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MODIFIED DECOMPOSITION METHOD FOR SOLVING INITIAL AND BOUNDARY VALUE PROBLEMS USING PADE APPROXIMANTS

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ABSTRACT. In this paper, we apply a new decomposition method for solving initial and boundary value problems, which is due to Noor and Noor [18]. The analytical results are calculated in terms of convergent series with easily computable components. The diagonal Pade approximants are applied to make the work more concise and for the better understanding of the solution behavior. The proposed technique is tested on boundary layer problem; Thomas-Fermi, Blasius and sixth-order singularly perturbed Boussinesq equations. Numerical results reveal the complete reliability of the suggested scheme. This new decomposition method can be viewed as an alternative of Adomian decomposition method and homotopy perturbation methods.

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1. Introduction

It is well known that a wide class of problems involving astrophysics, experimental and mathematical physics, nuclear charge in heavy atoms, thermal behavior of a spherical cloud of gas, thermodynamics, population models, chemical kinetics and fluid mechanics, can be formulated as initial and boundary value problems, see [1-32]. Several techniques including Adomian's decomposition, variational iteration, finite difference, polynomial spline and homotopy perturbation have been developed for solving such problems. Most of these methods have their inbuilt deficiencies like calculation of Adomian's polynomials, identification of Lagrange multiplier, divergent results and huge computational work. To overcome these difficulties and drawbacks, several new decomposition methods have been suggested for solving these problems. One of these methods is called the

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modified decomposition method, which was developed by Noor and Noor [18] to construct iterative-type methods for solving nonlinear equations of the type $f(x) = 0$. The main motivation of this paper is to apply this modified decomposition method for finding the solution of initial and boundary value problems. It is shown that the modified decomposition method provides the solution in a rapid convergent series with easily computable components. In the present study, we implement this technique to solve boundary layer problem; Thomas Fermi, Blasius and singularly perturbed sixth-order Boussinesq equations. The series solutions (in case of boundary layer problem; Thomas Fermi and Blasius equations) are replaced by the powerful Pade approximants in order to make the work more concise and to get a better understanding of the solution behavior $[5, 15, 16, 21-23, 29-31]$. The suggested method is free from round off errors, calculation of the so-called Adomian's polynomials and the identification of Lagrange multipliers. Moreover, the proposed MDM is easier to implement and reduces the computational work while still maintaining a higher level of accuracy. The modified decomposition method may be considered as an alternative to Adomian decomposition and homotopy perturbation techniques.

2. Modified decomposition method

Consider the following general functional equation

$$f(y) = 0.$$

To convey the main idea of the Noor and Noor decomposition method [18], we re-write the above equation as

$$y = N(y) + c, \tag{1}$$

where N is a nonlinear operator and c is a constant. We are looking for a solution of equation (1) having the series form:

$$y = \sum_{i=0}^{\infty} y_i. \tag{2}$$

The nonlinear operator N can be decomposed as:

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=0}^{\infty}\left(N\left(\sum_{j=0}^{i} y_j\right)\right). \tag{3}$$

From equations (1), (2) and (3),we have

$$\sum_{i=0}^{\infty} y_i = c + N(y_0) + \sum_{i=0}^{\infty}\left(N\left(\sum_{j=0}^{i} y_j\right)\right). \tag{4}$$

We define the recurrence relation:

$$y_0 = c,$$

$$\begin{aligned} y_1 &= N(y_0), \\ y_{m+1} &= N(y_0 + \cdots + y_m), \qquad m = 1, 2, 3, \ldots, \end{aligned}$$

then, for $m = 1, 2, 3, \ldots,$

$$(y_1 + \cdots + y_{m+1}) = N(y_0) + N(y_0 + y_1) + \cdots + N(y_0 + y_1 + \cdots + y_m) \tag{5}$$

and

$$y = c + \sum_{i=1}^{\infty} y_i \tag{6}$$

Using the Banach fixed-point theorem, one can study the convergence criteria of this decomposition method.

3. Pade approximants

A Pade approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(x)$. The $[L/M]$ Pade approximants to a function $y(x)$ are given by $[5, 15, 16, 21-23, 29-31]$.

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_m(x)}, \tag{7}$$

where$P_L(x)$ is polynomial of degree at most L and $Q_m(x)$ is a polynomial of degree at most M, the formal power series

$$y(x) = \sum_{i=1}^{\infty} a_i x^i, \tag{8}$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = O\left(x^{L+M+1}\right), \tag{8}$$

determine the coefficients of $P_L(x)$and $Q_m(x)$by the equation. Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we imposed the normalization condition

$$Q_M(0) = 1.0. \tag{9}$$

Finally, we require that $P_L(x)$and $Q_m(x)$have non common factors, if we write the coefficient of $P_L(x)$and $Q_m(x)$as

$$\begin{aligned} P_L(x) &= p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L, \\ Q_M(x) &= q_0 + q_1 x + q_2 x^2 + \cdots + p_M x^M \end{aligned} \tag{10}$$

Then by (10) and (11), we may multiply (7) by , which linearizes the coefficient equations. We can write out (9) in more details as

$$\begin{aligned} a_{L+1} + a_L q_1 + \cdots + a_{L-M} q_M &= 0, \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M &= 0, \\ &\vdots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M &= 0, \end{aligned} \tag{11}$$

$$\begin{aligned} a_0 &= p_0, \\ a_0 + a_0 q_1 + \cdots &= p_1, \\ &\vdots \\ a_L + a_{L-1} q_1 + \cdots + a_0 q &= p_L \end{aligned} \tag{12}$$

To solve these equations, we start with equation (12), which is a set of linear equations for all the unknown $q's$. Once the q's are known, then equation (13) gives and explicit formula fro the unknown $p's$, which complete the solution. If equations (12) and (13) are nonsingular, then we can solve them directly and obtain equation (14), (see [15]), where equation (14) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right] = \frac{\det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \cdots & \ddots & \cdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum\limits_{j=M}^{L} a_{j-M} x^j & \sum\limits_{j=M-1}^{L} a_{j-M+1} x^j & \cdots & \sum\limits_{j=0}^{L} a_j x^j \end{bmatrix}}{\det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \cdots & \ddots & \cdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}} \tag{13}$$

To obtain diagonal Pade approximants of different order such as [2/2], [4/4] or [6/6], we can use the symbolic calculus software, Mathematica or Maple.

4. Numerical Applications

In this section, we apply the modified decomposition method for solving boundary layer problem; Thomas-Fermi, Blasius and sixth-order singularly perturbed Boussinesq equations. We also introduce a slight modification in the selection of initial value which makes the application of the proposed algorithms simpler and improves the efficiency of the scheme. To make the work more concise and for the better understanding of the solution behavior series solutions are replaced by the powerful diagonal Pade approximants.

Example 4.1[23,29]. Consider the Thomas-Fermi (TF) equation which is used to model the effective nuclear charge in heavy atoms, to study the potentials and charge densities of atoms having numerous electrons

$$y''(x) = \frac{y^{\frac{3}{2}}}{x^{\frac{1}{2}}}, \tag{14}$$

with boundary conditions

$$y(0) = 1, \qquad \lim_{x\to\infty} y(x) = 0. \tag{15}$$

The above Thomas-Fermi problem can be written as the following integral equation

$$y(x) = 1 + Bx + \int_0^x \int_0^x x^{-\frac{1}{2}} (y(x))^{\frac{3}{2}} \, dx dx. \quad (16)$$

Applying the modified decomposition method

$$y_{n+1}(x) = 1 + Bx + \int_0^x \int_0^x x^{-\frac{1}{2}} (y_n(x))^{\frac{3}{2}} \, dx dx.$$

The given initial values admits the use of $y_0(x) = 1 + Bx$, where$B = y'(0)$ but we use the modified approach and take $y_0(x) = 1$.Consequently, the following approximants are obtained

$$\begin{aligned}
y_0(x) &= c, \\
y_0(x) &= 1, \\
y_1(x) &= N y_0(x), \\
y_1(x) &= Bx + \frac{4}{3}x^{\frac{3}{2}}, \\
y_2(x) &= N(y_0(x) + y_1(x)), \\
y_2(x) &= \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}Bx^{\frac{5}{2}} + \frac{1}{3}x^3, \\
y_3(x) &= N(y_0(x) + y_1(x) + y_2(x)), \\
y_3(x) &= \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}Bx^{\frac{5}{2}} + \frac{2}{3}x^3 + \frac{1}{20}Bx^4 + \frac{2}{63}x^{\frac{9}{2}}.
\end{aligned}$$

The series solution after three iterations is given as

$$y(x) = 1 + Bx + 4x^{\frac{3}{2}} + \frac{4}{5}Bx^{\frac{5}{2}} + x^3 + \frac{1}{20}Bx^4 + \frac{2}{63}x^{\frac{9}{2}}.$$

Setting $x^{\frac{1}{2}} = t$,the series solution is obtained as

$$y(t) = 1 + Bt + 4t^3 + \frac{4}{5}Bt^5 + t^6 + \frac{1}{20}Bt^8 + \frac{2}{63}t^9.$$

The diagonal Pade approximants can be applied [22, 28] in order to study the mathematical behavior of the potential and to determine the initial slope of the potential .

Table 4.1. Pade approximants and initial slopes $y'(0)$ [22, 28]

Pade approximants	Initial slope	Error (%)
[2/2]	-1.211413729	23.71
[4/4]	-1.550525919	2.36
[7/7]	-1.586021037	12.9×10^{-2}
[8/8]	-1.588076820	3.66×10^{-4}
[10/10]	-1.588076779	3.64×10^{-4}

Example 4.2[21,28,30]. Consider the following nonlinear third order boundary layer problem which appears mostly in the mathematical modeling of physical phenomena in fluid mechanics

$$f'''(x) + (k-1) f(x) f''(x) - 2k (f'(x))^2 = 0, \qquad k > 0,$$

with boundary conditions

$$f(0) = 0, \qquad f'(0) = 1, \qquad f'(\infty) = 0, \qquad k > 0.$$

The above boundary layer problem can be written as the following integral equation

$$f(x) = f(x) - \int_0^x \int_0^x \int_0^x \left((k-1) f(x) f''(x) - 2k \left(f'(x)\right)^2\right) dx dx dx. \qquad k > 0,$$

Applying the modified decomposition method, we have

$$f_{n+1}(x) = f_n(x) - \int_0^x \int_0^x \int_0^x \left((k-1) f_n(x) f_n''(x) - 2k \left(f_n'(x)\right)^2\right) dx dx dx, k > 0,$$

Using the initial conditions, we obtain

$$f_{n+1}(x) = x + \frac{1}{2!}\alpha x^2 - \int_0^x \int_0^x \int_0^x \left((k-1) f_n(x) f_n''(x) - 2k \left(f_n'(x)\right)^2\right) dx dx dx, k > 0,$$

where$f''(0) = \alpha < 0$. Consequently, the following approximants are obtained

$$\begin{aligned}
f_0(x) &= c, \qquad f_0(x) = x, \\
f_1(x) &= N f_0(x), \\
f_1(x) &= \frac{1}{2}\alpha x^2 + \frac{1}{3} k x^3, \\
f_2(x) &= N\left(f_0(x) + f_1(x)\right), \\
f_2(x) &= \frac{1}{3} k x^3 + \left(\frac{1}{24} + \frac{1}{8}\alpha k\right) x^4 + \left(\frac{1}{120}\alpha^2 + \frac{1}{30} k + \frac{1}{30} k^2 + \frac{1}{40} k \alpha^2\right) x^5 \\
&\quad \left(\frac{1}{90} k\alpha + \frac{1}{45}\alpha k^2\right) x^6 + \left(\frac{1}{315} k^2 + \frac{2}{315} k^3\right) x^7.
\end{aligned}$$

The series solution is given as

$$\begin{aligned}
f(x) &= x + \frac{1}{2}\alpha x^2 + \frac{2}{3} k x^3 + \left(\frac{1}{24} + \frac{1}{8}\alpha k\right) x^4 + \left(\frac{1}{120}\alpha^2 + \frac{1}{30} k + \frac{1}{30} k^2 + \frac{1}{40} k\alpha^2\right) x^5 \\
&\quad \left(\frac{1}{90} k\alpha + \frac{1}{45}\alpha k^2\right) x^6 + \left(\frac{1}{315} k^2 + \frac{2}{315} k^3\right) x^7.
\end{aligned}$$

Table 4.2. Numerical values for by using diagonal Pade approximants[22, 30, 32]

n	[2/2]	[3/3]	[4/4]	[5/5]	[6/6]
0.2	−0.3872983347	−0.3821533832	−0.3819153845	−0.3819148088	−0.3819121854
0.3	−0.5773502692	−0.5615999244	−0.5614066588	−0.5614481405	−0.561441934
0.4	−0.6451506398	−0.6397000575	−0.6389732578	−0.6389892681	−0.6389734794
0.6	−0.8407967591	−0.8393603021	−0.8396060478	−0.8395875381	−0.8396056769
0.8	−1.007983207	−1.007796981	−1.007646828	−1.007646828	−1.007792100

Table 4.3. Numerical values for for by using diagonal Pade approximants[22, 29, 31]

n	α
4	−2.483954032
10	−4.026385103
100	−12.84334315
1000	−40.65538218
5000	−104.8420672

Example 4.3[31]. Consider the two dimensional nonlinear inhomogeneous initial boundary value problem for the integro differential equation related to the Blasius problem

$$y''(x) = \alpha - \frac{1}{2}\int_0^x y(t)\, y''(t)\, dt, \qquad -\infty < x < 0$$

with boundary conditions$y(0) = 0$, $y'(0) = 1$ and $\lim_{x\to\infty} y'(x) = 0$, where the constant is positive and is defined by

$$y''(0) = \alpha, \qquad \alpha > 0.$$

The above Blasius problem can be written as the following integral equation

$$y(x) = x + \frac{1}{2}\alpha x^2 - \frac{1}{2}\int_0^x\int_0^x \left(y(x)\, y''(x)\right) dxdx. \qquad -\infty < x < 0$$

Applying the modified decomposition method, we have

$$y_{n+1}(x) = x + \frac{1}{2}\alpha x^2 - \frac{1}{2}\int_0^x\int_0^x \left(y_n(x)\, y_n''(x)\right) dxdx. \qquad -\infty < x < 0$$

Consequently, the following approximants are obtained

$$\begin{aligned}
y_0(x) &= c,\\
y_0(x) &= x + \frac{1}{2}\alpha x^2,\\
y_1(x) &= N y_0(x),\\
y_1(x) &= \frac{1}{2}\alpha x^2 + \frac{1}{6}\alpha^2 x^3,\\
y_2(x) &= N\left(y_0(x) + y_1(x)\right),\\
y_2(x) &= \left(\frac{1}{48}\alpha^2 - \frac{1}{24}\alpha\right) x^4 - \frac{7}{240}\alpha^2 x^5 + \left(\frac{1}{720}\alpha^2 - \frac{7}{1440}\alpha^3\right) x^6\\
&\quad + \frac{1}{1344}\alpha^3 x^7 + \frac{1}{10752}\alpha^4 x^8.
\end{aligned}$$

The series solution is given by

$$\begin{aligned} y(x) &= x+\frac{1}{2}\alpha x^{2}+\frac{1}{12}\alpha x^{3}+\left(\frac{1}{48}\alpha^{2}-\frac{1}{24}\alpha\right)x^{4}-\frac{7}{240}\alpha^{2}x^{5} \\ &+\left(\frac{1}{720}\alpha^{2}-\frac{7}{1440}\alpha^{3}\right)x^{6}+\frac{1}{1344}\alpha^{3}x^{7}. \end{aligned}$$

The diagonal Pade approximants can be applied [30] in order to study the mathematical behavior and to determine the constant

Table 4.4. Pade approximants and numerical value of α

Pade approximant	α
[2/2]	0.5778502691
[3/3]	0.5163977793
[4/4]	0.5227030798

Example 4.4[10,19,22]. Consider the following singularly perturbed sixth-order Boussinesq equation

$$u_{tt}=u_{xx}+(p(u))_{xx}+\alpha u_{xxxx}+\beta u_{xxxxxx},$$

taking $\alpha=1,\beta=0$and $p(u)=3u^{2}$,the model equation is given as

$$u_{tt}=u_{xx}+3\left(u^{2}\right)_{xx}+u_{xxxx},$$

with initial conditions

$$u(x,0)=\frac{2ak^{2}e^{kx}}{\left(1+ae^{kx}\right)^{2}},\qquad u_{t}(x,0)=\frac{2ak^{3}\sqrt{1+k^{2}}\left(1-ae^{kx}\right)e^{kx}}{\left(1+ae^{kx}\right)^{3}},$$

where a and k are arbitrary constants. The exact solution of the problem is given as [10, 19, 22]

$$u(x,t)=2\frac{ak^{2}\exp\left(kx+k\sqrt{1+k^{2}t}\right)}{\left(1+a\exp\left(kx+k\sqrt{1+k^{2}t}\right)\right)^{2}}$$

The above Boussinesq problem can be written as the following integral equation

$$\begin{aligned} u(x,t) &= \frac{2ak^{2}e^{kx}}{\left(1+ae^{kx}\right)^{2}} \\ &+ \frac{2ak^{3}\sqrt{1+k^{2}}\left(1-ae^{kx}\right)e^{kx}}{\left(1+ae^{kx}\right)^{3}}t+\int_{0}^{t}\int_{0}^{t}\left(\frac{\partial^{2}u}{\partial x^{2}}+\frac{\partial^{4}u}{\partial x^{4}}+3\left(u^{2}\right)_{xx}\right)dtdt. \end{aligned}$$

Applying the modified decomposition method, we obtain

$$\begin{aligned} u_{n+1}(x,t) &= \frac{2ak^{2}e^{kx}}{\left(1+ae^{kx}\right)^{2}}+\frac{2ak^{3}\sqrt{1+k^{2}}\left(1-ae^{kx}\right)e^{kx}}{\left(1+ae^{kx}\right)^{3}}t \\ &+ \int_{0}^{t}\int_{0}^{t}\left(\frac{\partial^{2}u_{n}}{\partial x^{2}}+\frac{\partial^{4}u_{n}}{\partial x^{4}}+3\left(u_{n}^{2}\right)_{xx}\right)dtdt. \end{aligned}$$

Consequently, the following approximants are obtained

$$u_0(x,t) = \frac{2ak^2e^{kx}}{(1+ae^{kx})^2},$$

$$u_1(x,t) = +\frac{2ak^3\sqrt{1+k^2}\left(1-ae^{kx}\right)e^{kx}}{(1+ae^{kx})^3}t - \frac{2ak^4e^{kx}\left(-a^2e^{2kx}-a^2k^2e^{2kx}+4ae^{kx}+4ak^2e^{kx}-1-k^2\right)}{(1+e^x)^4}t^2,$$

the series solution is given by

$$u(x,t) = \frac{2ak^2e^{kx}}{(1+ae^{kx})^2} + \frac{2ak^3\sqrt{1+k^2}\left(1-ae^{kx}\right)e^{kx}}{(1+ae^{kx})^3}t - \frac{2ak^4e^{kx}\left(-a^2e^{2kx}-a^2k^2e^{2kx}+4ae^{kx}+4ak^2e^{kx}-1-k^2\right)}{(1+e^x)^4}t^2,$$

Table 4.5. (Error estimates)

x	t_j			
	1	.5	.1	.05
−1	$2.97E-2$	$7.34E-3$	$8.21E-2$	9.42
−0.8	$4.04E-2$	$4.01E-3$	$4.39E-2$	$5.50E-2$
−0.6	$1.20E-1$	$1.23E-2$	$2.63E-2$	$1.19E-2$
−0.4	$2.00E-1$	$1.77E-2$	$3.88E-2$	$3.19E-2$
−0.2	$2.68E-1$	$2.11E-2$	$7.74E-2$	$7.35E-2$
0	$3.14E-1$	$2.39E-2$	$1.10E-1$	$1.10E-1$
0.2	$3.31E-1$	$2.74E-2$	$1.36E-1$	$1.39E-1$
0.4	$3.18E-1$	$3.24E-2$	$1.54E-1$	$1.60E-1$
0.6	$2.79E-1$	$3.88E-2$	$1.64E-1$	$1.72E-1$
0.8	$2.22E-1$	$4.61E-2$	$1.67E-1$	$1.76E-1$
1	$1.55E-1$	$5.33E-2$	$1.63E-1$	$1.73E-1$

Table 4.5 exhibits the absolute error between the exact and the series solutions. Higher accuracy can be obtained by introducing some more components of the series solution.

Example 4.5[10,17,19,22]. Consider the following singularly perturbed sixth order Boussinesq equation

$$u_{tt} = u_{xx} + \left(u^2\right)_{xx} - u_{xxxx} + \frac{1}{2}u_{xxxxxx},$$

with initial condition

$$u(x,0) = -\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right), u_t(x,0) = \frac{-210\sqrt{\frac{194}{13}}\sec h^4\left(\frac{x}{\sqrt{26}}\right)\tanh\left(\frac{x}{\sqrt{26}}\right)}{2197}$$

The exact solution of the problem is given as

$$u(x,t) = -\frac{105}{169}\sec h^4\left[\sqrt{\frac{1}{26}}\left(x-\sqrt{\frac{97}{169}}t\right)\right].$$

The above Boussinesq problem can be written as the following integral equation

$$u(x,t) = -\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right) + \frac{-210\sqrt{\frac{194}{13}}\sec h^4\left(\frac{x}{\sqrt{26}}\right)\tanh\left(\frac{x}{\sqrt{26}}\right)}{2197}t + \int_0^x\int_0^x\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} + \frac{1}{2}\left(\frac{\partial^6 u}{\partial x^6}\right) + \left(u^2\right)_{xx}\right)dtdt.$$

Applying the modified decomposition method, we have

$$u_{n+1}(x,t) = -\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right) + \frac{-210\sqrt{\frac{194}{13}}\sec h^4\left(\frac{x}{\sqrt{26}}\right)\tanh\left(\frac{x}{\sqrt{26}}\right)}{2197}t + \int_0^x\int_0^x\left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^4 u_n}{\partial x^4} + \frac{1}{2}\left(\frac{\partial^6 u_n}{\partial x^6}\right) + \left(u_n^2\right)_{xx}\right)dtdt.$$

Consequently, the following approximants are obtained

$$u_0(x,t) = -\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right),$$

$$u_1(x,t) = -\frac{105\sqrt{\frac{194}{13}}\sec h^6\left(\frac{x}{\sqrt{26}}\right)\sinh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)}{2197}t - \frac{105}{371293}\left(-291 + 194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right)\sec h^6\left(\frac{x}{\sqrt{26}}\right)t^2,$$

The series solution is obtained as

$$u(x,t)) = -\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right) - \frac{210\sqrt{\frac{194}{13}}\sec h^6\left(\frac{x}{\sqrt{26}}\right)\sinh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)}{2197}t - \frac{210}{371293}\left(-291 + 194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right)\sec h^6\left(\frac{x}{\sqrt{26}}\right)t^2$$

Table 4.6

x	t_j			
	1	.5	.1	.05
−1	$1.23E-3$	$2.48E-2$	$2.93E-2$	$2.95E-2$
−0.8	$8.91E-3$	$2.34E-2$	$2.85E-2$	$2.87E-2$
−0.6	$5.95E-3$	$2.22E-2$	$2.79E-2$	$2.81E-2$
−0.4	$3.55E-3$	$2.14E-2$	$2.75E-2$	$2.77E-2$
−0.2	$1.77E-3$	$2.08E-2$	$2.72E-2$	$2.74E-2$
0	$6.92E-4$	$2.06E-2$	$2.71E-2$	$2.73E-2$
0.2	$3.38E-4$	$2.06E-2$	$2.72E-2$	$2.74E-2$
0.4	$7.31E-4$	$2.10E-2$	$2.75E-2$	$2.77E-2$
0.6	$1.86E-3$	$2.17E-2$	$2.79E-2$	$2.81E-2$
0.8	$3.72E-3$	$2.27E-2$	$2.85E-2$	$2.87E-2$
1	$6.24E-3$	$2.40E-2$	$2.93E-2$	$2.95E-2$

Table 4.6 exhibits the absolute error between the exact and the series solutions. Higher accuracy can be obtained by introducing some more components of the series solution.

Conclusion

In this paper, we applied the modified decomposition method (MDM) for solving initial and boundary value problems. The proposed algorithm is successfully implemented on boundary layer problem, Thomas-Fermi; Blasius and singularly perturbed sixth-order Boussinesq equations. Moreover, the obtained series solutions were replaced by the powerful Pade approximants to make the work more concise and to get a better understanding of the solution behavior. The proposed technique is employed without using linearization, perturbation, discretization or restrictive assumptions, is free from round off errors, the complexities arising in the calculation of the so-called Adomian's polynomials and the identification of Lagrange multiplier. It may be concluded that the suggested algorithm is very powerful and efficient in finding the analytical solutions for a wide class of initial and boundary value problems.

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