y,x)\Big\rangle + f(y,x)\ \geq\ \alpha(y-x) \text{ for } v\in T(y).$$

It follows that $x\in G(y)$, so

$$F(y)\subset G(y),\ \text{ for all } y\in K,$$

which implies that G is also a KKM mapping.

On the other hand, let $\langle x_\beta\rangle$ be a net in $G(y)$ converging weakly to x_0, then for all $v\in T(y)$,

$$\Big\langle N(x_\beta,v),\ \eta(y,x_\beta)\Big\rangle + f(y,x_\beta)\ \geq\ \alpha(y-x_\beta).$$

Since $x\mapsto\eta(\cdot,x)$ and $x\mapsto f(\cdot,x)$ are upper semicontinuous, and α is weakly lower semicontinuous, it follows that

$$\begin{aligned}
&\Big\langle N(x_\beta,v),\ \eta(y,x_\beta)\Big\rangle + f(y,x_\beta)\\
&\geq \overline{\lim}_\beta\Big\langle N(x_\beta,v),\ \eta(y,x_\beta)\Big\rangle + \overline{\lim}_\beta f(y,x_\beta)\\
&\geq \overline{\lim}_\beta\Big(\langle N(x_\beta,v),\ \eta(y,x_\beta)\rangle + f(y,x_\beta)\Big)\\
&\geq \underline{\lim}_\beta\Big(\langle N(x_\beta,v),\ \eta(y,x_\beta)\rangle + f(y,x_\beta)\Big)\\
&\geq \underline{\lim}_\beta\alpha(y-x_\beta)\\
&\geq \alpha(y-x_0).
\end{aligned}$$

It follows that $x_0\in G(y)$, which shows that $G(y)$ is weakly closed for all $y\in K$. Since K is bounded closed and convex, K is weakly compact, so $G(y)$ is weakly compact in K for each $y\in K$. It follows from Fan-KKM Theorem and Theorem 2.1 that

$$\bigcap_{y\in K} F(y)\ =\ \bigcap_{y\in K} G(y)\ \neq\ \emptyset.$$

Hence, there exists at least one solution for (1.1). □

Theorem 2.3. *Let K be a nonempty unbounded closed convex subset of a reflexive Banach space E, and a set-valued mapping $T: K\to 2^{E^*}$ be η-hemicontinuous, relaxed η-f-α-pseudomonotone with respect to the second argument of a mapping $N: K\times E^*\to E^*$ and $f: K\times K\to\mathbb{R}\cup\{+\infty\}$ be a proper function. Assume that*

(i) $\eta(x,y)+\eta(y,x)=0$ *and* $f(x,y)+f(y,x)=0$ *for all* $x,y\in K$*;*
(ii) $x\mapsto\eta(x,\cdot)$ *and* $x\mapsto f(x,\cdot)$ *are convex and lower semicontinuous;*
(iii) $\alpha: E\to\mathbb{R}$ *is weakly lower semicontinuous;*
(iv) T *is* η*-coercive with respect to* f*.*

Then there exists at least one solution for (1.1).

Proof. For $r > 0$, let

$$B_r = \Big\{y \in E : \|y\| \le r\Big\}.$$

Consider the following problem: Find $x_r \in K \cap B_r$ such that

$$\Big\langle N(x_r, u_r),\ \eta(y, x_r)\Big\rangle + f(y, x_r) \ \ge\ 0 \text{ for } y \in K \cap B_r, \tag{2.4}$$

and $u_r \in T(x_r)$.

By Theorem 2.2, we know that problem (2.4) has one solution $x_r \in K \cap B_r$. Take x_0 whose norm $\|x_0\|$ is less than r in the coercive conditions (iv). Then $x_0 \in K \cap B_r$ and

$$\Big\langle N(x_r, u_r),\ \eta(x_0, x_r)\Big\rangle + f(x_0, x_r) \ \ge\ 0 \text{ for } u_r \in T(x_r). \tag{2.5}$$

By condition (i) we get

$$\begin{aligned}
&\Big\langle N(x_r, u_r),\ \eta(x_0, x_r)\Big\rangle + f(x_0, x_r)\\
&= -\Big\langle N(x_r, u_r) - N(x_r, u_0),\ \eta(x_r, x_0)\Big\rangle + f(x_0, x_r) + \langle N(x_r, u_0),\ \eta(x_0, x_r)\rangle\\
&\le -\Big\langle N(x_r, u_r) - N(x_r, u_0),\ \eta(x_r, x_0)\Big\rangle + f(x_0, x_r) + \|N(x_r, u_0)\|,\ \|\eta(x_r, x_0)\|\\
&\le \|\eta(x_r, x_0)\| \left(-\frac{\langle N(x_r, u_r) - N(x_r, u_0),\ \eta(x_r, x_0)\rangle + f(x_r, x_0)}{\|\eta(x_r, x_0)\|} + \|N(x_r, u_0)\|\right)
\end{aligned}$$

for $u_r \in T(x_r)$ and $u_0 \in T(x_0)$.

If $\|x_r\| = r$ for all r, we may choose r large enough so that the above inequality and the η-coercivity of T with respect to f imply that

$$\Big\langle N(x_r, u_r),\ \eta(x_0, x_r)\Big\rangle + f(x_0, x_r) < 0,$$

which contradicts (2.5). So there exists r such that $\|x_r\| < r$. For any $y \in K$, we can choose $0 < \epsilon < 1$ small enough such that

$$x_r + \epsilon(y - x_r) \in K \cap B_r.$$

It follows from (2.4) that

$$\begin{aligned}
0 &\le \Big\langle N(x_r, u_r),\ \eta(x_r + \epsilon(y - x_r), x_r)\Big\rangle + f(x_r + \epsilon(y - x_r), x_r)\\
&\le (1-\epsilon)\Big\langle N(x_r, u_r),\ \eta(x_r, x_r)\Big\rangle + (1-\epsilon) f(x_r, x_r) + \epsilon\langle N(x_r, u_r), \eta(y, x_r)\rangle\\
&\quad + \epsilon f(y, x_r)\\
&= \epsilon\Big[\langle N(x_r, u_r),\ \eta(y, x_r)\rangle + f(y, x_r)\Big],
\end{aligned}$$

which implies that

$$\langle N(x_r, u_r),\ \eta(y, x_r)\rangle + f(y, x_r) \ \ge\ 0,$$

for $y \in K$ and $u_r \in T(x_r)$. This completes the proof. □

Remark 2.1. (1) If T is continuous on finite dimensional subspaces, the conclusions of Theorems 2.1–2.3 are also true.

(2) Theorems 2.2 and 2.3 improve and generalize the known results of Hartman and Stampacchia [8] and the corresponding results of $[1, 3, 5, 7, 9 - 11, 14]$.

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