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ON THE CONVERGENCE OF PARALLEL *GAOR* METHOD FOR BLOCK DIAGONALLY DOMINANT MATRICES

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ABSTRACT. In [2] A.Hadjidimos proposed the generalized accelerated overrelaxation (*GAOR*) methods which generalize the basic iterative method for the solution of linear systems. In this paper we consider the convergence of the two parallel accelerated generalized *AOR* iterative methods and obtain some convergence theorems for the case when the coefficient matrix A is a block diagonally dominant matrix or a generalized block diagonally dominant matrix.

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1. Introduction

For solving the large nonsingular linear systems

$$Ax = b, \tag{1}$$

where $A \in R^{n\times n}$ is a nonsingular matrix with nonvanishing diagonal entries and $b \in R^n$. O'Leary and White [1,12] proposed the idea of matrix parallel multisplitting which provides a effective way for projecting parallel iterative methods. Recently, the parallel iterative methods of some families of matrices attracted researchers' attention and several significant results were proposed. The multisplitting method was further studied by Neumann and Plemmons [3], Frommer and Mayer [4,5], Wang [6], J.H.Yun et al.[13].

A.Hadjidimos [2,11] researched the generalized accelerated overrelaxation (G AOR) methods which generalize the basic iterative method for the solution of linear systems. Let

$$A = D_1 - D_2 - D_3 - C_L - C_U, \tag{2}$$

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where $D_i(i=1,2,3)$ are diagonal matrices and C_L, C_U are strictly lower and upper triangular matrices, $D_A = D_1 - D_2 - D_3 = diag(A)$. For the parameters $(\gamma,\omega), \omega \neq 0$, if $D_1 - \gamma D_2$ is nonsingular, then the generalized accelerated overrelaxation $(GAOR)$ method of linear systems (1) is as follows:

$$x^{(k+1)} = L_{\gamma,\omega}x^{(k)} + \omega\big(D_1 - \gamma(D_2 + C_L)\big)^{-1}b, \quad k = 0,1,\cdots, \tag{3}$$

where the generalized accelerated overrelaxation $(GAOR)$ matrix

$$L_{\gamma,\omega} = \big(D_1 - \gamma(D_2 + C_L)\big)^{-1}\big((1-\omega)D_1 + (\omega-\gamma)(D_2 + C_L) + \omega(D_3 + C_U)\big). \tag{4}$$

When the parameters (γ,ω) are transformed to $(\omega,\omega), (1,1), (0,\omega)$ and $(0,1)$, respectively. Then the generalized accelerated overrelaxation $(GAOR)$ method reduces to the generalized $SOR(GSOR)$, the generalized $Gauss-Seidel(GGS)$, the generalized $JOR(GJOR)$ method and the generalized $Jacobi$ (GJ) method.

Song [8] gave a example to show the merits of $GAOR$ method. By choosing the appropriate matrices $D_i(i=1,2,3)$, which can make $\rho(L_{\gamma,\omega})$ be small as soon as possible. For example

$$\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix},$$

Let $D_1 = diag(8,7,8), D_2 = 0, \omega = 2$, it shows that $\rho(L_{\omega,\omega}) = 0$.

In this paper, our main idea is to apply appropriate matrices $D_i(i=1,2,3)$ to the parallel generalized accelerated overrelaxation methods for the solution of linear systems. Then, we discuss the convergence of the two parallel multisplitting generalized AOR iterative methods and obtain some convergence theorems for the case when the coefficient matrix A is a block diagonally dominant matrix or a generalized block diagonally dominant matrix.

2. Multisplitting and algorithms

Consider an $n \times n$ real matrix A, which is partitioned as the following form:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix},$$

where A_{ii} is an $n_i \times n_i$ nonsingular principal submatrix of A, $i = 1,2,\cdots,s$, $\sum_{i=1}^{s} n_i = n$. For a positive integer n, $n_i(n_i \leq n, i = 1,2,\cdots,s, s \leq n)$ satisfy $\sum_{i=1}^{s} n_i = n$, we define $V_n(n_1,n_2,\cdots,n_s) = \{x \in R^n | x = (x_1^T, x_2^T, \cdots, x_s^T)^T, x_i \in R^{n_i}, i = 1,2,\cdots,s\}$,

$$L_n(n_1,n_2,\cdots,n_s) = \{A \in L(R^n) | A = (A_{ij}), A_{ij} \in L(R^{n_j}, R^{n_i}), i,j = 1,2,\cdots,s\}.$$

Specially, $L(R^{n_i}, R^{n_i})$ is further abbreviated to $L(R^{n_i})$, where $L(R^{n_j}, R^{n_i})$ denotes the set of all $n_j \times n_i$ real matrix.

Definition 1. For $k \in \{1, 2, \cdots, \tau\}$, if block matrices $M_k, N_k, E_k \in L_n(n_1, n_2, \cdots, n_s)$ satisfy

(i) $A = M_k - N_k, det(M_k) \neq 0, k = 1, 2, \cdots, \tau,$

(ii) $E_k = diag(E_{11}^{(k)}, \cdots, E_{ss}^{(k)}), k = 1, 2, \cdots, \tau, \sum\limits_{k=1}^{\tau} \|E_{ii}^{(k)}\| = 1, i = 1, 2, \cdots, s,$

(iii) $\sum\limits_{k=1}^{\tau} E_k = I$

The triad $(M_k, N_k, E_k), k = 1, 2, \cdots, \tau$, is called a mulsplitting of a block matrix A, where $\|\cdot\|$ denotes the compatible matrix norm such that $\|I\| = 1 (I \in L(R^m)$ is an identity matrix).

Given that $\tau (\tau \leq s)$, $J_k (k = 1, 2, \cdots, \tau)$ are excisive sets of the set $\{1, 2, \cdots, s\}$. Namely, $J_k \subseteq \{1, 2, \cdots, s\}, k = 1, 2, \cdots, \tau, \bigcup\limits_{k=1}^{\tau} J_k = \{1, 2, \cdots, s\}$. For a block matrix $A \in L_n(n_1, n_2, \cdots, n_s)$, we define

$$\begin{cases} D = diag(A_{11}, A_{22}, \cdots, A_{ss}) \in L_n(n_1, n_2, \cdots, n_s), detD \neq 0, \\ L_k = (\tilde{L}_{ij}^{(k)}) \in L_n(n_1, n_2, \cdots, n_s), \tilde{L}_{ij}^{(k)} = \begin{cases} L_{ij}^{(k)}, & if\ i, j \in J_k, i > j, \\ 0, & otherwise, \end{cases} \\ U_k = (\tilde{U}_{ij}^{(k)}) \in L_n(n_1, n_2, \cdots, n_s), \tilde{U}_{ij}^{(k)} = \begin{cases} U_{ij}^{(k)}, & if\ i \neq j, \\ 0, & if\ i = j, \end{cases} \\ i, j = 1, 2, \cdots, s; k = 1, 2, \cdots, \tau, \end{cases}$$

where $D, L_k, U_k \in L_n(n_1, n_2, \cdots, n_s) (k = 1, 2, \cdots, \tau)$ are a block diagonal, block strictly lower triangular and zero diagonal matrices, $A = D - L_k - U_k (k = 1, 2, \cdots, \tau)$, $E_k = diag(E_{11}^{(k)}, E_{22}^{(k)}, \cdots, E_{ss}^{(k)}) \in L_n(n_1, n_2, \cdots, n_s) (k = 1, 2, \cdots, \tau)$

$$\begin{cases} E_{ii}^{(k)} = \begin{cases} E_{ii}^{(k)}, & i \in J_k, \\ 0, & i \in \{1, 2, \cdots, s\} \backslash J_k, \end{cases} & i = 1, 2, \cdots, s; k = 1, 2, \cdots, \tau, \\ \sum\limits_{k=1}^{\tau} \|E_{ii}^{(k)}\| = 1, & i = 1, 2, \cdots, s. \end{cases}$$

By the above definitions, we now describe the two relaxed parallel accelerated generalized AOR methods.

Algorithm I. Choose $x^{(0)} \in V_n(n_1, n_2 \cdots, n_s)$, for $p = 0, 1, 2, \cdots$, until convergence, perform

$$x^{p,k} = L_{MBGAOR}^{(k)}(A) x^p + b_{MBGAOR}^{(k)}, k = 1, 2, \cdots, \tau, \tag{5}$$

$$x^{p+1} = \sum_{k=1}^{\tau} E_k x^{p,k}, \tag{6}$$

where

$$\begin{aligned} L^{(k)}_{MBGAOR}(A) &= \big(D_1 - \gamma(D_2 + L_k)\big)^{-1}\big((1-\omega)D_1 \\ &\quad + (\omega - \gamma)(D_2 + L_k) + \omega(D_3 + U_k)\big), \end{aligned}$$

$$b^{(k)}_{MBGAOR} = \omega\big(D_1 - \gamma(D_2 + L_k)\big)^{-1} b,$$

with $\omega > 0$.

If we define the matrix and the vector:

$$L_{MBGAOR}(A) = \sum_{k=1}^{\tau} E_k L^{(k)}_{MBGAOR}(A),$$

$$b_{MBGAOR} = \sum_{k=1}^{\tau} E_k b^{(k)}_{MBGAOR},$$

then Algorithm I can be equivalently written as

$$x^{p+1} = L_{MBGAOR}(A)x^p + b_{MBGAOR}. \tag{7}$$

It is obvious to see that Algorithm I is convergent if and only if $\rho(L_{MBGAOR}(A)) < 1$.

By introducing a appropriate positive relaxation parameter β to the Algorithm I, we then get the following relaxed Algorithm II.

Algorithm II. Choose $x^{(0)} \in V_n(n_1, n_2 \cdots, n_s)$, for $p = 0, 1, 2, \cdots$, until convergence, perform

$$y^{p,k} = L^{(k)}_{MBGAOR}(A)x^p + b^{(k)}_{MBGAOR}, k = 1, 2, \cdots, \tau, \tag{8}$$

$$x^{p,k} = \beta y^{p,k} + (1 - \beta)x^p \tag{9}$$

$$x^{p+1} = \sum_{k=1}^{\tau} E_k x^{p,k}, \tag{10}$$

where $L^{(k)}_{MBGAOR}(A)$ and $b^{(k)}_{MBGAOR}$ are similarly defined as (5), $\beta \in (0, +\infty)$.

Similarly, we can write the iterative matrix of Algorithm II as

$$\begin{aligned} L'_{MBGAOR}(A) &= \beta \sum_{k=1}^{\tau} E_k L^{(k)}_{MBGAOR}(A) + (1-\beta)\sum_{k=1}^{\tau} E_k \\ &= \beta L_{MBGAOR}(A) + (1-\beta)I, \end{aligned}$$

$$b'_{MBGAOR} = \beta \sum_{k=1}^{\tau} E_k b^{(k)}_{MBGAOR},$$

then Algorithm II can be equivalently written as:

$$x^{p+1} = L'_{MBGAOR}(A)x^p + b'_{MBGAOR}. \tag{11}$$

It is obvious to see that Algorithm II is convergent if and only if $\rho(L'_{MBGAOR}(A)) < 1$.

3. Preliminaries

For a matrix $G = (g_{ij}) \in L(R^m)$, $G \in L(R^m)$ is called an $M-$ matrix if $g_{ij} \leq 0(i \neq j)$, $i, j = 1, 2, \cdots, m$, G^{-1} exists and $G^{-1} \geq 0$. Let $D_G = diag(g_{11}, g_{22}, \cdots, g_{mm}), B_G = D_G - G$, G is an $M-$ matrix if and only if $D_G > 0$ is nonsingular and $\rho(J_G) = \rho(D_G^{-1}B_G) < 1$, where J_G denotes the matrix $D_G^{-1}B_G$. We define the set

$$L_{n,I}(n_1, n_2, \cdots, n_s) = \{M = (M_{ij}) \in L_n(n_1, n_2, \cdots, n_s) | M_{ii} \in L(R^{n_i}) \text{ is nonsingular}, i = 1, 2, \cdots, s\},$$
$$L^d_{n,I}(n_1, n_2, \cdots, n_s) = \{M^d = diag(M_{11}, M_{22}, \cdots, M_{ss}) | M_{ii} \in L(R^{n_i}) \text{ is nonsingular}, i = 1, 2, \cdots, s\}.$$

Definition 2 ([7]). Let $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$, a (I) block comparison matrix$\langle M \rangle = (\langle M \rangle_{ij}) \in L(R^s)$and a (II) block comparison matrix $\langle\langle M \rangle\rangle = (\langle\langle M \rangle\rangle_{ij}) \in L(R^s)$ are defined respectively as follows:

$$\langle M \rangle_{ij} = \begin{cases} \|M_{ii}^{-1}\|^{-1}, & i = j, \\ -\|M_{ij}\|, & i \neq j, \end{cases} \quad i, j = 1, 2, \cdots, s,$$

$$\langle\langle M \rangle\rangle_{ij} = \begin{cases} 1, & i = j, \\ -\|M_{ii}^{-1}M_{ij}\|, & i \neq j, \end{cases} \quad i, j = 1, 2, \cdots, s,$$

where $\|\cdot\|$ denotes the compatible matrix norm such that $\|I\| = 1$.

For block matrices $L \in L_n(n_1, n_2, \cdots, n_s)$ and $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$, we define $D(L) = diag(L_{11}, L_{22}, \cdots, L_{ss}), B(L) = D(L) - L$, $J(M) = D(M)^{-1}B(M)$, $\mu_1(M) = \rho(J_{\langle M \rangle}), \mu_2(M) = \rho(I - \langle\langle M \rangle\rangle)$. Then from definition 2, it is easy to verify that $\langle I - J(M) \rangle = \langle\langle I - J(M) \rangle\rangle = \langle\langle M \rangle\rangle, \mu_2(M) \leq \mu_1(M)$.

Definition 3 ([7]). Let $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$. A matrix M is said to be a (I) block $H-$ matrix ($H_B^{(I)}(P, Q)$-matrix) relative to nonsingular matrices P and Q if there exists two matrices $P, Q \in L^d_{n,I}(n_1, n_2, \cdots, n_s)$ such that $\langle PMQ \rangle$ is an $M-$ matrix; A matrix M is said to be a (II) block $H-$ matrix ($H_B^{(II)}(P, Q)$-matrix) relative to nonsingular matrices P and Q if $\langle\langle PMQ \rangle\rangle$ is an $M-$ matrix.

Remark. From definition 3, we know an $H_B^{(I)}(P, Q)-$ matrix must be an $H_B^{(II)}(P, Q)-$ matrix, but not conversely.

Definition 4 ([9]). Let $A = (a_{ij}) \in R^{n \times n}$. $A = B - C$ is said to be a normal splitting of a matrix A if $B^{-1} \geq 0, C \geq 0$; $A = B - C$ is said to be an $M-$splitting of a matrix A if B is an $M-$matrix and $C \geq 0$.

Definition 5. Let $M \in L_n(n_1, n_2, \cdots, n_s)$. $[M] = (\|M_{ij}\|) \in L(R^s)$ is said to be a block absolute value matrix of a block matrix M. Similarly, $[x] \in R^s$ is said to be a block value vector of a block vector $x \in V_n(n_1, n_2, \cdots, n_s)$.

Lemma 1 ([9]). *Let $A, B \in R^{n \times n}$, and $|A| \leq B$. Then $\rho(A) \leq \rho(B)$.*

Lemma 2 ([10]). *Let* $L, M \in L_n(n_1, n_2, \cdots, n_s)$, $x, y \in V_n(n_1, n_2, \cdots, n_s)$, $\gamma \in R$. *Then*

(1) $|[L]-[M]| \le [L+M] \le [L]+[M]$ $(|[x]-[y]| \le [x+y] \le [x]+[y])$.

(2) $[LM] \le [L][M]$ $([Mx] \le [M][x])$.

(3) $[\gamma M] \le |\gamma|[M]$ $([\gamma x] \le |\gamma|[x])$.

(4) $\rho(M) \le \rho(|M|) \le \rho([M])$ *(where* $\|\cdot\|$ *is* $\|\cdot\|_\infty$ *or* $\|\cdot\|_1$*).*

Lemma 3 ([10]). *Let* $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$ *be an* $H_B^{(I)}(P, Q)$*-matrix. Then*

(1) M *is nonsingular.*

(2) $[(PMQ)^{-1}] \le \langle PMQ \rangle^{-1}$.

(3) $\mu_1(PMQ) < 1$.

Lemma 4 ([10]). *Let* $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$ *be an* $H_B^{(II)}(P, Q)$*-matrix. Then*

(1) M *is nonsingular.*

(2) $[(PMQ)^{-1}] \le \langle\langle PMQ \rangle\rangle^{-1}[D(PMQ)^{-1}]$.

(3) $\mu_2(PMQ) < 1$.

Finally, we use

$$\Omega_B^{(I)}(M) = \{F = (F_{ij}) \in L_{n,I}(n_1, n_2, \cdots, n_s) | \|F_{ii}^{-1}\| = \|M_{ii}^{-1}\|, \|F_{ij}\| = \|M_{ij}\|, \quad i, j = 1, 2, \cdots, s\},$$

and

$$\Omega_B^{(II)}(M) = \{F = (F_{ij}) \in L_{n,I}(n_1, n_2, \cdots, n_s) | \|F_{ii}^{-1}F_{ij}\| = \|M_{ii}^{-1}M_{ij}\|, \quad i, j = 1, 2, \cdots, s\}$$

to denote respectively the set of (I) and (II) matrices such that the absolute values of whose elements are equal to absolute values of corresponding elements of the matrix M. In the next discussion, $\|\cdot\|$ denotes generally $\|\cdot\|_\infty$ or $\|\cdot\|_1$.

4. Convergence of the algorithms

Theorem 1. *Let* $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$ *be an* $H_B^{(I)}(P, Q)$*-matrix.* $A \in \Omega_B^{(I)}(PMQ)$, (D_A, L_k, U_k, E_k), $k = 1, 2, \cdots, \tau$, *is a multisplitting of the block matrix* A. *Suppose that for* $k = 1, 2, \cdots, \tau$, *we have*

$$\langle A \rangle = \langle D_A \rangle - [L_k] - [U_k] = D_{\langle A \rangle} - B_{\langle A \rangle}.$$

If γ, ω *satisfy* $0 \le \gamma \le \omega, 0 < \omega < max\{\omega_1, 2/(1+\mu_1(PMQ))\}$, *where* $\omega_1 = 2/min\|D_1^{-1}(\alpha([D_2]+[D_3])+[A])\|_\alpha$ *(*α *is* ∞ *or* 1*). Then the sequence* $x^{(p)} \subset V_n(n_1, n_2, \cdots, n_s)$ *generated by Algorithm I converges to the solution vector of system* (1) *for any starting vector* $x^{(0)} \in V_n(n_1, n_2, \cdots, n_s)$.

Proof. To verify that the sequence $x^{(p)} \subset V_n(n_1, n_2, \cdots, n_s)$ generated by Algorithm I converges to the solution vector of linear systems (1), it suffices to prove $\rho(L_{MBGAOR}(A)) < 1$, where $\rho(L_{MBGAOR}(A))$ denotes the spectral of matrix $L_{MBGAOR}(A)$.

Since $A \in \Omega_B^{(I)}(PMQ)$, then $\langle A\rangle = \langle PMQ\rangle$. Because $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$ is an $H_B^{(I)}(P,Q)$-matrix, so $A \in L_{n,I}(n_1, n_2, \cdots, n_s)$ is an $H_B^{(I)}(I,I)$-matrix and $\mu_1 = \rho(J_{\langle A\rangle}) = \rho(J_{\langle PMQ\rangle}) = \mu_1(PMQ)$. From Lemma 3, we have $\mu_1 = \rho(J_{\langle A\rangle}) = \mu_1(PMQ) < 1$.

Let us first consider

$$L^{(k)}_{MBGAOR}(A) = \big(D_1-\gamma(D_2+L_k)\big)^{-1}\big((1-\omega)D_1+(\omega-\gamma)(D_2+L_k)+\omega(D_3+U_k)\big).$$

It is clear that $D_1 - \gamma(D_2 + L_k)$ is an $H_B^{(I)}(I,I)$-matrix. From Definition 5 and Lemma 3, we have

$$[\big(D_1-\gamma(D_2+L_k)\big)^{-1}] \le \langle D_1-\gamma(D_2+L_k)\rangle^{-1} = ([D_1]-\gamma([D_2]+[L_k]))^{-1}, \quad (12)$$

and

$$[L^{(k)}_{MBGAOR}(A)] \le [\big(D_1-\gamma(D_2+L_k)\big)^{-1}][\big((1-\omega)D_1+(\omega-\gamma)(D_2+L_k)+\omega(D_3+U_k)\big)]$$

$$\le ([D_1]-\gamma([D_2]+[L_k]))^{-1}\{[1-\omega][D_1]+(\omega-\gamma)([D_2]+[L_k])+\omega([D_3]+[U_k])\}.$$

Let

$$M_k(\gamma) = [D_1]-\gamma([D_2]+[L_k]),$$
$$N_k(\omega,\gamma) = [1-\omega][D_1]+(\omega-\gamma)([D_2]+[L_k])+\omega([D_3]+[U_k]).$$

Thus, we have

$$[L_{MBGAOR}(A)] \le \sum_{k=1}^{\tau} E_k M_k(\gamma) N_k(\omega,\gamma).$$

From lemma 1, we have

$$\rho(L_{MBGAOR}(A)) \le \rho([L_{MBGAOR}(A)]) \le \rho(\sum_{k=1}^{\tau} E_k M_k(\gamma) N_k(\omega,\gamma)).$$

To prove that $\rho(L_{MBGAOR}(A)) < 1$, we consider

$$\begin{aligned} A_k(\omega,\gamma) &= M_k(\gamma)-N_k(\omega,\gamma) = [D_1]-\gamma([D_2]+[L_k]) - \\ &\quad \{[1-\omega][D_1]+(\omega-\gamma)([D_2]+[L_k])+\omega([D_3]+[U_k])\} \\ &= (1-[1-\omega])\big([D_A]-\frac{\omega}{1-[1-\omega]}[B_A]\big). \end{aligned}$$

Since $A \in \Omega_B^{(I)}(PMQ)$, we know that $\mu_1 = \rho(J_{\langle A\rangle}) = \mu_1(PMQ) < 1$ and

$$[D_1^{-1}]\big(2([D_2]+[D_3])+[A]\big) = I + [D_1^{-1}]\big(([D_2]+[D_3])+[L_k]+[U_k]\big).$$

It is easy to show that

$$\min_{\alpha=\infty,1} \|[D_1^{-1}]\big(([D_2]+[D_3])+[L_k]+[U_k]\big)\|_\alpha < 1.$$

Then we have

$$\min_{\alpha=\infty,1} \|[D_1^{-1}]\big(2([D_2]+[D_3])+[A]\big)\|_\alpha > 2.$$

Since $0 < \omega < max\{\omega_1, 2/(1+\mu_1(PMQ))\}$, we have $0 < \omega < 2$ and $1-[1-\omega] > 0$. Thus, we have

$$\left([D_A] - \frac{\omega}{1-[1-\omega]}[B_A]\right)^{-1} \geq 0,$$

then $A_k(\omega,\gamma)^{-1} \geq 0$. It is easy to know that $M_k(\gamma)$ is an M-matrix, $N_k(\omega,\gamma) \geq 0$. Hence $M_k(\gamma) - N_k(\omega,\gamma)$ is an M-splitting of $A_k(\omega,\gamma)$. According to the lemma 2 of M.Neuman [3], we know that the block matrix

$$\overline{A}_k(\omega,\gamma) = \overline{M}_k(\gamma) - \overline{N}_k(\omega,\gamma)$$

where

$$\overline{M}_k(\gamma) = \begin{bmatrix} M_1(\gamma) & 0 & \cdots & 0 \\ 0 & M_2(\gamma) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_\tau(\gamma) \end{bmatrix},$$

$$\overline{N}_k(\omega,\gamma) = \begin{bmatrix} N_1(\omega,\gamma) & 0 & \cdots & 0 \\ 0 & N_2(\omega,\gamma) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & N_\tau(\omega,\gamma) \end{bmatrix} \begin{bmatrix} E_1 & E_2 & \cdots & E_\tau \\ E_1 & E_2 & \cdots & E_\tau \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_\tau \end{bmatrix}.$$

is also an M-matrix, $\overline{M}_k(\gamma) - \overline{N}_k(\omega,\gamma)$ is an M-splitting of $\overline{A}_k(\omega,\gamma)$. From lemma 1, we have

$$\rho\left(\overline{M}_k(\gamma)^{-1}\overline{N}_k(\omega,\gamma)\right) < 1.$$

On the other hand, we can use the method of [8] to obtain

$$\begin{aligned}
&\rho\left(\sum_{k=1}^{\tau} E_k M_k(\gamma)^{-1} N_k(\omega,\gamma)\right) \\
&= \rho\left(\begin{bmatrix} E_1 & E_2 & \cdots & E_\tau \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} M_1(\gamma)^{-1}N_1(\omega,\gamma) & 0 & \cdots 0 \\ M_2(\gamma)^{-1}N_2(\omega,\gamma) & 0 & \cdots 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_\tau(\gamma)^{-1}N_\tau(\omega,\gamma) & 0 & \cdots 0 \end{bmatrix}\right) \\
&= \rho\left(\begin{bmatrix} M_1(\gamma)^{-1}N_1(\omega,\gamma) & 0 & \cdots 0 \\ M_2(\gamma)^{-1}N_2(\omega,\gamma) & 0 & \cdots 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_\tau(\gamma)^{-1}N_\tau(\omega,\gamma) & 0 & \cdots 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 & \cdots & E_\tau \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \rho\left(\begin{bmatrix} M_1(\gamma)^{-1}N_1(\omega,\gamma)E_1 & M_1(\gamma)^{-1}N_1(\omega,\gamma)E_2 & \cdots & M_1(\gamma)^{-1}N_1(\omega,\gamma)E_\tau \\ M_2(\gamma)^{-1}N_2(\omega,\gamma)E_1 & M_2(\gamma)^{-1}N_2(\omega,\gamma)E_2 & \cdots & M_2(\gamma)^{-1}N_2(\omega,\gamma)E_\tau \\ \vdots & \vdots & \ddots & \vdots \\ M_\tau(\gamma)^{-1}N_\tau(\omega,\gamma)E_1 & M_\tau(\gamma)^{-1}N_\tau(\omega,\gamma)E_2 & \cdots & M_\tau(\gamma)^{-1}N_\tau(\omega,\gamma)E_\tau \end{bmatrix}\right) \\
&= \rho\left(\overline{M}_k(\gamma)^{-1}\overline{N}_k(\omega,\gamma)\right).
\end{aligned}$$

Hence, we have

$$\rho(L_{MBGAOR}(A)) \leq \rho(\sum_{k=1}^{\tau} E_k M_k(\gamma) N_k(\omega,\gamma)) = \rho\left(\overline{M}_k(\gamma)^{-1}\overline{N}_k(\omega,\gamma)\right) < 1. \quad \square$$

Theorem 2. *Let $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$ be an $H_B^{(II)}(P,Q)$-matrix. $A \in \Omega_B^{(II)}(PMQ)$, (D_A, L_k, U_k, E_k), $k = 1, 2, \cdots, \tau$, is a multisplitting of the block matrix A. Suppose that for $k = 1, 2, \cdots, \tau$, we have*

$$\langle A\rangle = \langle D_A\rangle - [L_k] - [U_k] = D_{\langle A\rangle} - B_{\langle A\rangle}.$$

If γ, ω satisfy $0 \leq \gamma \leq \omega, 0 < \omega < max\{\omega_1, 2/(1+\mu_2(PMQ))\}$, where $\omega_1 = 2/min\|D_1^{-1}(\alpha([D_2]+[D_3]) + [A])\|_\alpha$ (α is ∞ or 1). Then the sequence $x^{(p)} \subset V_n(n_1, n_2, \cdots, n_s)$ generated by Algorithm I converges to the solution vector of linear systems (1) for any starting vector $x^{(0)} \in V_n(n_1, n_2, \cdots, n_s)$.

Proof. Since $A \in \Omega_B^{(II)}(PMQ)$, then $\langle\langle A\rangle\rangle = \langle\langle PMQ\rangle\rangle$. Because $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$ is an $H_B^{(II)}(P,Q)$-matrix, so $A \in L_{n,I}(n_1, n_2, \cdots, n_s)$ is an $H_B^{(II)}(I,I)$-matrix and $\mu_2 = \rho(J_{\langle\langle A\rangle\rangle}) = \rho(J_{\langle\langle PMQ\rangle\rangle}) = \mu_2(PMQ)$. From Lemma 4, we have $\mu_2 = \rho(J_{\langle\langle A\rangle\rangle}) = \mu_2(PMQ) < 1$. The proof of the residual part is same as that given in Theorem 1, which completes the proof. $\square$

Theorem 3. *Let $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$ be an $H_B^{(I)}(P,Q)$-matrix. $A \in \Omega_B^{(I)}(PMQ)$, (D_A, L_k, U_k, E_k), $k = 1, 2, \cdots, \tau$, is a multisplitting of the block matrix A. Suppose that for $k = 1, 2, \cdots, \tau$, we have*

$$\langle A\rangle = \langle D_A\rangle - [L_k] - [U_k] = D_{\langle A\rangle} - B_{\langle A\rangle}.$$

If parameters γ, β, ω satisfy

$$0 \leq \gamma \leq \omega, 0 < \omega < max\{\omega_1, 2/(1+\mu_1(PMQ))\},$$
$$0 < \beta < 2/(1+\eta^2(PMQ)), \eta^2(PMQ) = \max\{|1-\omega| + \omega\mu_1(PMQ)\}$$

where $\omega_1 = 2/min\|D_1^{-1}(\alpha([D_2]+[D_3])+[A])\|_\alpha$ (α is ∞ or 1). Then the sequence $x^{(p)} \subset V_n(n_1, n_2, \cdots, n_s)$ generated by Algorithm II converges to the solution vector of linear systems (1) for any starting vector $x^{(0)} \in V_n(n_1, n_2, \cdots, n_s)$.

Proof. In order to prove theorem 3, from the proof of theorem 2, we have

$$\begin{aligned}
[L'_{MBGAOR}(A)] &\leq \beta\sum_{k=1}^{\tau} E_k[(D_1 - \gamma(D_2 + L_k))^{-1}][((1-\omega)D_1 \\
&\quad + (\omega-\gamma)(D_2+L_k) + \omega(D_3+U_k))] + [1-\beta]I \\
&\leq \beta\sum_{k=1}^{\tau} E_k([D_1] - \gamma([D_2]+[L_k]))^{-1}\{[1-\omega][D_1] \\
&\quad + (\omega-\gamma)([D_2]+[L_k]) + \omega([D_3]+[U_k])\} + [1-\beta]I.
\end{aligned}$$

Let

$$M_k(\gamma) = [D_1] - \gamma([D_2]+[L_k]),$$

$$N_k(\omega,\gamma)=[1-\omega][D_1]+(\omega-\gamma)([D_2]+[L_k])+\omega([D_3]+[U_k]).$$

Thus, we have

$$[L'_{MBGAOR}(A)]\le \beta\sum_{k=1}^{\tau}E_kM_k^{-1}(\gamma)N_k(\omega,\gamma)+[1-\beta]I$$

$$\le \beta\sum_{k=1}^{\tau}E_kM_k^{-1}(\gamma)(M_k(\gamma)-A_k(\omega,\gamma))+[1-\beta]I$$

$$=\beta I+[1-\beta]I-\beta(1-[1-\beta])\sum_{k=1}^{\tau}E_kM_k^{-1}(\gamma)[D_A](I-\frac{\omega}{1-[1-\omega]}[D_{\langle A\rangle}]^{-1}[B_{\langle A\rangle}]).$$

Since M is an $H_B^{(I)}(P,Q)$-matrix, $A\in\Omega_B^{(I)}(PMQ)$ and $1\le\omega<max\{\omega_1,2/(1+\mu_1(PMQ))$, we have $\mu_1=\rho(J_{\langle A\rangle})=\rho(J_{\langle PMQ\rangle})=\mu_1(PMQ)<1$, and there exists $\varepsilon>0$, if let

$$J_\varepsilon=J_{\langle A\rangle}+\varepsilon ee^T,\quad e=(1,1,\cdots,1)\in R^s. \tag{13}$$

Then

$$\rho_\varepsilon=\rho(J_\varepsilon)<1,\ \ [1-\omega]+\omega\rho_\varepsilon<1. \tag{14}$$

By properties of the spectral radius of nonnegative matrices, if ε is monotone decreasing, then the strict inequality of (15) is invariant. Since $0<\beta<2/(1+\eta^2(PMQ))$, we know that if ε is sufficient small, then not only (15) holds, but we have

$$|1-\beta|+\beta\rho_A<1,\rho_A=[1-\omega]+\omega\rho_\varepsilon<1. \tag{15}$$

Obviously, J_ε is a positive matrix, by Perron-Frobenius theorem, there exists a positive vector $x_\varepsilon\in R^s$ such that $J_\varepsilon x_\varepsilon=\rho_\varepsilon x_\varepsilon$. Thus, we have

$$[L'_{MBGAOR}(A)]\ \le\ \beta I+[1-\beta]I-\beta\left(1-[1-\omega]\right)\sum_{k=1}^{\tau}E_kM_k^{-1}(\gamma)[D_A]$$
$$\left[I-\frac{\omega}{1-[1-\omega]}\left([D_{\langle A\rangle}]^{-1}[B_{\langle A\rangle}]+\varepsilon ee^T\right)\right].$$

Hence, we have

$$[L'_{MBGAOR}(A)]x_\varepsilon\ \le\ (\beta I+[1-\beta]I)x_\varepsilon-\beta\left(1-[1-\omega]\right)\sum_{k=1}^{\tau}E_kM_k^{-1}(\gamma)[D_A]$$
$$\left[I-\frac{\omega}{1-[1-\omega]}\left(M_k^{-1}(\gamma)^{-1}[B_{\langle A\rangle}]+\varepsilon ee^T\right)\right]x_\varepsilon.$$

Since $M_k^{-1}(\gamma)\ge[D_A]^{-1}$, we have

$$[L'_{MBGAOR}(A)]x_\varepsilon\le\left([1-\beta]+\beta\left([1-\omega]+\omega\rho_\varepsilon\right)\right)x_\varepsilon.$$

Observe that $0\le\gamma\le\omega,0<\omega<max\{\omega_1,2/(1+\mu_1(PMQ))\},0<\beta<2/(1+\eta^2(PMQ))$, we have

$$[L'_{MBGAOR}(A)]x_\varepsilon<x_\varepsilon.$$

Hence, we have

$$\rho(L'_{MBGAOR}(A)) \le \rho([L'_{MBGAOR}(A)]) < 1.$$

This is complete the proof. □

Theorem 4. *Let $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$ be an $H_B^{(II)}(P,Q)$-matrix. $A \in \Omega_B^{(II)}(PMQ)$, (D_A, L_k, U_k, E_k), $k = 1, 2, \cdots, \tau$, is a multisplitting of the block matrix A. Suppose that for $k = 1, 2, \cdots, \tau$, we have*

$$\langle A \rangle = \langle D_A \rangle - [L_k] - [U_k] = D_{\langle A \rangle} - B_{\langle A \rangle}.$$

If parameters γ, β, ω satisfy

$$0 \le \gamma \le \omega, 0 < \omega < max\{\omega_1, 2/(1 + \mu_2(PMQ))\},$$
$$0 < \beta < 2/(1 + \eta^2(PMQ)), \eta^2(PMQ) = \max\{|1 - \omega| + \omega\mu_2(PMQ)\},$$

where $\omega_1 = 2/min\|D_1^{-1}(\alpha([D_2]+[D_3])+[A])\|_\alpha$ (α is ∞ or 1). Then the sequence $x^{(p)} \subset V_n(n_1, n_2, \cdots, n_s)$ generated by Algorithm II converges to the solution vector of linear systems (1) for any starting vector $x^{(0)} \in V_n(n_1, n_2, \cdots, n_s)$.

Proof. Since $A \in \Omega_B^{(II)}(PMQ)$, then $\langle\langle A \rangle\rangle = \langle\langle PMQ \rangle\rangle$. Because $M \in L_{n,I}(n_1, n_2, \cdots, n_s)$ is an $H_B^{(II)}(P,Q)$-matrix, so $A \in L_{n,I}(n_1, n_2, \cdots, n_s)$ is an $H_B^{(II)}(I,I)$-matrix and $\mu_2 = \rho(J_{\langle\langle A \rangle\rangle}) = \rho(J_{\langle\langle PMQ \rangle\rangle}) = \mu_2(PMQ)$. From Lemma 4, we have $\mu_2 = \rho(J_{\langle\langle A \rangle\rangle}) = \mu_2(PMQ) < 1$. The proof of the residual part is same as that given in Theorem 3, which completes the proof. □

Remark. If we let $D_2 = D_3 = 0$, then Theorem in this paper is the special condition of [9] for the case when A is a block H-matrix.

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