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VARIATIONAL DECOMPOSITION METHOD FOR SOLVING SIXTH-ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we implement a relatively new analytical technique by combining the traditional variational iteration method and the decomposition method which is called as the variational decomposition method (VDM) for solving the sixth-order boundary value problems. The proposed technique is in fact the modification of variatioanal iteration method by coupling it with the so-called Adomian's polynomials. The analytical results of the equations have been obtained in terms of convergent series with easily computable components. Comparisons are made to verify the reliability and accuracy of the proposed algorithm. Several examples are given to check the efficiency of the proposed algorithm. We have also considered an example where the VDM is not reliable.

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1. Introduction

In this paper, we consider the general sixth-order boundary value problems of the type

$$y^{(vi)}(x) = f(x, y), \quad a \leq x \leq b \tag{1}$$

with boundary conditions

$$\begin{aligned} y(a) &= A_{1}, \ y''(a) = A_{2}, \ y^{(iv)}(a) = A_{3}, \\ y(b) &= B_{1}, \ y''(b) = B_{1}, \ y^{(iv)}(b) = B_{1}, \end{aligned}$$

where $f = f(x, y)$, is a given continuous, linear or nonlinear function, $f(x, y) \in C[a.b]$ is real and $A_i$and B_i are real finite constants. The sixth-order boundary value problems are known to arise in astrophysics; the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be

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modeled by sixth-order boundary-value problems [4-13, 27, 28, 30-34]. Glatzmaier also noticed that dynamo action in some stars may be modeled by such equations see, [12]. Moreover, when an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in [30-34], when this instability is of ordinary convection than the governing ordinary differential equation is of sixth-order, see [4-13, 27, 28, 30-34] and the references therein. The literature of numerical analysis contains little on the solution of the sixth-order boundary value problems [3-12, 34]. Theorems which list the conditions for the existence and uniqueness of solutions of such problems are thoroughly discussed i by Agarwal [1]. Baldwin [5, 6] developed non numerical techniques for solving such problems. However, numerical methods of solutions were introduced implicitly by Chawla and Katti [8]; although the authors focused their attention on fourth-order boundary value problems. Finite difference methods of solutions for such problems were also developed by Boutayeb and Twizell [7]. A second-order method was developed in [33] for solving special and general sixth-order boundary value problem. In a later work [32], finite difference method of order two was established to handle such problems. Sextic spline solutions of linear sixth order boundary-value problems were derived by Siddiqi and Twizell [30] using polynomial splines of degree six where the spline function values at the mid knots of the interpolation interval and the corresponding values of the even order derivatives are related through consistency relations. However, the performance of the techniques used so far is well-known that it provides solutions at grid points only. Moreover, the existing techniques require huge computational work.

He [14-18] developed the variational iteration method (VIM) for solving linear, nonlinear, initial and boundary value problems. It is worth mentioning that the origin of variational iteration method is traced back to Inokuti, Sekine and Mura [19], but the real potential of the VIM was explored by He. Since the beginning of 1980s, the Adomian's decomposition method has been applied to a wide class of functional equations [34, 35]. In these methods the solution is given in an infinite series usually converging to an accurate solution, see [1, 2, 14-19, 21-25, 29, 34-36] and the references therein. Recently, decomposition method [34], Ritz's method based on variational theory [13], non-polynomial spline technique [4] and Sinc-Galerkin method [11] have been applied for the solution of sixth-order boundary value problems. Noor and Mohyud-Din [23-28] employed homotopy perturbation method, variational iteration method and the variational iteration decomposition method for solving sixth-order and some other higher-order boundary value problems. developed techniques.

In this paper, we apply the variational decomposition method (VDM) which is an elegant combination of variational iteration and the Adomian's decomposition methods to solve the sixth-order boundary value problems. The proposed technique is in fact the modified variational iteration method where the variatioanal iteration method is coupled with Adomian's polynomials. This idea has been used by Abbasbandy [1, 2] for solving quadratic Riccati differential equation and Klein-Gordon equation and by Noor and Mohyud-Din [24] for solving the eighth-order boundary value problems. The basic motivation of this paper is to apply the variational decomposition method (VDM) for solving the sixth-order boundary value problems. It is shown that the variational decomposition method provides the solution in a rapid convergent series. We write the correct functional for the sixth-order boundary value problem and calculate the Lagrange multiplier optimally via variational theory. The Adomian's polynomials are introduced in the correct functional and evaluated by using the specific algorithms. See also [34, 35] and the references therein). Finally, the approximants are calculated by employing the Lagrange multipliers and the Adomian's polynomial scheme simultaneously. The use of Lagrange multiplier reduces the successive application of the integral operator and minimizes the computational work. Moreover, the selection of the initial value is done by exploiting the concept of modified decomposition method. The VDM solves effectively, easily and accurately a large class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions. Several examples are given to illustrate the reliability and performance of the proposed method. We have also considered an example where the proposed variational decomposition method (VDM) is not reliable.

2. Variational iteration method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{2}$$

where L is a linear operator, N a nonlinear operator and g(x) is the inhomogeneous term. According to variational iteration method [1, 2, 14-19, 21-25, 29, 36], we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left(Lu_n(s) + N\widetilde{u_n}(s) - g(s) \right) ds, \tag{3}$$

where λ is a Lagrange multiplier [14-19], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\widetilde{u_n}$ is considered as a restricted variation. i.e. $\delta\widetilde{u_n} = 0$; (3) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [14-19]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation $u_{n+1},\ n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier λ and any selective function u_0, consequently, the solution is given by$u = \lim_{n\to\infty} u_n$. For the convergence criteria and error estimates of variational iteration method, see Ramos [29].

3. Adomian's decomposition method

Consider the differential equation [34, 35] of the type

$$Lu + Ru + Nu = g, \tag{4}$$

where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order lesser order than L,$N u$represents the nonlinear terms and g is the source term. Applying the inverse operator $L^{-1}$to both sides of (4) and using the given conditions, we obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Nu), \tag{5}$$

where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian's decomposition method [34, 35] defines the solution $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{6}$$

where the components $u_n(x)$are usually determined recurrently by using the relation

$$\begin{aligned} u_0 &= f, \\ u_{k+1} &= f - L^{-1}(Ru_k) - L^{-1}(Nu_k), \quad k \geq 0. \end{aligned}$$

The nonlinear operator $F(u)$can be decomposed into an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n,$$

where are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in [34, 35] which yields

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right) F\left(\sum_{i=0}^{n} (\lambda^i u_i)\right)_{\lambda=0} \qquad n = 0, 1, 2, \cdots$$

4. Variational decomposition method (VDM)

To illustrate the basic concept of the variational decomposition method, we consider the following general differential equation (1)

$$Lu + Nu = g(x)$$

where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to variational iteration method [1, 2, 14-19, 21-25, 29, 36], we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left(Lu_n(s) + N\widetilde{u_n}(s) - g(s)\right) ds,$$

where λ is a Lagrange multiplier [14-19], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\widetilde{u_n}$ is considered as a restricted variation. i.e., $\delta\widetilde{u_n} = 0$; (3) is called as a correct functional. We define the solution $u(x)$ by the series$u(x) = \sum_{i=0}^{\infty} u_i(x)$, and the nonlinear term $N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_i)$, where A_n are the so-called Adomian's polynomials and can be generated for all type of nonlinearities according to the algorithm developed, in [34, 35] which yields the following

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right) F\left(u\left(\lambda\right)\right)_{\lambda=0}.$$

Hence, we obtain the following iterative scheme for finding the approximate solution

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left(Lu_n(s) + \sum_{n=0}^{\infty} A_n - g(s) \right) ds, \tag{7}$$

which is the variatioanal decomposition method (VDM) and is formulated by the coupling of variational iteration and the decomposition methods.

5. Numerical applications

In this section, we apply the variational decomposition method (VDM) for solving the sixth-order boundary value problems. We write the correct functional for the sixth-order boundary value problem and carefully select the initial value because the approximants are heavily dependent on the initial value. The Adomian's polynomials are introduced in the correct functional for the nonlinear terms. The results are very encouraging indicating the reliability and efficiency of the proposed method. For the sake of comparison, we take the same examples as discussed in [20, 27, 28, 34].

Example 5.1 Consider the following nonlinear boundary value problem of sixth-order of the type

$$y^{(vi)}(x) = e^{-x} y^2(x), \quad 0 < x < 1$$

with boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = 1, \quad y(1) = y''(1) = y^{(iv)}(1) = e$$

The exact solution for this problem is $y(x) = e^x$.

The correct functional for the boundary value problem is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left(\frac{d^6\widetilde{y_n}}{dx^6} - e^{-s} {y_n}^2(s) \right) ds.$$

Making the correct functional stationary, the Lagrange multiplier can be identified as $\lambda = \frac{1}{5!}(s-x)^5$ [14-19, 36], we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{5!}(s-x)^5 \left(\frac{d^6 y_n}{dx^6} - e^{-s} {y_n}^2(s) \right) ds.$$

which implies that

$$y_{n+1}(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4}x^4 + \frac{1}{5!}Cx^5 + \int_0^x \frac{1}{5!}(s-x)^5 \left(\frac{d^6 y_n}{dx^6} - e^{-x} {y_n}^2(s)\right) ds.$$

where $A = y'(0)$, $B = y'''(0)$, $C = y^{(v)}(0)$. Applying the variational decomposition method, we have

$$y_{n+1}(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \int_0^x \frac{1}{5!}(s-x)^5 \left(\frac{d^6 y_n}{dx^6} - e^{-s}\sum_{n=0}^{\infty} A_n\right) ds,$$

where A_n are Adomian's polynomials for nonlinear operator $F(y) = y^2(x)$ and can be generated for all type ofnonlinearities according to the algorithm developed in [34, 35] which yields the following

$$A_0 = y_0^2(x).,\quad A_1 = 2y_0(x)\, y_1(x),\quad A_2 = y_2 F'(y_0) + \frac{y_1^2}{2!}F''(y_0),$$
$$A_3 = 2y_0(x)\, y_2(x) + y_1^2(x).$$

Consequently, we obtain the following approximants

$$y_0(x) = 1,$$

$$y_1(x) = 1 + (A+1)x - 1 + \left(\frac{1}{3!}B + \frac{1}{3!}\right)x^3 + \left(\frac{1}{5!}C + \frac{1}{5!}\right)x^5 + e^{-x},$$

$$\begin{aligned} y_2(x) = {} & 1 + (A+1)x - 1 + \left(\frac{1}{3!}B + \frac{1}{3!}\right)x^3 + \left(\frac{1}{5!}C + \frac{1}{5!}\right)x^5 + e^{-x} \\ & + \frac{1}{2520}Ax^7 + + \left(-\frac{1}{10080}A + \frac{1}{20160}\right)x^8 \\ & + \left(\frac{1}{60480}A + \frac{1}{181440}B - \frac{1}{60480}\right)x^9 \\ & + \left(-\frac{1}{453600}A - \frac{1}{453600}B + \frac{1}{259200}\right)x^{10} \\ & + \left(\frac{1}{3991680}A + \frac{1}{1995840}B + \frac{1}{19958400}C - \frac{1}{1330560}\right)x^{11} \\ & + \left(-\frac{1}{39916800}A - \frac{1}{11975040}B - \frac{1}{39916800}C + \frac{1}{239500800}\right)x^{12}. \end{aligned}$$

The series solution is given as

$$y(x) = 1 + Ax + \frac{1}{2}x^2 + \frac{1}{6}Bx^3 + \frac{1}{24}x^4 + \frac{1}{120}Cx^5 + \frac{1}{720}x^6$$

$$
\begin{aligned}
&+\left(\frac{1}{2520}A-\frac{1}{5040}\right)x^{7}+\left(-\frac{1}{10080}A+\frac{1}{13440}\right)x^{8}\\
&+\left(\frac{1}{60480}A+\frac{1}{181440}B-\frac{1}{51840}\right)x^{9}\\
&+\left(-\frac{1}{453600}A-\frac{1}{453600}B+\frac{1}{241920}\right)x^{10}\\
&+\left(\frac{1}{3991680}A+\frac{1}{1995840}B+\frac{1}{19958400}C-\frac{31}{39916800}\right)x^{11}\\
&+\left(-\frac{1}{39916800}A-\frac{1}{11975040}B-\frac{1}{39916800}C+\frac{1}{7603200}\right)x^{12}\\
&+o\left(x\right)^{13}
\end{aligned}
$$

Imposing the boundary conditions at $x=1$ leads to the following system

$$
\begin{bmatrix} \frac{39929261}{39916800} & \frac{9979423}{59875200} & \frac{332641}{39916800} \\ \frac{5501}{453600} & \frac{45371}{45360} & \frac{75601}{453600} \\ \frac{2089}{10080} & \frac{43}{5040} & \frac{10080}{10080} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} e-\frac{11731097}{7603200} \\ e-\frac{5575343}{3628800} \\ e-\frac{57103}{40320} \end{bmatrix}
$$

The solution of above system is given as

$$A=1.001252684, B=0.988483055, C=1.085993892.$$

The series solution is given as

$$
\begin{aligned}
y\left(x\right) &= 1+1.0012526840x+0.500x^{2}+0.16474717590x^{3}+0.0416666667x^{4}\\
&+0.0001989097x^{5}+0.0013888888x^{6}+0.0001989097x^{7}+.000001x^{8}\\
&+0.0000271296x^{9}-0.0000002529x^{10}+0.0000000239x^{11}\\
&-0.0000000033x^{12}+o\left(x^{13}\right),
\end{aligned}
$$

which is in full agreement with [27, 28, 34].

Table 5.1 (Error estimates)

	*Error			
x	Exact Solution	VDM	HPM	ADM
0.0	1.000000000	0.000000	0.000000	0.000000
0.1	1.105170918	-1.233E − 4	-1.233E − 4	-1.233E − 4
0.2	1.221402758	-2.354E - 4	-2.354E - 4	-2.354E - 4
0.3	1.349858808	-3.257E - 4	-3.257E - 4	-3.257E - 4
0.4	1.491824698	-3.855E - 4	-3.855E - 4	-3.855E - 4
0.5	1.648721271	-4.086E - 4	-4.086E - 4	-4.086E - 4
0.6	1.822118800	-3.919E - 4	-3.919E - 4	-3.919E - 4
0.7	2.013752707	-3.361E - 4	-3.361E - 4	-3.361E - 4
0.8	2.225540928	-2.459E - 4	-2.459E - 4	-2.459E - 4
0.9	2.459603111	-1.299E - 4	-1.299E - 4	-1.299E - 4
1.0	2.718281828	2.000E - 9	2.000E - 9	2.000E - 9

*Error = Exact solution − Series solution

Table 5.1 exhibits the errors obtained by using the variational decomposition method (VDM), the homotopy perturbation method (HPM) and the Adomian's decomposition method (ADM). It is obvious that evaluation of more components of $y(x)$ will reasonably improve the accuracy of series solution.

Example 5.2. Consider the following nonlinear boundary value problem of sixth-order

$$y^{(vi)}(x) = e^x y^2(x), \quad 0 < x < 1$$

with boundary conditions

$$\begin{aligned} y(0) &= 1, \;\; y'(0) = -1, \;\; y''(0) = 1; \;\; y(1) = e^{-1}, \;\; y''(1) = -e^{-1}, \\ & y^{(iv)}(1) = e^{-1} \end{aligned}$$

The exact solution for this problem is $y(x) = e^{-x}$.

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left(\frac{d^6 y_n}{dx^6} - e^s \widetilde{y_n}^2(s) \right) ds.$$

The correct functional for this boundary value problem is given as $\lambda = \frac{1}{5!}(s-x)^5$

Making the correct functional stationary, the Lagrange multiplier can be identified as [14-19, 36], we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{5!}(s-x)^5 \left(\frac{d^6 y_n}{dx^6} - e^s \widetilde{y_n}^2(s) \right) ds,$$

where $A = y'''(0)$, $B = y^{(iv)}(0)$, $C = y^{(v)}(0)$. Applying the variational iteration decomposition method, we obtain

$$\begin{aligned} y_{n+1}(x) &= 1 - x + \frac{1}{2!}x^2 + \frac{1}{3!}Ax^3 + \frac{1}{4!}Bx^4 + \frac{1}{5!}Cx^5 \\ &+ \int_0^x \frac{1}{5!}(s-x)^5 \left(\frac{d^6 y_n}{dx^6} - e^s \sum_{n=0}^{\infty} A_n \right) ds. \end{aligned}$$

where A_n are Adomian's polynomials for nonlinear operator $F(y) = y^2(x)$and can be generated for all type of nonlinearities according to the algorithm developed in [34, 35] which yields the following

$$\begin{aligned} A_0 &= y_0^2(x), \;\; A_1 = 2y_0(x)y_1(x), \;\; A_2 = y_2 F'(y_0) + \frac{y_1^2}{2!}F''(y_0), \\ A_3 &= 2y_0(x)y_2(x) + y_1^2(x) \end{aligned}$$

Consequently, we obtain the following approximants $y_0(x) = 1$,

$$y_1(x) = 1 - 2x - 1 + \left(\frac{1}{3!}A - \frac{1}{3!} \right)x^3 + \left(\frac{1}{4!}B - \frac{1}{4!} \right)x^4 + \left(\frac{1}{5!}C - \frac{1}{5!} \right)x^5 + e^{-x},$$

$$
\begin{aligned}
y_2(x) \quad = \quad & 1-2x-1+\left(\frac{1}{3!}A-\frac{1}{3!}\right)x^3+\left(\frac{1}{4!}B-\frac{1}{4!}\right)x^4+\left(\frac{1}{5!}C-\frac{1}{5!}\right)x^5 \\
& +e^{-x}-\frac{1}{2520}Ax^7 \\
& -\frac{1}{20160}x^8+\frac{1}{181440}Ax^9+\left(\frac{1}{453600}A+\frac{1}{1814400}B+\frac{1}{907200}\right)x^{10} \\
& +\left(\frac{1}{19958400}A+\frac{1}{3991680}B+\frac{1}{19958400}C+\frac{1}{13391680}\right)x^{11} \\
& +\left(\frac{1}{11975040}A+\frac{1}{15966720}B+\frac{1}{39916800}C+\frac{1}{239500800}\right)x^{12}.
\end{aligned}
$$

The series solution is given as

$$
\begin{aligned}
y(x) & = 1-x+\frac{1}{2}x^2+\frac{1}{6}Ax^3+\frac{1}{24}Bx^4 \\
& +\frac{1}{120}Cx^5+\frac{1}{720}x^6-\frac{1}{5040}x^7-\frac{1}{40320}x^8 \\
& +\left(\frac{1}{181440}A+\frac{1}{362880}\right)x^9+\left(1453600A+\frac{1}{181440}B+\frac{1}{725760}\right)x^{10} \\
& +\left(\frac{1}{1995840}A+\frac{1}{3991680}B+\frac{1}{19958400}C+\frac{1}{3628800}\right)x^{11} \\
& +\left(\frac{1}{11975040}A+\frac{1}{15969720}B+\frac{1}{3916800}C+\frac{1}{22809600}\right)x^{12}+o(x)^{13}
\end{aligned}
$$

Imposing the boundary conditions at $x=1$,leads to the following system

$$
\begin{bmatrix}
\frac{1425671}{8553600} & \frac{1108823}{26611200} & \frac{110881}{13305600} \\
\frac{83173}{166320} & \frac{18481}{110880} & \frac{831617}{19958400} \\
\frac{1513}{1512} & \frac{181471}{362880} & \frac{18901}{113400}
\end{bmatrix}
\begin{bmatrix} A \\ B \\ C \end{bmatrix}
=
\begin{bmatrix}
e^{-1}-\frac{8891159}{17740800} \\
e^{-1}-\frac{135481}{19958400} \\
e^{-1}-\frac{3746021}{3628800}
\end{bmatrix}
$$

The solution of this algebraic system gives

$$A=-0.99816409, B=0.98167470, C=-0.93907310.$$

The series solution is given as

$$
\begin{aligned}
y(x) \quad = \quad & 1-x+0.500x^2-0.1663606817x^3+0.0409031125x^4 \\
& -0.007825609x^5+0.00013888889x^6 \\
& -0.000198412x^7-.000024801x^8-2.74561337\times 10^{-6}x^9 \\
& -0.0000001557x^{10}-0.0000000256x^{11}-0.0000000015x^{12}+o\left(x^{13}\right),
\end{aligned}
$$

which is in full agreement with [27, 28, 34].

Table 5.2 exhibits the errors obtained by using the variational decomposition method (VDM), the homotopy perturbation method (HPM) and the Adomian's decomposition method (ADM). It is obvious that evaluation of more components of y(x) will reasonably improve the accuracy of series solution.

Table 5.2 (Error estimates)

	*Error			
x	Exact Solution	VDM	HPM	ADM
0.0	1.000000000	0.000000	0.000000	0.000000
0.1	0.9048374180	-2.347E - 7	-2.347E - 7	-2.347E - 7
0.2	0.8187307531	-1.389E - 6	-1.389E - 6	-1.389E - 6
0.3	0.7408182207	-3.307E - 6	-3.307E - 6	-3.307E - 6
0.4	0.6703200460	-5.203E - 6	-5.203E - 6	-5.203E - 6
0.5	0.6065306597	-6.198E - 6	-6.198E - 6	-6.198E - 6
0.6	0.5488116361	-5.780E - 6	-5.780E - 6	-5.780E - 6
0.7	0.4965853038	-4.082E - 6	-4.082E - 6	-4.082E - 6
0.8	0.4493289641	-1.903E - 6	-1.903E - 6	-1.903E - 6
0.9	0.4065696597	-3.570E - 7	-3.570E - 7	-3.570E - 7
1.0	0.3678794412	-5.000E - 10	-5.000E - 10	-5.000E - 10

*Error=Exact solution – Series solution

Example 5.3. Consider the following special sixth-order boundary value problem involving a parameter c

$$u^{(6)}(x) = (1+c)\,u^{(4)}(x) - cu^{(2)}(x) + cx$$

with boundary conditions

$$\begin{aligned} u(0) &= 1, \quad u'(0) = 1, \quad u''(0) = 0, u(1) = \frac{7}{6} + \sinh(1) \\ u'(1) &= \frac{1}{2} + \cosh(1), \quad u''(1) = 1 + \sinh(1). \end{aligned}$$

The exact solution of the problem is$u(x) = 1 + \frac{1}{6}x^3 + \sinh(x)$.

The correct functional for this boundary value problem is given as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left(\frac{d^6 u_n}{dx^6} - \left((1+c)\,\widetilde{u_n}^{(4)}(s) - c\widetilde{u_n}^{(2)}(s) + cs\right)\right)ds.$$

Making the correct functional stationary, the Lagrange multiplier can be identified as $\lambda = \frac{1}{5!}(s-x)^5$[14-19, 36], we get the following iterative formula

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{5!}(s-x)^5\left(\frac{d^6 u_n}{dx^6} - \left((1+c)\,\widetilde{u_n}^{(4)}(s) - c\widetilde{u_n}^{(2)}(s) + cs\right)\right)ds$$

Applying the variational decomposition method, we obtain

$$\begin{aligned} u_{n+1}(x) &= 1 + x + \frac{1}{3!}Ax^3 + \frac{1}{4!}Bx^4 + \frac{1}{5!}Dx^5 \\ &+ \int_0^x \frac{1}{5!}(s-x)^5\left(\frac{d^6 u_n}{dx^6} - \left((1+c)\sum_{n=0}^{\infty}\widetilde{u_n}^{(4)}(s) - c\sum_{n=0}^{\infty}\widetilde{u_n}^{(2)}(s) + cs\right)\right)ds, \end{aligned}$$

where $A = y'''(0)$, $B = y^{(iv)}(0)$, $D = y^{(v)}(0)$. Proceeding as before, the series solution is given as

$$u(x) = 1+x+\frac{1}{3!}Ax^3+\frac{1}{4!}Bx^4+\frac{1}{5!}Dx^5+\frac{1}{6!}Bx^6+\frac{1}{6!}Bcx^6+\frac{1}{7!}cx^7$$
$$+\frac{1}{7!}Dx^7-\frac{1}{7!}Acx^7+\frac{1}{7!}Dcx^7-\frac{1}{8!}Bcx^8+\frac{1}{9!}cx^9-\frac{1}{11!}c^2x^{11}$$

Table 5.3

	$c=1$	$c=10$	$c=100$	$c=1000$	$c=100000$
A	2.0000060289	1.9905263769	1.8454775798	-6.852614584	1099168.307
B	-0.0000558706	0.09564140157	1.0137308481	-8.390834496	-1018.591684
C	1.00017512540	0.60597366252	-5.6392050771	-71.57411568	1106809.16

Table 5.4

	$c=1$				
x	Exact Solution	*E(DTM)	*E(HPM)	*E(ADM)	*E(VDM)
0.0	1.000000000	1.00000	0.0000	0.0000	0.0000
0.1	1.1003334166	-4.5 E-6	-7.8E-10	-7.8E-10	-7.8E-10
0.2	1.2026693358	-2.5 E-5	-4.7E-9	-4.7E-9	-4.7E-9
0.3	1.3060202934	-5.9 E-5	-1.7E-8	-1.7E-8	-1.7E-8
0.4	1.4214189924	-9.1 E-5	-1.9E-8	-1.9E-8	-1.9E-8
0.5	1.5419286388	-1.0 E-4	-2.4E-8	-2.4E-8	-2.4E-8
0.6	1.6726535821	-9.6 E-5	-2.3E-8	-2.3E-8	-2.3E-8
0.7	1.8157503685	-6.6 E-5	-1.7E-8	-1.7E-8	-1.7E-8
0.8	1.9734383155	-3.0 E-5	-8.6E-8	-8.6E-8	-8.6E-8
0.9	2.1480167257	-5.5 E-6	-1.7E-9	-1.7E-9	-1.7E-9
1.0	2.3418678603	0.0000	0.0000	0.0000	0.0000

*E=Exact solution–series solution

Table 5.5

	$c=10$				
x	Exact Solution	*E(DTM)	*E(HPM)	*E(ADM)	*E(VDM)
0.0	1.000000000	1.00000	0.0000	0.0000	0.0000
0.1	1.1003334166	-2.9 E-5	1.2 E-6	1.2 E-6	1.2 E-6
0.2	1.2026693358	-1.6 E-4	7.2 E-6	7.2 E-6	7.2 E-6
0.3	1.3060202934	-3.6 E-4	1.7 E-5	1.7 E-5	1.7 E-5
0.4	1.4214189924	-5.3 E-4	2.7 E-5	2.7 E-5	2.7 E-5
0.5	1.5419286388	-6.0 E-4	3.4 E-5	3.4 E-5	3.4 E-5
0.6	1.6726535821	-5.3 E-4	3.2 E-5	3.2 E-5	3.2 E-5
0.7	1.8157503685	-3.5 E-4	2.3 E-5	2.3 E-5	2.3 E-5
0.8	1.9734383155	-1.5 E-4	1.1 E-5	1.1 E-5	1.1 E-5
0.9	2.1480167257	-2.7 E-5	2.2 E-6	2.2 E-6	2.2 E-6
1.0	2.3418678603	0.0000	0.0000	0.0000	0.0000

*E= Exact Solution – Series Solution

which is exactly the same as obtained in [20, 27] by using Adomian's decomposition method and homotopy perturbation method.

Imposing the boundary conditions at x =1, we have

$$A\approx 2+g(c),\quad B\approx 0+h(c),\quad D\approx 1+p(c),$$

where the functions g(c), h(c) and p(c) grow rapidly with c. In reality, they should go to zeros as the number of terms in the series goes to infinity. Table 3.3 shows the vales of A, B and D, for different values of the parameter c. It is easy to notice that the approximate solution obtained by the variational decomposition method is in good agreement with the exact solution for the small values of the parameter c and continuously depends on the parameter c.

Table 5.3 exhibits the values of the constants A, B and D for different values of c.

Table 5.6

	$c = 100$				
x	Exact Solution	*E(DTM)	*E(HPM)	*E(ADM)	*E(VDM)
0.0	1.000000000	1.00000	0.0000	0.0000	0.0000
0.1	1.1003334166	4.7 E-5	2.1E-5	2.1E-5	2.1E-5
0.2	1.2026693358	3.2 E-4	1.4E-4	1.4E-4	1.4E-4
0.3	1.3060202934	8.7 E-4	4.1E-4	4.1E-4	4.1E-4
0.4	1.4214189924	1.5 E-3	7.5E-4	7.5E-4	7.5E-4
0.5	1.5419286388	2.1 E-3	1.0E-3	1.0E-3	1.0E-3
0.6	1.6726535821	2.2 E-4	1.1E-3	1.1E-3	1.1E-3
0.7	1.8157503685	1.7 E-3	9.2E-3	9.2E-3	9.2E-3
0.8	1.9734383155	9.1 E-4	4.9E-3	4.9E-3	4.9E-3
0.9	2.1480167257	1.9 E-4	1.0E-4	1.0E-4	1.0E-4
1.0	2.3418678603	0.0000	0.0000	0.0000	0.0000

*E= Exact solution – Series solution

Table 5.7

	$c = 1000$				
x	Exact Solution	*E(DTM)	*E(HPM)	*E(ADM)	*E(VDM)
0.0	1.000000000	1.00000	0.0000	0.0000	0.0000
0.1	1.1003334166	5.9 E-5	1.4 E-3	1.4 E-3	1.4 E-3
0.2	1.2026693358	4.0 E-4	1.0 E-2	1.0 E-2	1.0 E-2
0.3	1.3060202934	1.1 E-3	3.2 E-2	3.2 E-2	3.2 E-2
0.4	1.4214189924	1.9 E-3	6.3 E-2	6.3 E-2	6.3 E-2
0.5	1.5419286388	2.6 E-3	9.3 E-2	9.3 E-2	9.3 E-2
0.6	1.6726535821	2.8 E-3	1.0 E-1	1.0 E-1	1.0 E-1
0.7	1.8157503685	2.2 E-3	8.6 E-2	8.6 E-2	8.6 E-2
0.8	1.9734383155	1.1 E-3	4.7 E-2	4.7 E-2	4.7 E-2
0.9	2.1480167257	2.5 E-4	1.0 E-2	1.0 E-2	1.0 E-2
1.0	2.3418678603	0.0000	0.0000	0.0000	0.0000

*E= Exact Solution – Series Solution

Table 5.4 – 5.8 exhibit the numerical results for small and large values of c. The tables show that solution obtained using ADM, HPM and VDM is in good agreement with the exact solution for small values of c only, whereas the approximate solution obtained using DTM are in good agreement with the exact solution for all values of the parameter c. Consequently, one can say that the variational decomposition method (VDM) is not reliable for solving such problems.

Example 5.4 Consider the following linear boundary value problem of sixth-order

$$y^{(vi)}(x) = -6e^x + y(x), \quad 0 < x < 1$$

with boundary conditions

$$y(0) = 1, y''(0) = -1, y^{(iv)}(0) = -3, \quad y(1) = 0, y''(1) = -2e, y^{(iv)}(1) = -4e.$$

Table 5.8

	$c = 10000$				
x	Exact Solution	*E(DTM)	*E(HPM)	*E(ADM)	*E(VDM)
0.0	1.000000000	1.00000	0.0000	0.0000	0.0000
0.1	1.1003334166	6.0 E-5	-1.8 E+2	-1.8 E+2	-1.8 E+2
0.2	1.2026693358	4.1 E-4	-1.3 E+3	-1.3 E+3	-1.3 E+3
0.3	1.3060202934	1.1 E-3	-4.2 E+3	-4.2 E+3	-4.2 E+3
0.4	1.4214189924	2.0 E-3	-8.4 E+3	-8.4 E+3	-8.4 E+3
0.5	1.5419286388	2.7 E-3	-1.2 E+4	-1.2 E+4	-1.2 E+4
0.6	1.6726535821	2.9 E-3	-1.4 E+4	-1.4 E+4	-1.4 E+4
0.7	1.8157503685	2.3 E-3	-1.1 E+4	-1.1 E+4	-1.1 E+4
0.8	1.9734383155	1.2 E-3	6.5 E+4	6.5 E+4	6.5 E+4
0.9	2.1480167257	2.5 E-4	-1.4 E+3	-1.4 E+3	-1.4 E+3
1.0	2.3418678603	0.0000	0.0000	0.0000	0.0000

*E= Exact solution – Series solution

The exact solution of the problem is $y(x) = (1 - x)e^x$. The correct functional for this boundary value problem is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(\frac{d^6 y_n}{dx^6} - (y_n(s) + 6e^s)\right) ds.$$

Making the correct functional stationary, the Lagrange multiplier can be identified as $\lambda = \frac{1}{5!}(s - x)^5$[14-19, 36], we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{5!}(s - x)^5 \left(\frac{d^6 y_n}{dx^6} - (\widetilde{y_n}(s) + 6e^s)\right) ds.$$

Applying the variational decomposition method, we have

$$\begin{aligned} y_{n+1}(x) &= 1 + Ax - \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 - \frac{3}{4!}x^4 + \frac{1}{5!}Cx^5 \\ &+ \int_0^x \frac{1}{5!}(s - x)^5 \left(\frac{d^6 y_n}{dx^6} - \left(\sum_{n=0}^{\infty} y_n(s) + 6e^s\right)\right) ds. \end{aligned}$$

where $A = y'(0)$, $B = y'''(0)$, $C = y^{(v)}(0)$. Consequently, we obtain the following approximants $y_0(x) = 1$,

$$y_1(x) = 7 + (A + 6)x + \frac{5}{2!}x^2 + \left(\frac{1}{3!}B + 1\right)x^3 + \frac{1}{8!}x^4 + \left(\frac{1}{5!}C - \frac{1}{20}\right)x^5 - 6e^x,$$

$$\begin{aligned} y_2(x) &= 13 + (A+12)\,x + \frac{11}{2!}x^2 + \left(\frac{1}{3!}B + 2\right)x^3 + \frac{3}{8}x^4 + \left(\frac{1}{5!}C + \frac{2}{20}\right)x^5 \\ &\quad + \frac{1}{120}x^6 - 12e^x + \left(\frac{1}{5040}A + \frac{1}{840}\right)x^7 + \frac{1}{8064}x^8 \\ &\quad + \left(\frac{1}{362880}B + \frac{1}{60840}\right)x^9 \\ &\quad + \frac{1}{1209600}x^{10} + \left(\frac{1}{6652800} + \frac{1}{39916800}C\right)x^{11} \end{aligned}$$

The series solution is given as

$$\begin{aligned} y(x) &= 1 + Ax - \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 - \frac{3}{4!}x^4 + \frac{1}{5!}Cx^5 - \frac{1}{120}x^6 \\ &\quad + \left(\frac{1}{5040}A - \frac{1}{840}\right)x^7 - \frac{1}{5760}x^8 \\ &\quad + \left(\frac{1}{362880}B - \frac{1}{60840}\right)x^9 - \frac{1}{403200}x^{10} \\ &\quad + \left(-\frac{1}{6652800} + \frac{1}{39916800}C\right)x^{11} \\ &\quad - \frac{1}{39916800}x^{12} + \left(-\frac{1}{518918400} + \frac{1}{6227020800}A\right)x^{13} + 0\left(x^{14}\right) \end{aligned}$$

Imposing the boundary conditions at x = 1 leads to the following system

$$\begin{bmatrix} \frac{889750903}{889574400} & \frac{60481}{362880} & \frac{332641}{39916800} \\ \frac{332641}{39916800} & \frac{5041}{5040} & \frac{60481}{362880} \\ \frac{60481}{362880} & \frac{1}{120} & \frac{5041}{5040} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -\frac{189552271}{518918400} \\ -2e - \frac{18702059}{6652800} \\ -4e - \frac{55609}{7560} \end{bmatrix}$$

The solution of the above system gives

$$A = 0.0041622709, \quad B = -2.041623366, \quad C = -3.500425047$$

The series solution is given as

$$\begin{aligned} y(x) &= 1 + 0.0041622709x - 0.500x^2 - 0.3402705611x^3 - .125x^4 \\ &\quad - 0.0291702095x^5 - 0.0083333333x^6 - 0.0011896503x^7 \\ &\quad - .00017358x^8 - 0.221605582 \times 10^{-4}x^9 \\ &\quad - 0.0000024800x^{10} - 0.2380056805 \times 10^{-6}x^{11} - 0.0000000025x^{12} + o\left(x^{13}\right), \end{aligned}$$

which is in full agreement with [27, 28, 34].

Table 5.9 exhibits the errors obtained by using the variational decomposition method (VDM), the homotopy perturbation method (HPM) and the Adomian's decomposition method (ADM). It is obvious that evaluation of more components of y(x) will reasonably improve the accuracy of series solution.

Table 5.9 (Error estimates)

	*Errors			
x	Exact Solution	VDM	HPM	ADM
0.0	1.000000000	0.000000	0.000000	0.000000
0.1	0.99465383	-0.00040933	-0.00040933	-0.00040933
0.2	0.97712221	-0.00077820	-0.00077820	-0.00077820
0.3	0.94490117	-0.00107048	-0.00107048	-0.00107048
0.4	0.89509482	-0.00125787	-0.00125787	-0.00125787
0.5	0.82436064	-0.00132238	-0.00132238	-0.00132238
0.6	0.72884752	-0.00125787	-0.00125787	-0.00125787
0.7	0.60412581	-0.00107048	-0.00107048	-0.00107048
0.8	0.44510819	-0.00077820	-0.00077820	-0.00077820
0.9	0.24596031	-0.00040933	-0.00040933	-0.00040933
1.0	0.000000	0.000000	0.000000	0.000000

*Error=Exact solution − Series solution

Conclusion

In this paper, we have used the variational decomposition method (VDM) by combining the traditional variational iteration method and the decomposition method for finding the solution of boundary value problems for sixth-order. The method is applied in a direct way without using linearization, perturbation or restrictive assumptions. It may be concluded that VDM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result, the size reduction amounts to the improvement of performance of approach. It is worth mentioning that we also considered an example where the proposed VDM is not reliable.

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