J. Appl. Math. & Informatics Vol. **27***(2009), No. 5 - 6, pp. 1361 - 1370*
Website: http://www.kcam.biz

POSITIVE SOLUTIONS OF SINGULAR FOURTH-ORDER TWO POINT BOUNDARY VALUE PROBLEMS

JIEMEI LI

ABSTRACT. In this paper, we consider singular fourth-order two point boundary value problems

$$u^{(4)}(t) = f(t, u), \quad 0 < t < 1,$$
$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where $f : (0,1) \times (0, +\infty) \to [0, +\infty)$ may be singular at $t = 0$, 1 and $u = 0$. By using the upper and lower solution method, we obtained the existence of positive solutions to the above boundary value problems. An example is also given to illustrate the obtained theorems.

AMS Mathematics Subject Classification : 34B15; 34B18
Key words and phrases : Singular boundary value problems, lower and upper solutions, maximum principle, schauder fixed point theorem.

1. Introduction

In this paper, we consider singular fourth-order boundary value problems

$$u^{(4)}(t) = f(t, u), \quad 0 < t < 1, \tag{1}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0, \tag{2}$$

where $f : (0,1) \times (0, +\infty) \to [0, +\infty)$ may be singular at $t = 0$, 1 and $u = 0$.

It is well known that the deformation of an elastic beam in equilibrium state, whose both ends clamped, can be described by (1)-(2). Important though it is, there are few results obtained for (1)-(2). The likely reason is that fewer techniques are available to this kind of problems. Recently, Ma and Tisdell in [6] considered singular boundary value problems

$$y^{(4)}(t) = p(t)y^{\lambda}(t), \quad 0 < t < 1, \tag{3}$$

Received January 30, 2008. Revised March 15, 2008. Accepted May 4, 2008.
Supported by LZJTU-ZXKT-40728 and by "Qing Lan" Talent Engineering Funds (QL-05-164) by Lanzhou Jiaotong University.

$$y(0) = y(1) = y'(0) = y'(1) = 0, \tag{4}$$

where $\lambda \in (0,1)$, $p(t)$ may be singular at $t = 0, 1$. Under the assumption:
(A) $p : (0,1) \to [0, +\infty)$ is continuous and λ is given.

They established the following

Theorem 1.1. *Let (A) hold. Then (3)-(4) has $C^3[0,1]$ positive solutions if and only if*

$$0 < \int_0^1 t^{2\lambda}(1-t)^{2\lambda}p(t)dt < +\infty$$

Theorem 1.2. *Let (A) hold. Then (3)-(4) has $C^2[0,1]$ positive solutions if and only if*

$$0 < \int_0^1 t^{1+2\lambda}(1-t)^{1+2\lambda}p(t)dt < +\infty$$

Theorem 1.3. *Let (A) hold. Then (3)-(4) has $C^1[0,1]$ positive solutions provided*

$$0 < \int_0^1 t^2(1-t)^2p(t)dt < +\infty$$

Remark 1.1. According to $\lambda \in (0,1)$, we easily know that the nonlinearity item in (3) implies that it is nondecreasing in y.

Remark 1.2. The nonlinearity item in (3) doesn't allow y to be singular at $y = 0$.

Of course an interesting question is what would happen if the nonlinearity item in (3) is decreasing in y and it may be singular at $y = 0$?

In order to fill this gap, we will consider singular boundary value problems (1)-(2), whose nonlinearity item is more general than in (3). By using lower and upper solution method, we establish the existence of positive solutions to (1)-(2). For other singular boundary value problems concerning the existence and multiplicity of solutions, we refer the reader to [1, 2, 4, 5, 7-12] and references therein.

The rest of this paper are organized as follows. In section 2, we recall some definitions and lemmas which are used throughout this paper. Section 3 is the main results of this paper. In section 4, we give an example to check the validity of the main results.

2. Preliminaries

Definition 2.1. A function u is said to be a solution of the boundary value problems (1)-(2) if $u \in C^1[0,1] \cap C^4(0,1)$ satisfies (1)-(2). In addition, u is said to be a positive solution if $u(t) > 0$ for $t \in (0,1)$ and u is a solution of (1)-(2).

Definition 2.2. Suppose $\alpha \in C^1[0,1] \cap C^4(0,1)$, if α satisfies

$$\alpha^{(4)} \le f(t, \alpha(t)), \ t \in (0,1),$$
$$\alpha(0) \le 0, \ \ \alpha(1) \le 0, \ \ \alpha'(0) \le 0, \ \ \alpha'(1) \ge 0.$$

Then α is called a lower solution of singular problems (1)-(2).

Definition 2.3. Suppose $\beta \in C^1[0,1] \cap C^4(0,1)$, if β satisfies

$$\beta^{(4)} \ge f(t, \beta(t)), \ \ t \in (0,1),$$
$$\beta(0) \ge 0, \ \ \beta(1) \ge 0, \ \ \beta'(0) \ge 0, \ \ \beta'(1) \le 0.$$

Then β is called a upper solution of the singular problems (1)-(2).

Let $G(t,s)$ be the Green's function of the linear problems

$$x^{(4)}(t) = 0, \quad 0 < t < 1,$$
$$x(0) = x(1) = x'(0) = x'(1) = 0,$$

which can be explicitly given by

$$G(t,s) = \frac{1}{6}\begin{cases} t^2(1-s)^2[(s-t)+2(1-t)s], & 0 \le t \le s \le 1, \\ s^2(1-t)^2[(t-s)+2(1-s)t], & 0 \le s \le t \le 1, \end{cases} \tag{5}$$

Lemma 2.1. *Suppose that $e : (0,1) \to [0,+\infty)$ is continuous and satisfies $\int_0^1 s^2(1-s)^2 e(s)ds < +\infty$, then linear boundary value problems*

$$u^{(4)}(t) = e(t), \quad 0 < t < 1,$$
$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

has a unique solution $u(t) \in C^1[0,1] \cap C^4(0,1)$ and it can be expressed by

$$u(t) = \int_0^1 G(t,s)e(s)ds, \quad t \in [0,1],$$

where $G(t,s)$ is defined by (5).

Proof. It's the direct results of lemmas 2.3-2.7 in [6]. □

Lemma 2.2. (Maximum principal [6]) *Let $e : (0,1) \to [0,+\infty)$ is continuous, $e(t) \ge 0$ and $\int_0^1 s^2(1-s)^2 e(s)ds < +\infty$. $a \ge 0$, $b \ge 0$, $c \ge 0$, $d \le 0$ are given constants. Then the unique solution u of singular problems*

$$u^{(4)}(t) = e(t), \quad 0 < t < 1,$$
$$u(0) = a, \ u(1) = b, \ u'(0) = c, \ u'(1) = d,$$

satisfies

$$u(t) \ge 0, \quad t \in [0,1].$$

Lemma 2.3. *The Green's function $G(t,s)$ defined by (5) have the following simple properties :*

(1) *For any $(t,s) \in [0,1] \times [0,1]$, we have $G(t,s) \le \frac{1}{2}t^2(1-t)^2$.*

(2) *For any $t_0 \in (0,1)$,*

$$\frac{G(t,s)}{G(t_0,s)} \geq \frac{2}{3}t^2(1-t)^2,\ \forall t,\ s \in (0,1) \tag{6}$$

Proof. (1) If $0 \leq t \leq s \leq 1$,

$$G(t,s) = \frac{1}{6}t^2(1-s)^2[(s-t)+2(1-t)s] \leq \frac{1}{6}t^2(1-t)^2[(1-t)+2(1-t)] \leq \frac{1}{2}t^2(1-t)^2.$$

If $0 \leq s \leq t \leq 1$,

$$G(t,s) = \frac{1}{6}s^2(1-t)^2[(t-s)+2(1-s)t] \leq \frac{1}{6}t^2(1-t)^2[t+2t] \leq \frac{1}{2}t^2(1-t)^2.$$

(2) Let $t_0 \in (0,1)$. For any $t,\ s \in (0,1)$,
If $t_0,\ t \leq s$,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{t^2(1-s)^2[(s-t)+2(1-t)s]}{t_0^2(1-s)^2[(s-t_0)+2(1-t_0)s]} \geq \frac{2t^2(1-t)t_0}{3t_0^2(1-t_0)} \geq \frac{2}{3}t^2(1-t)^2,$$

If $t \leq s \leq t_0$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)} &= \frac{t^2(1-s)^2[(s-t)+2(1-t)s]}{s^2(1-t_0)^2[(t_0-s)+2(1-s)t_0]} \geq \frac{2t^2(1-s)^2(1-t)s}{3s^2(1-t_0)^2(1-s)} \\ &\geq \frac{2t^2(1-s)(1-t)}{3s(1-t_0)^2} \geq \frac{2t^2(1-t)(1-t_0)}{3t_0(1-t_0)^2} \\ &= \frac{2t^2(1-t)}{3t_0(1-t_0)} \geq \frac{2}{3}t^2(1-t)^2,\end{aligned}$$

If $t_0 \leq s \leq t$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)} &= \frac{s^2(1-t)^2[(t-s)+2(1-s)t]}{t_0^2(1-s)^2[(s-t_0)+2(1-t_0)s]} \geq \frac{2s^2(1-t)^2t(1-s)}{3t_0^2(1-s)^2(1-t_0)} \\ &\geq \frac{2t(1-t)^2}{3(1-s)(1-t_0)} \geq \frac{2t(1-t)^2}{3(1-t_0)^2} \geq \frac{2}{3}t^2(1-t)^2,\end{aligned}$$

If $s \leq t_0,\ t$,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{s^2(1-t)^2[(t-s)+2(1-s)t]}{s^2(1-t_0)^2[(t_0-s)+2(1-s)t_0]} \geq \frac{2(1-t)^2(1-s)t}{3(1-t_0)^2t_0} \geq \frac{2}{3}t^2(1-t)^2.$$

□

3. Main results

Theorem 3.1. *Suppose that*

(A1) $f(t,u) \in C((0,1)\times(0,+\infty),[0,+\infty))$ and $f(t,u)$ is nonincreasing with respect to u.

(A2) For any constant $\lambda > 0$,

$$0 < \int_0^1 s^2(1-s)^2 f(s,\lambda s^2(1-s)^2) < +\infty.$$

Then the boundary value problems (1)-(2) has at least one $C^1[0,1]$ positive solution u which satisfies $u(t) \geq mt^2(1-t)^2$ for some $m > 0$.

Proof. Let

$$X = \{u \mid u \in C^1[0,1] \cap C^4(0,1)\},$$
$$P = \{u \in C[0,1] : \text{ there exists a positive constant } k_u \text{ such that } u(t) \geq k_u t^2(1-t)^2,\ t \in [0,1]\}.$$

Obviously, P is nonempty because of $t^2(1-t)^2 \in P$. Now, define an operator A on X by

$$Au(t) = \int_0^1 G(t,s) f(s,u(s))ds, \quad \forall u \in P.$$

For any $u \in P$, by the definition of P, there exists a positive constant k_u, such that $u(t) \geq k_u t^2(1-t)^2,\ t \in [0,1]$.

By the nonincreasing property of $f(t,u)$ and $u(t) \geq k_u t^2(1-t)^2,\ t \in [0,1]$, we have

$$f(t,u(t)) \leq f(t, k_u t^2(1-t)^2), \quad t \in (0,1).$$

So, combining Lemma 2.3 (2), we get

$$Au(t) = \int_0^1 G(t,s) f(s,u(s))ds \leq \frac{1}{2}\int_0^1 s^2(1-s)^2 f(s, k_u s^2(1-s)^2)ds < +\infty \quad (7)$$

On the other hand, choose fixed $t_0 \in (0,1)$ such that $Au(t_0) = \frac{3}{2}k_{Au} > 0$. It follows from Lemma 2.3 that

$$\begin{aligned} Au(t) &= \int_0^1 \frac{G(t,s)}{G(t_0,s)} G(t_0,s) f(s,u(s))ds \\ &\geq \frac{2}{3} t^2(1-t)^2 \int_0^1 G(t_0,s) f(s,u(s))ds \\ &= k_{Au} t^2(1-t)^2, \quad \forall t \in [0,1] \end{aligned} \quad (8)$$

From (7), (8), we conclude that operator A is well-defined and $A(P) \subset P$.

According to the lemmas 2.3-lemma 2.7 in [6], we immediately obtain

$$(Au)^{(4)}(t) = f(t,u(t)), \quad t \in (0,1), \quad (9)$$

and

$$Au(0) = Au(1) = (Au)'(0) = (Au)'(1) = 0 \quad (10)$$

Let

$$e(t) = t^2(1-t)^2$$
$$a(t) = \min\{e(t),\ Ae(t)\}, \quad b(t) = \max\{e(t),\ Ae(t)\} \quad (11)$$

Obviously, $a(t),\ b(t) \in P$ and $a(t) \leq b(t)$. Moreover, $Aa(t),\ Ab(t) \in P$, and

$$Ab(t) \leq Aa(t) \quad (12)$$

So, by (11), (12) and the nonincreasing property of the operator A, we have

$$Ab(t) \leq Ae(t) \leq b(t), \quad Aa(t) \geq Ae(t) \geq a(t) \quad (13)$$

From (9), (13) and $(A1)$, we get

$$(Ab)^{(4)}(t) - f(t, Ab(t)) \leq (Ab)^{(4)}(t) - f(t, b(t)) = 0 \quad (14)$$

$$(Aa)^{(4)}(t) - f(t, Aa(t)) \geq (Aa)^{(4)}(t) - f(t, a(t)) = 0 \tag{15}$$

Let $\alpha(t) = Ab(t)$, $\beta(t) = Aa(t)$. It is easy to see that $\alpha(t)$, $\beta(t) \in C^1[0,1] \cap C^4(0,1)$ satisfy the boundary condition (2). Then from (14), (15), $\alpha(t)$, $\beta(t)$ are lower and upper solutions of (1)-(2), respectively.

In the following, we shall show that (1)-(2) has a $C^1[0,1]$ positive solution $\tilde{u}$ such that

$$0 < \alpha(t) \leq \tilde{u}(t) \leq \beta(t) \quad t \in (0,1) \tag{16}$$

Let $C[0,1]$ be a Banach Space with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$.

Firstly, we define an auxiliary function

$$f^*(t,u) = \begin{cases} f(t,\alpha(t)), & u < \alpha(t) \\ f(t,u), & \alpha(t) \leq u(t) \leq \beta(t) \\ f(t,\beta(t)), & u > \beta(t) \end{cases} \tag{17}$$

By $(A1)$, $f^* : (0,1) \times R \to [0,+\infty)$ is continuous.

Consider the singular problems

$$u^{(4)}(t) = f^*(t,u(t)) \quad 0 < t < 1, \tag{18}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0. \tag{19}$$

We claim that if $u(\cdot)$ is any solution of (18)-(19), and $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in [0,1]$, then $u(\cdot)$ is a solution of (1)-(2) which satisfies (16).

In fact, let $z(t) = u(t) - \alpha(t)$, we have

$$z^{(4)}(t) = u^{(4)}(t) - \alpha^{(4)}(t) \geq f^*(t,u(t)) - f(t,\alpha(t)) \geq 0$$

and

$$z(0) = 0, \quad z(1) = 0, \quad z'(0) = 0, \quad z'(1) = 0$$

By lemma 2.2, we conclude that $z(t) \geq 0$, which means $u(t) \geq \alpha(t)$, $t \in [0,1]$.

Similarly, we can prove $\beta(t) \geq u(t)$, $t \in [0,1]$.

Now it is sufficient to prove (18)-(19) has at least one solution which can be drawn by Schauder fixed point theorem [3].

Since $\alpha(t) \in P$, there exists a positive constant k_α such that $\alpha(t) \geq k_\alpha t^2(1-t)^2$, $t \in [0,1]$. It follows from $(A2)$ that

$$\begin{aligned} \int_0^1 G(t,s) f^*(s,u(s))ds &\leq \frac{1}{2}\int_0^1 s^2(1-s)^2 f(s,\alpha(s))ds \\ &\leq \frac{1}{2}\int_0^1 s^2(1-s)^2 f(s,k_\alpha s^2(1-s)^2)ds \\ &< +\infty \end{aligned} \tag{20}$$

For any $u \in C[0,1]$,

$$Au(t) = \int_0^1 G(t,s) f^*(s,u(s))ds \leq \frac{1}{2}\int_0^1 s^2(1-s)^2 f(s,k_\alpha s^2(1-s)^2)ds < +\infty$$

therefore, $A(C[0,1])$ is a bounded set.

Let $u_n,\ u_0 \in C[0,1],\ \ u_n \to u_0(n \to +\infty)$, then

$$|Au_n(t) - Au_0(t)| \le \int_0^1 G(t,s)|f^*(s,u_n(s)) - f^*(s,u_0(s))|ds.$$

Combining (20), the continunity of f^* and Lebesgue control convergence theorem,

$$\|Au_n - Au_0\| \to 0, (n \to +\infty),$$

so $A : C[0,1] \to C[0,1]$ is continuous. Furthermore

$$\begin{aligned}|(Au)'(t)| &\le \frac{1}{6}\int_0^t \{2s^2(1-s)[(1-s)+2(1-s)] + s^2(1-s)^2(1+2)\}f^*(s,u(s))ds\\ &+ \frac{1}{6}\int_t^1 \{2s(1-s)^2(s+2s) + s^2(1-s)^2(1+2)\}f^*(s,u(s))ds\\ &\le \frac{9}{6}\int_0^t s^2(1-s)^2 f(s,\alpha(s))ds + \frac{9}{6}\int_t^1 s^2(1-s)^2 f(s,\alpha(s))ds\\ &\le \frac{3}{2}\int_0^1 s^2(1-s)^2 f(s,k_\alpha s^2(1-s)^2)ds := D\end{aligned}$$

so, any $\epsilon > 0$, choice $\delta = \dfrac{\epsilon}{D}$, for any $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$, any $u \in C[0,1]$, we have

$$|Au(t_1) - Au(t_2)| = |\int_{t_1}^{t_2}(Au)'(s)ds| \le |\int_{t_1}^{t_2}|(Au)'(s)|ds| \le D|t_1 - t_2| < \epsilon.$$

This implies A is equi-continuous.

By Schauder fixed point theorem, A has at least one solution $\tilde{u}(\cdot) \in C^1[0,1] \cap C^4(0,1)$ satisfies

$$\alpha(t) \le \tilde{u}(t) \le \beta(t), \quad t \in [0,1]$$

i.e., (1)-(2) has at least one $C^1[0,1]$ positive solution. Furthermore,

$$\tilde{u}(t) \ge \alpha(t) \ge k_\alpha t^2(1-t)^2 > 0, \quad t \in (0,1).$$

Thus, $\tilde{u}(t)$ is the positive solution of (1)-(2) and satisfy $\tilde{u}(t) \ge mt^2(1-t)^2$, where $m = k_\alpha$ is a constant. □

Theorem 3.2. *Suppose that*

(A1) $f(t,u) \in C((0,1) \times (0,+\infty), [0,+\infty))$ and $f(t,u)$ is nonincreasing with respect to u.

(A2)′ For any constant $\lambda > 0$,

$$0 < \int_0^1 s(1-s)f(s,\lambda s^2(1-s)^2) < +\infty.$$

Then the boundary value problems (1)-(2) has at least one $C^2[0,1]$ positive solution u which satisfies $u(t) \ge mt^2(1-t)^2$ for some $m > 0$.

Proof. Since $(A2)'$ hold, we get $(A2)$ hold. So, all conditions of theorem 3.1 are satisfied. By theorem 3.1, the boundary value problems (1)-(2) has at least one $C^1[0,1]$ positive solution u satisfying $u(t) \geq mt^2(1-t)^2$ for some $m > 0$.

In the following, we prove that u is $C^2[0,1]$ positive solution of (1)-(2). By the nonincreasing property of $f(t,u)$ and $u(t) \geq mt^2(1-t)^2$, we have

$$f(t,u(t)) \leq f(t,mt^2(1-t)^2), \quad t \in (0,1).$$

i.e.,

$$|u^{(4)}(t)| \leq f(t,mt^2(1-t)^2), \quad t \in (0,1).$$

According to (2), there exist $t_0 \in (0,1)$, such that $u'''(t_0) = 0$, and for $t \in (0,t_0)$, $u'''(t) < 0$; for $t \in (t_0,1)$, $u'''(t) > 0$. Therefore,

$$\begin{aligned}
\int_0^1 |u'''(s)|ds &= \int_0^{t_0} |u'''(s)|ds + \int_{t_0}^1 |u'''(s)|ds \\
&\leq \int_0^{t_0}\int_s^{t_0} u^{(4)}(\tau)d\tau ds + \int_{t_0}^1\int_{t_0}^s u^{(4)}(\tau)d\tau ds \\
&\leq \int_0^{t_0} su^{(4)}(s)ds + \int_{t_0}^1 (1-s)u^{(4)}(s)ds \\
&\leq \int_0^{t_0} sf(s,ms^2(1-s)^2)ds + \int_{t_0}^1 (1-s)f(s,ms^2(1-s)^2)ds \\
&\leq \max\{\frac{1}{1-t_0},\frac{1}{t_0}\}\int_0^1 s(1-s)f(s,ms^2(1-s)^2)ds \\
&< +\infty
\end{aligned}$$

This implies that $u'''(t)$ is absolutely integrable on $[0,1]$, hence $u \in C^2[0,1]$. □

Theorem 3.3. *Suppose that*

$(A1)$ $f(t,u) \in C((0,1)\times(0,+\infty),[0,+\infty))$ *and* $f(t,u)$ *is nonincreasing with respect to* u.

$(A2)''$ *For any constant* $\lambda > 0$,

$$0 < \int_0^1 f(s,\lambda s^2(1-s)^2) < +\infty$$

Then the boundary value problems (1)-(2) has at least one $C^3[0,1]$ *positive solution* u *which satisfies* $u(t) \geq mt^2(1-t)^2$ *for some* $m > 0$.

Proof. Since $(A2)''$ hold, we get $(A2)$ hold. So, all conditions of theorem 3.1 are satisfied. By theorem 3.1, the boundary value problems (1)-(2) has at least one $C^1[0,1]$ positive solution u satisfying $u(t) \geq mt^2(1-t)^2$ for some $m > 0$.

In the following, we prove that u is $C^3[0,1]$ positive solution of (1)-(2). By the nonincreasing property of $f(t,u)$ and $u(t) \geq mt^2(1-t)^2$, we have

$$f(t,u(t)) \leq f(t,mt^2(1-t)^2), \quad t \in (0,1).$$

i.e.,

$$|u^{(4)}(t)| \le f(t, mt^2(1-t)^2), \quad t \in (0,1).$$

By $(A2)''$, $u^{(4)}(t)$ is absolutely integrable on $[0,1]$. This implies that $u'''(0+)$ and $u'''(1-)$ both exist, therefore, u is a $C^3[0,1]$ positive solution of (1)-(2). □

4. Examples

Example 4.1. The singular boundary value problems

$$u^{(4)}(t) = t^{-\frac{1}{2}}(1-t)^{-\frac{3}{2}}u^{-\frac{1}{4}}, \quad 0 < t < 1,$$

$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

has a $C^1[0,1]$ positive solution u satisfying $u(t) \ge mt^2(1-t)^2$ for some constant $m > 0$.

Proof. To show this, we only need to verify that conditions $(A1)$, $(A2)$ hold in Theorem 3.1. Here $f(t,u) = t^{-\frac{1}{2}}(1-t)^{-\frac{3}{2}}u^{-\frac{1}{4}}$. Obviously, $(A1)$ hold.

For any constant $\lambda > 0$,

$$\int_0^1 s^2(1-s)^2 f(s, \lambda s^2(1-s)^2) = \int_0^1 s^2(1-s)^2 s^{-\frac{1}{2}}(1-s)^{-\frac{3}{2}}(\lambda s^2(1-s)^2)^{-\frac{1}{4}}$$

$$= \lambda^{-\frac{1}{4}} \int_0^1 s ds = \frac{1}{2}\lambda^{-\frac{1}{4}} < +\infty.$$

Thus, $(A2)$ hold. Consequently, the above conclusion is guaranteed by Theorem 3.1. □

Remark 4.1. Example 4.1 indicates that $f(t,u)$ can be singular at $t = 0, 1$ and $u = 0$. Particularly, when $f(t,u) = p(t)u^{\rho}(t)$, $(A1)$ shows $\rho < 0$. These are different from [6]. Furthermore, the conditions of the main results in this paper are easy to verify.

Remark 4.2. Comparing with theorems in [11], we do not require the existence of upper and lower solutions. $C^2[0,1]$, $C^3[0,1]$ positive solutions are also studied.

References

1. Y. Cui and Y. Zou, *Positive solutions of singular fourth-order boundary value problems*, Electron. J. Differential Equations(2006), 1-10.
2. X. Du and Z. Zhao, *Existence and uniqueness of positive solutions to a class of singular m-point boundary value problems*, Appl. Math. Comput. doi:10.1016/j.amc.2007.08.080
3. J. Dugundij and A. Grans, *Fixed point theory*, Mongrafie mat. Warsaw: PWN, 1982.
4. P. Korman, *Uniqueness and exact multiplicity of solutions for a class of fourth-order semilinear problems*, Proc. Roy. Soc. Edinburgh Sect.A**134**(2004), 179-190.
5. R. Ma, H.Wang, *On the existence of positive solutions of fourth-order ordinary differential equations*, Appl. Anal **59**(1995), 225-231.
6. R. Ma and C. C.Tisdell, *Positive solutions of singular sublinear fourth-order boundary value problems*, Appl.Anal. **84**(2005), 1199–1221.

7. D. O'Regan, *Solvability of some fourth (and higher) order singular boundary value problems*, J. Math. Anal. Appl. **161**(1991), 78-116.
8. S. D.Taliaferro, *A nonlinear singular boundary value problem*, Nonlinear Anal. TMA **3**(1979), 897-904.
9. Zh. Wei, *Positive solutions of singular boundary value problems of fourth order differential equations*, Acta Math.Sini. (in chinese)**42**(1999), 715-722
10. Q. Yao, *Positive solutions for eigenvalue problems of fourth-order elastic beam equations*, Appl. Math. Lett. **17**(2004), 237-243.
11. X. Zhang and L. Liu, *Positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator*, J. Math. Anal. Appl. **336**(2007), 1414-1423.
12. Y. Zhang, *Positive solutions of singular sublinearEmden-Fowler boundary value problems*, SIAM J. Math. Anal. Appl. **26**(1995), 329-339.

Jiemei Li received her MS from Northwest Normal University, Lanzhou, China. Since 2005 she has been at Lanzhou Jiaotong University, Lanzhou, China. Her research interests focus on boundary value problems.

Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, P R China
e-mail: jmli@mail.lzjtu.cn