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# SPANNING 3-FORESTS IN BRIDGES OF A TIGHT SEMIRING IN AN LV-GRAPH

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ABSTRACT. An infinite locally finite plane graph is an LV-graph if it is 3-connected and VAP-free. In this paper, as a preparatory work for solving the problem concerning the existence of a spanning 3-tree in an LV-graph, we investigate the existence of a spanning 3-forest in a bridge of type 0,1 or 2 of a tight semiring in an LV-graph satisfying certain conditions.

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## 1. Introduction

All graphs considered in this paper are undirected and have neither multiple edges nor loops. We use standard terminology and notation as used in [2]. The reader may overview infinite graph theory concerning the problems about the existence of spanning subgraphs in [3] or [9].

Let $G$ be a graph. If $S \subseteq V(G)$, $G[S]$ is the subgraph induced by $S$ in $G$. The *degree* and *neighborhood* of a vertex $v$ of $G$ are respectively denoted by $d_G(v)$ and $N_G(v)$. A *spanning subgraph* $H$ of $G$ is a subgraph of $G$ with $V(H) = V(G)$. For a positive integer $k$, a $k$-subgraph $H$ of $G$ is a subgraph of $G$ with $d_H(v) \leq k$ for all $v \in V(H)$. A *spanning tree*, *spanning $k$-tree*, *spanning forest* and *spanning $k$-forest* are similarly defined.

Let $H$ be a subgraph of $G$. We define a relation $\sim$ on $E(G) \setminus E(H)$ by the condition that $e_1 \sim e_2$ if there exists a path $P$ such that

(i) the first and last edges of $P$ are $e_1$ and $e_2$, respectively, and
(ii) $P$ and $H$ are edge-disjoint.

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A subgraph of $G - E(H)$ induced by an equivalence class under the relation $\sim$ is called a *bridge* of $H$ in $G$. If $B$ is a bridge of $H$ in $G$, then the elements of $V(H) \cap V(B)$ are called the *vertices of attachment* of $B$. In particular, if $H$ and $H'$ are disjoint subgraphs of $G$ and $B$ is a bridge of $H \cup H'$, we say that $B$ *connects* $H$ *with* $H'$ if both $V(B) \cap V(H) \neq \emptyset$ and $V(B) \cap V(H') \neq \emptyset$ hold.

Let $G$ be a finite connected plane graph. A *block* of $G$ is either a cut-edge which will be called *trivial*, or a maximally 2-connected subgraph of $G$ which is called *nontrivial*. For a block $Q$, we may denote $\partial Q$ the subgraph of $G$ constituted by the vertices and edges incident to the unbounded face of $G$. If a block $Q$ contains at most one cut-vertex, we say that $Q$ is an *endblock* of $G$. Thus, in particular, if $G$ is a nontrivial block (i.e., $G$ is 2-connected), $\partial G$ is the outer cycle of $G$. On the other hand, if $G$ is a trivial block, then $\partial G = G$. A plane graph $G$ is a *circuit graph* (following D. Barnette [1]), if there exists a cycle $C$ in a 3-connected plane graph such that $G$ is isomorphic to the subgraph consisted of the vertices and edges of $C$ and in the interior of $C$; or equivalently for every vertex cut $S$ of $G$ with $|S| = 2$, every component of $G - S$ contains a vertex of $\partial G$. (See [8] for a more thorough exposition of the theory of circuit graphs.)

Let $G$ be an infinite connected planar graph. A separating path in $G$ is said to be *unbounded* if each of the two endvertices of the path is incident to an unbounded face. A finite set of unbounded separating paths $\mathcal{P} = \{P_1, \dots, P_n\}$ in $G$ will be called a *semicycle* if there exist connected subgraphs $G_0, G_1, \dots, G_n$ of $G$ such that

[S1] $G = \bigcup_{i=0}^{n} G_i, \quad G_0 \cap G_i = P_i$ for all $i \in \{1, \dots, n\}$
and $G_i \cap G_j = \emptyset$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$, and

[S2] $G_0$ is finite, but $G_i$ $(i = 1, \dots, n)$ are infinite.

In this case, the finite subgraph $G_0$ of $G$ is called the *center* of the semicycle $\mathcal{P}$, which will be denoted by $C(\mathcal{P})$. A semicycle $\mathcal{P}$ is *induced* if all paths in $\mathcal{P}$ are induced. Two semicycles $\mathcal{P}$ and $\mathcal{P}'$ are *disjoint* if $V(\mathcal{P}) \cap V(\mathcal{P}') = \emptyset$; for convenience, the set of vertices $V(\mathcal{P})$ (respectively, the set of edges $E(\mathcal{P})$) of $\mathcal{P}$ will be understood to be the union of all vertices (respectively, edges) of the paths in $\mathcal{P}$.

Let $\mathcal{P}$ and $\mathcal{P}'$ be disjoint semicycles with $\mathcal{P} \subseteq C(\mathcal{P}')$ in a connected planar graph $G$. A $(\mathcal{P}, \mathcal{P}')$*-semiring* in $G$ is a subgraph of $G$ consisting of not only the cycles in $\mathcal{P}$ and $\mathcal{P}'$ but also all vertices and edges lying between $\mathcal{P}$ and $\mathcal{P}'$. *Bridges* of a $(\mathcal{P}, \mathcal{P}')$-semiring $\mathcal{R}$ are defined by the bridges connecting $P$ with $P'$ in $\mathcal{R}$. A bridge $B$ is of *type* $k$ $(k = 0, 1, 2, \dots)$ if the number of vertices of $B$ on $\mathcal{P}'$ is $k$. A $(\mathcal{P}, \mathcal{P}')$-semiring $\mathcal{R}$ is said to be *tight* if it satisfies following conditions:

[T1] $\mathcal{P}$ and $\mathcal{P}'$ are induced.

[T2] For each infinite component $H$ of $G - C(\mathcal{P})$, there exists exactly one path $P$ in $\mathcal{P}'$ such that the endvertices of $P$ are adjacent to the endvertices of the foot of $H$.

[T3] $|V(B) \cap V(\mathcal{P}')| \leq 2$ for all bridges $B$ of $\mathcal{R}$.
[T4] If $B$ is a bridge of $\mathcal{R}$ with $V(B) \cap V(\mathcal{P}') = \{z, z'\}$, $z \neq z'$, then $zz' \in E(\mathcal{P}')$.

An infinite locally finite plane graph is an *LV-graph* if it is 3-connected and VAP-free (=vertex-accumulation-point free). It is not hard to verify that an LV-graph can contain continuum many ends, and therefore it can have continuum many unbounded faces. In such a point of view, we define a *3LV-graph* to be an LV-graph containing no unbounded faces, as introduced in [7].

In finite graph theory, many research papers have been published on the problem of the existence of spanning subgraphs in a graph with lower degrees (see [5], [6] or [12]) since the appearance of the theorem of Barnette [1], who showed that every circuit graph contains a spanning 3-tree; but in case of infinite graph theory the history is not so long. As to the existence of end-faithful spanning trees or end-faithful spanning forests, since the first thesis was written in 1988 by Širáň [11], there have been several theses published, for example, by Diestel [3] or Polat [10], etc. On the other hand, the first paper on the research concerning the existence of spanning subgraphs or spanning $k$-tree in a graph, particularly in a planar graph, was written by Jung [7] in connection with the $[2,3]$-factors. In fact he showed that the theorem of Enomoto et al. [4] can be extended to 3LV-graphs; i.e., he showed the existence of $[2,3]$-factors in such a graph under certain conditions. However, it seems to be quite difficult to extend this theorem to general LV-graphs which contains continuum many ends, and this is obviously one of the critical problems in infinite graph theory that we should solve in the future. As shown in [9], a key to the solution of this problem is the analysis of the three types of the bridges (see [T3] above) of a tight semiring in such a graph.

In this paper, as a preparatory work for solving this problem, we investigate the existence of a spanning 3-forest in a bridge of type 0,1 or 2 of a tight semiring in an LV-graph which satisfies certain conditions that we need.

## 2. Preliminaries

Let $\mathcal{R}$ be a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph $G$ and let $B$ be a nontrivial bridge of $\mathcal{R}$. Further set $\{x_1, \cdots, x_r = \overline{x}\}$ be the set of the vertices of attachment on $\mathcal{P}$ in the clockwise order. Then we may say that $x_1$ (and r $\overline{x}$) is *the first* (and *the last*, respectively) vertex of attachment of $B$ on $\mathcal{P}$. We set further $H := B - (\mathcal{P} \cup \mathcal{P}')$. If $r = 1$, then $B$ must be of type 2, and in this case $H$ contains at most 2 endblocks.

We now assume that $r \geq 2$. Since $H$ is still connected, there is a path in $B$ connecting $x_1$ and $\overline{x}$, such that the intersection of the path and $\mathcal{P}$ is $\{x_1, \overline{x}\}$. We choose such a path $P_B$ with

$$V(P_B) = \{x_1, z_1, \cdots, z_s, \overline{x}\},$$

where $z_1$ and $z_s$ denote the first and the last vertices adjacent to $x_1$ and $\overline{x}$ in the natural order, respectively. Now we may denote $\Delta_B$ the set of all blocks of $H$ containing at least one edge of $P_B$. Then we can easily see that the set $\Delta_B$ is unique for a given bridge $B$.

On the other hand, if the $x_1, \overline{x}$-path on $\mathcal{P}$ is denoted by $P'_B$, we have a cycle $J := P_B \cup P'_B$. The blocks of $H$ lying entirely in the interior or exterior of $J$ are said to be the *inner blocks* or *outer blocks* of $H$, respectively. The *inner endblocks* and *outer endblocks* of $H$ are analogously defined. By the 3-connectedness of $G$, if $B$ is of type 0 or 1, there exist no outer blocks, while there can be at most one such block if $B$ is of type 2.

**Proposition 2.1.** *Let $B$ be a bridge of a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph, and let further $Q_1, \cdots, Q_t$ be the inner endblocks of $\widetilde{B} = B - (\mathcal{P} \cup \mathcal{P}')$ with the articulations $u_1, \cdots, u_t$. Then there exist pairwise distinct vertices $x_1, \cdots, x_t \in V(B) \cap (V(\mathcal{P}) \setminus \{\overline{x}\})$ such that $x_i$ is adjacent to a vertex of $Q_i - u_i$ $(i = 1, \dots, t)$.*

*Proof.* Let $V(B) \cap V(\mathcal{P}) = \{x'_1, \dots, x'_r = \overline{x}\}$. We easily see that

(a) every inner endblock is adjacent to at least 2 vertices of $\mathcal{P}$; and

(b) if $x'_{i_1}, \cdots, x'_{i_{k_i}}$, $(k_i \geq 2)$ denote the vertices of $\mathcal{P}$ adjacent to $Q_i$ $(i = 1, \cdots, t)$ in the natural order, then

$$\underbrace{1 = 1_1 < \cdots < 1_{k_1}}_{\text{for } Q_1} \leq \underbrace{2_1 < \cdots < 2_{k_2}}_{\text{for } Q_2} \leq \cdots \leq \underbrace{t_1 < \cdots < t_{k_t}}_{\text{for } Q_t}.$$

If we set

$$x_1 := x'_{1_1}, \quad x_2 := x'_{2_1}, \quad \cdots, \quad x_t := x'_{t_1}$$

then $x_1, \dots, x_t$ are pairwise disjoint, and further, since $k_i \geq 2$ for all $i = 1, \dots, t$, we obtain $\overline{x} \notin \{x_1, \dots, x_t\}$. □

We may say that the set of vertices $\{x_1, \dots, x_t\}$ constructed in Proposition 2.1 a system of *representatives* for the endblocks $Q_1, \dots, Q_t$.

Now we consider the circuit graphs. As is defined in the preceding section, a 2-connected plane graph $G$ is a circuit graph, if, for each vertex cut $S$ with $|S| = 2$, every component of $G - S$ contains a vertex of $\partial G$. We say that a graph $G$ is a *block-circuit* graph if each block of $G$ is a circuit graph. We begin with the theorem of Barnette [1]; even if our results are a little stronger than those of his, we can prove them by almost similar method; i.e., by induction on the number of edges lying in the interior of the given circuit graph as he did. We will therefore omit to prove.

**Lemma 2.2.** *Let $G$ be a circuit graph.*

(1) *If $u, v \in V(\partial G)$, then there exists a spanning 3-tree $T$ in $G$ with $d_T(u) = 1$ and $d_T(v) \leq 2$.*

(2) *If $u, v, w \in V(\partial G)$, then there exists a spanning 3-tree $T$ in $G$ with $d_T(u) \leq 2$, $d_T(v) \leq 2$ and $d_T(w) \leq 2$.*

**Corollary 2.3.** *Let $G$ be a connected block-circuit graph containing 2 endblocks $Q$ and $Q'$ whose articulations are $u$ and $u'$, respectively. Further let $v \in V(\partial Q) \setminus \{u\}$ and $v' \in V(\partial Q') \setminus \{u'\}$ be arbitrary given. Then there exists a spanning 3-tree $T$ in $G - v'$ with $d_T(v) \leq 2$.*

**Corollary 2.4.** *Let $G$ be a connected block-circuit graph containing 3 endblocks $Q$, $Q'$ and $Q''$ whose articulations are $u$, $u'$ and $u''$, respectively. Further let $v \in V(\partial Q) \setminus \{u\}$, $v' \in V(\partial Q') \setminus \{u'\}$ and $v'' \in V(\partial Q'') \setminus \{u''\}$ be arbitrary given. Then there exists a spanning 3-tree $T$ in $G$ with $d_T(v) \leq 2$, $d_T(v') \leq 2$ and $d_T(v'') \leq 2$.*

Now we can give the final result in this section, which is a generalization of Corollary 2.4 and plays crucial role in the next section.

**Proposition 2.5.** *Let $G$ be a connected block-circuit graph containing $t$ endblocks $Q_1, \ldots, Q_t$ whose articulations are $u_1, \ldots, u_t$, respectively. Further let $v_i \in V(\partial Q_i) \setminus \{u_i\}$, $(i = 1, \ldots, t)$. Then, for arbitrary given 3 vertices $v_l, v_m, v_n \in \{v_1, \ldots, v_t\}$, there exists a spanning 3-forest $F$ in $G$ with $d_F(v_i) \leq 2$ for all $i \in \{1, \ldots, t\}$ containing exactly $t - 2$ components, such that $v_l$, $v_m$ and $v_n$ lie on a component of $F$ and each of the remaining components contains exactly one vertex of $\{v_1, \ldots, v_t\} \setminus \{v_l, v_m, v_n\}$.*

*Proof.* We assume without loss of generality that $\{l, m, n\} = \{t-2, t-1, t\}$, and we construct iteratively $t - 2$ connected induced subgraphs $G_0, G_1, \ldots, G_{t-3}$ of $G$ with

$$G = \bigcup_{j=0}^{t-3} G_j \quad \text{and} \quad \left[ \bigcup_{\substack{i=0 \\ i \neq j}}^{t-3} G_i \right] \cup G_j = \{x_j\} \quad (j = 1, \ldots, t-3) \qquad (*)$$

such that $G_j$ $(j = 1, \ldots, t-3)$ has a linear decomposition.

Obviously there exists exactly one minimally connected induced subgraph (denoted by $G_0$) with $v_l, v_m, v_n \in V(G_0)$ and $v_j \notin G_0$ for all $j \in \{1, \ldots, t\} \setminus \{l, m, n\}$, such that if $E(Q) \cap E(G_0) \neq \emptyset$ for a block $Q$ of $G$, then it must be hold $Q \subseteq G_0$.

Now we assume that for $j \in \{1, \ldots, t-3\}$, $G_0, \ldots, G_{j-1}$ are already constructed. From the decomposition property and the connectedness number of $G$ there exists exactly one connected induced subgraph $G_j$ satisfying the properties (*) above. Then we easily see that $G_0, G_1, \ldots, G_{t-3}$ satisfy the desired properties.

By Corollary 2.4 there exists a spanning 3-tree $T_0$ in $G_0$ with $d_{T_0}(v_l) \leq 2$, $d_{T_0}(v_m) \leq 2$ and $d_{T_0}(v_n) \leq 2$. For $j = 1, \ldots, t-3$ we use Corollary 2.3 to obtain a spanning 3-tree $T_j$ in $G_j - x_j$ with $d_{T_j}(v_j) \leq 2$. By setting $F = \bigcup_{j=1}^{t-3} T_j$ we get a spanning 3-forest in $G$ which satisfies the properties as we wanted. □

## 3. Spanning 3-forests in a bridge

In this section, a spanning 3-forest in a bridge of a tight $(\mathcal{P}, \mathcal{P}')$-semiring of type $k$ $(k = 0, 1, 2)$ will be constructed, such that neither of its components contains a vertex of $\mathcal{P}$ and that of $\mathcal{P}'$ simultaneously. In order to simplify to describe the following successive theorems or formulations, it will be assumed that every bridge contains at least one inner block. However we can similarly verify that the assertions are also true for the bridges containing no inner blocks.

**Theorem 3.1.** *Let $B$ be a bridge of type 0 of a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph, and let $x_0$ and $\overline{x}$ be the first and the last vertex of attachment of $B$ on $\mathcal{P}$, respectively. Then there exists a spanning 3-forest $F$ in $B$ such that:*

(1) *Each component of $F$ contains exactly one vertex of attachment of $B$ on $\mathcal{P}$.*
(2) *$d_F(x) \leq 1$ for each vertex $x$ of attachment of $B$ on $\mathcal{P}$.*
(3) *$d_F(x_0) = d_F(\overline{x}) = 0$.*

*Proof.* Let us denote $\Delta_B$ the set of blocks of $B - \mathcal{P}$ defined at the beginning of this section. From Proposition 2.1 each block of $B - \mathcal{P}$ is either an inner block or is contained in $\Delta_B$. Now let $Q_1, \dots, Q_t$ be the inner endblocks of $B - \mathcal{P}$ with representatives $x_1, \dots, x_t$. Let further $v_i$ $(i = 1, \dots, t)$ be a vertex of $Q_i - u_i$ adjacent to $x_i$, where $u_i$ is the articulation of $Q_i$. Finally we set $\widetilde{H} = H \cup \{z\overline{z}\}$, where $z$ is the last vertex of $H$ adjacent to $\overline{x}$ and $z\overline{z}$ is a new edge, and we denote $Q_0$ the first block in $\Delta_B$. Now we first consider the case that $Q_0$ is an endblock of $H$.

In this case we see that $Q_i$ $(i = 0, 1, \dots, t)$, $z\overline{z}$ are the endblocks of $\widetilde{H}$. Since $V(B) \cap V(\mathcal{P}') = \emptyset$, $Q_0$ contains at least 2 vertices adjacent to $\mathcal{P}$, and we have $x_0 \neq x_1$. By Proposition 2.5 there exists a spanning 3-forest $F'$ in $\widetilde{H}$ with $d_{F'}(v_i) \leq 2$ for all $i = 1, \dots, t$ which contains exactly $t$ components, such that $v_t$, $v$ and $\overline{z}$ lie on a common component of $F'$ and each of the remaining components contains exactly one $v_i$ $(i = 1, \dots, t-1)$. Then, by setting

$$F = [F' \cup \{x_i v_i \,|\, i = 1, \dots, t\} \setminus \{z\overline{z}\}] \cup [V(\mathcal{P}) \cap V(B)]$$

we obtain a spanning 3-forest in $B$ satisfying desired conditions.

Now consider the case that $Q_0$ is not an endblock of $H$. Note in this case that $Q_i$ $(i = 1, \dots, t)$, $z\overline{z}$ are the endblocks of $\widetilde{H}$. Since $|V(B) \cap V(\mathcal{P})| \geq 3$, there exists a vertex $\hat{x}$ of attachment of $B$ with $\hat{x} \neq x_0$ and $\hat{x} \neq \overline{x}$. First consider the case that $H$ contains only one endblock; i.e., $H$ is 2-connected. By choosing a vertex $\hat{v}$ of $H$ adjacent to $\hat{x}$, we obtain a spanning 3-tree $T$ in $\widetilde{H} \cup \{\hat{x}\hat{v}\}$, since it contains at most 3 endblocks. Then, clearly $F = [T - \overline{z}] \cup [V(B) \cap V(\mathcal{P})]$ satisfies the assertion (1)–(3).

Now assume that $H$ contains at least 2 endblocks. Then, since $\widetilde{H}$ has at least 3 endblocks, it follows that there exists a spanning 3-forest $F'$ in $\widetilde{H}$ with $d_{F'}(v_i) \leq 2$ for all $i = 1, \dots, t$ which contains exactly $t - 1$ components, such that one of the components of $F'$ contains the vertices $v_1, v_t$ and $\overline{z}$ and each of the remaining $t-2$ components contains one of the vertices $\{v_2, \dots, v_{t-1}\}$ Then

$$F = [F' \cup \{x_i v_i \,|\, i = 2, \dots, t\} \setminus \{z\overline{z}\}] \cup [V(\mathcal{P}) \cap V(B)]$$

obviously is a spanning 3-forest in $B$ satisfying the conditions in the theorem as we wanted. □

Next, we consider the existence of a spanning 3-forest in a bridge of type 1.

**Theorem 3.2.** *Let $B$ be a bridge of type 1 of a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph, and let $x_0$ be the first vertex of attachment of $B$ on $\mathcal{P}$. Let further $V(B) \cap V(\mathcal{P}') = \{y\}$. Then there exists a spanning 3-forest $F$ in $B$ such that:*

(1) *Each component of $F$ contains exactly one vertex of attachment of $B$ on $\mathcal{P} \cup \mathcal{P}'$.*
(2) *$d_F(x_0) = 0$ and $d_F(x) \leq 1$ for each vertex $x$ of attachment of $B$ on $\mathcal{P}$.*
(3) *$d_F(y) = 0$.*

*Proof.* Let $H$, $\{Q_1, \dots, Q_t\}$, $\{x_1, \dots, x_t\}$ and $\{v_1, \dots, v_t\}$ be vertices or subgraphs defined in the proof of Theorem 3.1. Let us further denote $\Delta_B$ the set of blocks defined at the beginning of this section, whose first and last endblocks are denoted by $Q$ and $\overline{Q}$ (in the clockwise) with the articulations $u$ and $\overline{u}$, respectively. Then there are 3 cases to consider:

(a) $Q$ is an endblock of $H$, and $y$ is adjacent to $\partial Q - u$.
(b) $Q$ is an endblock of $H$, but $y$ is not adjacent to $\partial Q - u$.
(c) $Q$ is not an endblock of $H$.

Assume first that $Q$ is an endblock of $H$, and $y$ is adjacent to $\partial Q - u$. Let us denote $v \in V(\partial Q) \setminus \{u\}$ which is adjacent to $y$, and set $\widetilde{H} = H \cup \{yv\}$. If $\overline{Q}$ is an endblock of $\widetilde{H}$, then we see that $\overline{Q}$, $Q_1, \dots, Q_t$ and $yv$ are all endblocks of $\widetilde{H}$. (Notice that in this case $Q$ is not an endblock of $\widetilde{H}$.) Using Proposition 2.5, we obtain a spanning 3-forest $F'$ in $\widetilde{H}$ with $d_{F'}(v_i) \leq 2$ for all $i \in \{1, \dots, t\}$ which has exactly $t$ components, such that one of the components of $F'$ contains the vertices $x_1, v', y$ and each of the remaining components contains one vertex of $\{x_2, \dots, x_t\}$, where $v'$ is an arbitrary vertex of $\partial Q - u$. Then

$$F = [F' \cup \{x_i v_i \,|\, i = 1, \dots, t\} \setminus \{yv\}] \cup [V(\mathcal{P}) \cap V(B)]$$

is a spanning 3-forest in $B$ as desired.

On the other hand, if $\overline{Q}$ is not an endblock of $\widetilde{H}$, we add a new edge $z\overline{z}$ as in the proof of Theorem 3.1. Then, in this case we can verify that $Q_1, \dots, Q_t$, $yz$, $z\overline{z}$ are all endblocks of $\widetilde{H}$. We can also use Proposition 2.5 to find a 3-forest $F'$ of $\widetilde{H}$ satisfying the corresponding properties above. By setting

$$F = [F' \cup \{x_i v_i \,|\, i = 1, \dots, t\} \setminus \{yv,\ z\overline{z}\}] \cup [V(\mathcal{P}) \cap V(B)]$$

we obtain a spanning 3-forest in $B$ satisfying the desired properties.

Now we consider the case (b); i.e., $Q$ is an endblock of $H$, but $y$ is not adjacent to $\partial Q - u$. Since $G$ is 3-connected, there exists a vertex $v_0 \in V(\partial Q) \setminus \{u\}$ with $x_0v_0 \in E(B)$, where $x_0$ is the first vertex of attachment of $B$ on $\mathcal{P}$. Let $v$ be an arbitrary vertex of $B$ adjacent to $y$, and set $\widetilde{H} = H \cup \{yv, x_0v_0\}$. Then in this case $x_0v_0$, $Q_1, \dots, Q_t$, $yv$ are all endblocks of $\widetilde{H}$. Thus we can also use Proposition 2.5 to obtain a spanning 3-forest $F'$ in $\widetilde{H}$ with $d_{F'}(v_i) \leq 2$ for all $i \in \{1, \dots, t\}$ which has exactly $t$ components, such that one of the components of $F'$ contains the vertices $x_0, v', y$ and each of the remaining components contains one vertex of $\{x_1, \dots, x_t\}$. Then, by setting

$$F = [F' \cup \{x_iv_i \mid i = 1, \dots, t\} \setminus \{yv\}] \cup [V(\mathcal{P}) \cap V(B)]$$

we obtain a spanning 3-forest in $B$.

It remains to consider the case (c); i.e., $Q$ is not an endblock of $H$. Without loss of generality we may assume that $Q'$ is an endblock of $H$, for otherwise it is suffices to add a new edge, and then we use the argument similar to the case 1). Now choose an arbitrary vertex $v$ adjacent to $y$ in $H$, and set $\widetilde{H} = H \cup \{yv\}$. Then we can clearly see that $Q_1, \dots, Q_t$, $yv$ and $Q'$ are all endblocks of $\widetilde{H}$. By defining a 3-forest $F$ spanning $B$ as described in the case 1), we also obtain a subgraph satisfying the desired conditions. □

**Corollary 3.3.** *Let $B$, $x_0$ and $y$ as in Theorem 3.2 be given. Further let $\overline{x}$ be the last vertex of attachment of $B$ on $\mathcal{P}$. Then there exists a spanning 3-forest $F$ in $B$ such that:*

(1) *One component of $F$ contains the vertices $\overline{x}$ and $y$, and each of the remaining components of $F$ contains exactly one vertex of attachment of $\mathcal{P}$.*
(2) *$d_F(x_0) = 0$, $d_F(\overline{x}) = 1$ and $d_F(x) \leq 1$ for each vertex $x$ of attachment of $B$ on $\mathcal{P}$.*
(3) *$d_F(y) = 1$.*

*Proof.* Let us designate by $F'$ the spanning 3-forest in $B$ satisfying the conditions described in Theorem 3.2. If $v$ is the vertex chosen in the proof of the theorem, $F = F' \cup \{vy\}$ is also a spanning 3-forest in $B$ with $d_F(x_0) = 0$ and $d_F(x) \leq 1$ for each vertex $x \in V(B) \cap V(\mathcal{P})$. Note in this case that $d_F(\overline{x}) = d_{F'}(\overline{x}) = 1$ from the construction of $F$ and $F'$. In particular, since the edge $vy$ is added, we have $d_F(y) = d_{F'}(y) + 1 = 0 + 1 = 2$. Therefore the constructed 3-forest $F$ satisfies the assertions in this corollary. □

Finally we investigate the bridges of type 2.

**Theorem 3.4.** *Let $B$ be a bridge of type 2 of a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph. Let $x_0$ be the first vertex of attachment of $B$ on $\mathcal{P}$ and let $\{y_1, y_2\} =$*

$V(B) \cap V(\mathcal{P}')$. *Then there exists a spanning 3-forest $F$ in $B$ such that:*

(1) *A component $T$ of $F$ contains both $y_1$ and $y_2$, but it does not contain a vertex of attachment of $B$ on $\mathcal{P}$.*
(2) *Each component of $F - T$ contains exactly one vertex of attachment of $B$ on $\mathcal{P}$.*
(3) $d_F(x_0) = 0$ *and* $d_F(x) \leq 1$ *for each vertex $x$ of attachment of $B$ on $\mathcal{P}$.*
(4) $d_F(y_1) = d_F(y_2) = 1$.

*Proof.* Set $H = B - (\mathcal{P} \cup \mathcal{P}')$, and let $\Delta_B$ be the subgraph of $B$ defined at the beginning of this section. We will decompose the graph $H$ into 2 induced subgraphs $H_1$ and $H_2$ as follows:

If $H$ contains no outer block, then we set $H_2 = \{y_1, y_2\}$. Otherwise we may denote $Q'$ the outer endblock of $H$, and further the outer blocks of $H$ by $Q' = Q'_1, Q'_2, \ldots, Q'_m$ with the articulations $u_1, \ldots, u_{m-1}$, respectively. In addition we denote by $u'_m$ the articulation in $B$ connecting $Q'_m$ with $\Delta_B$. We set $H_2 = \bigcup\{Q'_i \mid i = 1, \ldots, m\} - u_m$ and then $H_1 = H - H_2$. Then we can easily verify that the graphs $H_1$ and $H_2$ are connected, and moreover they are induced subgraphs with $V(H_1) \cup V(H_2) = V(H)$ and $V(H_1) \cap V(H_2) = \emptyset$. Now, for $i = 1, 2$, we designate by $B_i$ the subgraph of $B$ induced by $H_i$ and the vertices of attachment of $B_i$ on $\mathcal{P}$ and $\mathcal{P}'$. For the subgraph $B_1$, by using the arguments similar to those in Theorem 3.2, we can find a spanning 3-forest $F_1$ with $d_{F_1}(x_0) = 0$ and $d_{F_1}(x) \leq 1$ for all $x \in V(B_1) \cap V(\mathcal{P})$, such that each component of $F_1$ contains exactly one vertex of attachment of $B_1$ (or $B$) on $\mathcal{P}$.

Now we construct a spanning 3-tree $T_2$ in $B_2$. If $H_2 = \{y_1 y_2\}$, then set $B_2 = H_2 = \{y_1 y_2\}$. On the other hand, if $H$ contains an outer block (i.e., $H_2 \neq \emptyset$), we divide into two cases, which deal with the connectedness number of $Q'_m - u_m$. Consider the case that $Q'_m - u_m$ is connected, but not 2-connected. Since $G$ is 3-connected and $Q'_m - u_m$ has a linear decomposition, they follow that there exist a vertex $v \in V(\partial S) \setminus \{w\}$ and $v' \in V(\partial S') \setminus \{w'\}$ with $y_1 v \in E(B)$ and $y_2 v' \in E(B)$, where $S$ and $S'$ are the endblocks of $Q'_m - u_m$ with the articulations $w$ and $w'$, respectively. But, since $H_2$ can in this case contain at most 3 endblocks, we obtain a 3-tree (say $T'$) in $H_2$ with $d_{T'}(v) \leq 2$ and $d_{T'}(v') \leq 2$. Then we see that $T_2 = T' \cup \{y_1 v, y_2 v'\}$ is a spanning 3-tree in $B_2$ with $d_{T_2}(y_1) = 1$ and $d_{T_2}(y_2) = 1$.

Assume now that $Q'_m - u_m$ is 2-connected. From the connectedness number of $G$ we can find a vertex $v \in V(\partial Q'_m) \setminus \{u_{m-1}, u_m\}$ such that $v$ is adjacent to either $y_1$ or $y_2$. We may without loss of generality assume that $y_1 v \in E(B)$. Then, in this case we can also obtain a vertex $v' \in V(\partial Q'_1) \setminus \{u_1\}$ with $y_2 v' \in E(B)$, and therefore there exists a 3-tree (say $T'$) in $H_2$ with $d_{T'}(v) \leq 2$ and $d_{T'}(v') \leq 2$. We set $T_2 = T' \cup \{y_1 v, y_2 v'\}$, which is a spanning 3-tree in $B_2$ with $d_{T_2}(y_1) = 1$ and $d_{T_2}(y_2) = 1$.

Finally, by setting $F = F_1 \cup T_2$, we obtain a spanning 3-forest which satisfies the assertions of this theorem. Our proof is complete. □

**Corollary 3.5.** *Let $B$, $x_0$, $y_1$ and $y_2$ as in Theorem 3.4 be given. Further assume $|V(B) \cap V(\mathcal{P})| \geq 2$. Then there exists a spanning 3-forest $F$ in $B$ such that:*

(1) *A component $T$ of $F$ contains $y_2$, but it does not contain a vertex of attachment of $B$ on $\mathcal{P}$.*
(2) *Each component of $F - T$ contains exactly one vertex of attachment of $B$ on $\mathcal{P}$, and moreover one of them contains the vertex $y_1$.*
(3) *$d_F(x_0) = 0$ and $d_F(x) \leq 1$ for each vertex $x$ of attachment of $B$ on $\mathcal{P}$.*
(4) *$d_F(y_1) = 1$ and $d_F(y_2) \leq 1$.*

*Proof.* Let $H$, $\Delta_B$ and $\{Q_1, \ldots, Q_t\}$ be defined in the proof of Theorem 3.4. Analogously we define a set of representatives $\{x_1, \ldots, x_t\}$ of $\{Q_1, \ldots, Q_t\}$ and a set of vertices $\{v_1, \ldots, v_t\}$ of $B$ with $v_i \in \partial Q_i - u_i$ and $x_i v_i \in E(B)$, where $u_i$ is the articulation of $Q_i$ $(i = 1, \ldots, t)$. Since it can be similarly verified for the remaining cases, we will prove the most general case that $H$ contains 2 endblocks (say $Q$ and $Q'$) in $\Delta_B$ and an endblock (say $\widetilde{Q}$. Then

$$Q_1, \ldots, Q_t,\ Q,\ Q' \text{ and } \widetilde{Q}$$

are all endblocks of $\Delta_B$. We first consider the case that there exists a vertex $v \in V(\partial Q)$ with $y_1 v \in E(B)$. In this case we use Proposition 2.5 to obtain a spanning 3-forest $F'$ in $H$ with $d_{F'}(v) \leq 2$ and $d_{F'}(v_i) \leq 2$ $(i = 1, \ldots, t)$, such that $F'$ contains exactly $t+1$ components $T_1, \ldots, T_t$, $T_{t+1}$ with $v_1, v' \in V(T_1)$, $v \in V(T_{t+1}$ and $v_i \in V(T_i)$ $(i = 2, \ldots, t)$, where $v'$ is an arbitrary vertex of $\partial Q'$ adjacent to $x'$. Then it is not hard to verify that

$$F = F' \cup \{x_1 v_1, \ldots, x_t v_t,\ x'v',\ y_1 v\} \cup [V(B) \cap V(\mathcal{P} \cup \mathcal{P}')]$$

is a spanning 3-forest in $B$ satisfying the assertions in this corollary.

In order to consider the case that there exists no vertex of $Q$ adjacent to $y_1$, note first that $x_0 \neq x_1$ and $x'v' \in E(B)$, where $x'$ is a vertex of attachment on $\mathcal{P}$ and $v'$ is a vertex of $\partial Q'$. In this case we use the similar argument above to obtain a spanning 3-forest $F'$ in $H$ satisfying the corresponding properties. By setting

$$F = F' \cup \{x_1 v_1, \ldots, x_t v_t,\ x'v',\ y_1 v\} \cup [V(B) \cap V(\mathcal{P} \cup \mathcal{P}')]$$

we clearly obtain a spanning 3-forest in $B$ as desired, where $v$ is the vertex of $\partial Q$ selected above. □

**Corollary 3.6.** *Let $B$, $y_1$ and $y_2$ as in Theorem 3.4 be given. Further assume $V(B) \cap V(\mathcal{P}) = \{x_0\}$. Then there exists a spanning 3-forest $F$ in $B$ which contains exactly two components $T_1$ and $T_2$, such that:*

(1) *$V(T_1) = \{y_1\}$ and $x_0, y_2 \in V(T_2)$.*
(2) *$d_F(x_0) = d_F(y_2) = 1$.*

*Proof.* By the assumption $B$ contains exactly 3 vertices $x_0$, $y_1$ and $y_2$ of attachment on $\mathcal{P} \cup \mathcal{P}'$, and therefore from the fact that $G$ is 3-connected $H = B - (\mathcal{P} \cup \mathcal{P}')$ has at most 3 endblocks. We will only prove the case that $H$ contains 3 endblocks $Q$, $Q'$ and $\widetilde{Q}$ with the articulation $u$, $u'$ and $\overline{u}$, respectively. (The remaining cases can be similarly verified.)

From the planarity of $G$ each endblock of $H$ (except for its articulation) is adjacent to exactly 2 vertices of attachment of $\mathcal{P} \cup \mathcal{P}'$. Without loss of generality we may assume that $Q - u$ is adjacent to $\overline{x}$ and $y_1$, $Q' - u'$ adjacent to $\overline{x}$ and $y_2$, and $\widetilde{Q} - \overline{u}$ to $y_1$ and $y_2$. Further denote $v$ and $\overline{v}$ the vertices of $Q - u$ adjacent to $\overline{x}$ and $y_2$, respectively. By Corollary 2.4 there exists a spanning 3-tree $T'$ in $H$ with $d_{T'}(v) \leq 2$ and $d_{T'}(\overline{v}) \leq 2$. Then, by setting $T_1 = \{y_1\}$ and $T_2 = T' \cup \{v\overline{x}, \overline{v}y_2\}$, we obtain s spanning 3-forest $T = T_1 \cup T_2$ which satisfies the assertions (1) and (2). □

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