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ERROR ESTIMATES OF FULLY DISCRETE DISCONTINUOUS GALERKIN APPROXIMATIONS FOR LINEAR SOBOLEV EQUATIONS†

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ABSTRACT. In this paper, we construct fully discrete discontinuous Galerkin approximations to the solution of linear Sobolev equations. We apply a symmetric interior penalty method which has an interior penalty term to compensate the continuity on the edges of interelements. The optimal convergence of the fully discrete discontinuous Galerkin approximations in $\ell^\infty(L^2)$ norm is proved.

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1. Introduction

Discontinuous Galerkin methods using interior penalties for elliptic and parabolic equations were first introduced by Douglas, Dupont and Wheeler [4, 15] and Arnold [1] in 1970s. Compared over the conventional Galerkin method, discontinuous Galerkin methods allow meshes which are more flexible in their decompositions and degree of nonuniformity both in time and space.

A new type of discontinuous Galerkin method for diffusion problems which is elementwise conservative was introduced and analyzed by Oden, Babuška and Baumann [7]. In general, the interior penalty terms which are composed of the weighted L^2 inner product of the jumps of the function values across element edges are added to impose the continuity to the approximate solution, indirectly.

New applications of discontinuous Galerkin method with interior penalties to elliptic and nonlinear parabolic equations were introduced and analyzed by

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Riviere, Wheeler, Girault and Banas. It was shown that the methods in [9, 10, 11, 12] were elementwise conservative and a priori and a posteriori estimates in higher dimensions were derived. In [8], the authors developed a symmetric interior penalty method for nonlinear parabolic equations and prove the optimal convergence of the approximate solution in $L^\infty(L^2)$ norm.

The type of Sobolev equations is one of important partial differential equations. It is known that the equations of this type arise in many areas of mathematical physics and fluid mechanics. In [5, 6], the authors constructed Galerkin approximations for Sobolev equations and analyzed theier error estimate. Recently, in [13, 14] Sun and Yang formulated the discontinuous Galerkin approximate schemes and obtained error estimates in $L^\infty(H^1)$ and $\ell^2(H^1)$ norms. In this paper, we apply a discontinuous symmetric interior penalty Galerkin method to construct the fully discrete approximations and analyze the error estimate in $\ell^\infty(L^2)$ norm.

In section 2, we describe the model problem and state some assumptions for later use. In section 3, we introduce some notations, fully discrete discontinuous Galerkin approximations to the model problem using a SIPG method and state some preliminary lemmas. In section 4, we analyze the optimal convergence of the approximate solution.

2. Model problem and assumptions

In this paper we consider the following linear Sobolev equation:

$$u_t - \nabla \cdot (\nabla u + \nabla u_t) = f(x, u) \quad \text{in } \Omega \times (0, T], \tag{2.1}$$

with the boundary condition

$$(\nabla u + \nabla u_t) \cdot n = 0 \quad \text{on } \partial\Omega \times (0, T] \tag{2.2}$$

and the initial condition

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \tag{2.3}$$

where Ω is a convex polygonal domain in $\mathbb{R}^d$, $d = 2, 3$, and n is the unit outward normal vector to $\partial\Omega$.

Assume that the following conditions are satisfied.

1. f is uniformly Lipschitz continuous with respect to its second variable.
2. The model problem has a unique solution satisfying the following regularity condition:

$$u_{tt} \in L^\infty([0, T], H^2).$$

3. Notations and the formulation of fully discrete schemes

Let $\mathcal{E}_h = \Big\{E_1, E_2, \cdots, E_{N_h}\Big\}$ be a regular quasi-uniform subdivision of Ω, where E_j is a triangle or a quadrilateral if $d = 2$ and E_j is a 3-simplex or 3-rectangle if $d = 3$. Let $h_j = \mathrm{diam}(E_j)$ be the diameter of E_j and $h = \max\Big\{h_j : j = 1, 2, \cdots, N_h\Big\}$. We assume that $\mathcal{E}_h$ satisfies the non-degeneracy requirement, i.e., there exists a constant γ such that each E_j contains a ball of radius of γh_j and assume that there exists a constant $\rho > 0$ such that

$$\frac{h}{h_j} < \rho \ \text{ for all } \ j = 1, 2, \cdots, N_h.$$

This quasi-uniformity assumption is used for driving error estimates in terms of the degree of polynomials. We denote the set of all edges of the elements by $\Big\{e_1, e_2, \cdots, e_{P_h}, e_{P_h+1}, \cdots, e_{M_h}\Big\}$, where $e_k \subset \Omega$ for $1 \leq k \leq P_h$ and $e_k \subset \partial\Omega$ for $P_h + 1 \leq k \leq M_h$.

The L^2 inner product in $L^2(E)$ is denoted by $(\cdot, \cdot)_E$. For an $s \geq 0$ and a domain $E \subset \mathbb{R}^d$, the usual norm of the Sobolev space $H^s(E)$ is denoted by $\|\cdot\|_{s,E}$ and the usual seminorm is denoted by $|\cdot|_{s,E}$. When these notations are applied, we write simply $\|\cdot\|_s$ and $|\cdot|_s$ instead of $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ if $E = \Omega$.

For an $s \geq 0$ and a subdivision $\mathcal{E}_h$ of Ω, we define the following space

$$H^s(\mathcal{E}_h) = \Big\{v \in L^2(\Omega) : v|_{E_j} \in H^s(E_j), j = 1, 2, \cdots, N_h\Big\}.$$

And for $\phi \in H^s(\mathcal{E}_h)$, $s > \dfrac{1}{2}$, we define the average function $\{\phi\}$ and the jump function $[\phi]$ on $e_k = E_i \cap E_j$, $i < j$ as follows;

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}, \quad \forall x \in e_k, \quad 1 \leq k \leq P_h,$$
$$[\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}, \quad \forall x \in e_k, \quad 1 \leq k \leq P_h.$$

Now, we define the following broken norms for $\phi \in H^s(\mathcal{E}_h)$, $s \geq 2$

$$|||\phi|||_0^2 = \sum_{j=1}^{N_h} \|\phi\|_{0,E_j}^2,$$
$$|||\phi|||_1^2 = \sum_{j=1}^{N_h} \Big(\|\phi\|_{1,E_j}^2 + h_j^2|\phi|_{2,E_j}^2\Big) + J^\sigma(\phi, \phi),$$
$$|||\phi|||_2^2 = \sum_{j=1}^{N_h} \|\phi\|_{2,E_j}^2,$$

where

$$J^{\sigma}(\phi,\phi)=\sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|}\int_{e_k}[\phi][\psi]ds$$

is an interior penalty term and σ is a positive function which takes the constant value σ_k on the edge e_k and is bounded below by $\sigma_0>0$ and above by σ^*.

Let r be a positive integer. The finite element space is taken to be

$$\mathcal{D}_r(\mathcal{E}_h)=\left\{v\in L^2(\Omega):v|_{E_j}\in P_r(E_j),\quad j=1,2,\cdots,N_h\right\}$$

where $P_r(E_j)$ is the set of polynomials of degree less than or equal to r on E_j.

From now on, the symbol C indicates a generic positive constant independent of h and is not necessarily the same in any two places. The following *hp* approximation properties are proved in [2, 3].

Lemma 3.1. *Let $E_j\in\mathcal{E}_h$ and $\phi\in H^s(E_j)$. Then there exist a positive constant C depending on s, γ, ρ, but independent of ϕ, r, and h and a sequence $z_r^h\in P_r(E_j)$, $r=1,2,\cdots$, such that for any $0\le q\le s$,*

$$\|\phi-z_r^h\|_{q,E_j}\le C\frac{h_j^{\mu-q}}{r^{s-q}}\|\phi\|_{s,E_j}\quad s\ge 0,$$

$$\|\phi-z_r^h\|_{0,e_j}\le C\frac{h_j^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}}\|\phi\|_{s,E_j}\quad s>\frac{1}{2},$$

$$\|\phi-z_r^h\|_{1,e_j}\le C\frac{h_j^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}}\|\phi\|_{s,E_j}\quad s>\frac{3}{2},$$

where $\mu=\min(r+1,s)$ and e_j is an edge or a face of E_j.

Lemma 3.2. *For each $E_j\in\mathcal{E}_h$, there exists a positive constant C depending only on γ and ρ such that the following two trace inequalities hold:*

$$\|\phi\|_{0,e_j}^2\le C\left(\frac{1}{h_j}|\phi|_{0,E_j}^2+h_j|\phi|_{1,E_j}^2\right),\quad\forall\phi\in H^1(E_j),$$

$$\left\|\frac{\partial\phi}{\partial\eta_j}\right\|_{0,e_j}^2\le C\left(\frac{1}{h_j}|\phi|_{1,E_j}^2+h_j|\phi|_{2,E_j}^2\right),\quad\forall\phi\in H^1(E_j),$$

where e_j is an edge or a face of E_j and η_j is the unit outward normal vector to e_j.

Now, we define a bilinear functional A on $H^2(\mathcal{E}_h)\times H^2(\mathcal{E}_h)$ by

$$A(\phi,\psi)=\sum_{k=1}^{N_h}(\nabla\phi,\nabla\psi)_{E_k}-\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\phi\cdot n_k\}[\psi]-\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\psi\cdot n_k\}[\phi]+J^{\sigma}(\phi,\psi).$$

Then from (2.1), u satisfies the following weak formulation

$$(u_t, v) + A(u, v) + A(u_t, v) = (f(u), v). \tag{3.1}$$

For a positive integer N, let $\Delta t = \dfrac{T}{N}$ and $t_j = j\Delta t$, $0 \le j \le N$. And for a function $g(x,t)$, we use the following notations:

$$\begin{aligned}
g_j &= g(x, t_j), \quad \text{for } \ 0 \le j \le N,\\
g_{j,\theta} &= \frac{1+\theta}{2} g_{j+1} + \frac{1-\theta}{2} g_j, \quad 0 \le j \le N-1,\\
\Delta_t g_j &= \frac{g_{j+1} - g_j}{\Delta t}, \quad 0 \le j \le N-1,\\
g_{t,j,\theta} &= \left(\frac{\partial g}{\partial t}\right)_{j,\theta}, \quad 0 \le j \le N-1
\end{aligned}$$

where $\theta \in [0,1]$. Now we define a norm as follows

$$\|g\|_{\ell^2(H^1)} = \left(\sum_{j=0}^{N-1} \|g_j\|_1^2\right)^{1/2}.$$

The fully-discrete DG scheme to the problem (2.1) is formulated in the following way: Find $\{U_j\}_{j=0}^{N} \in D_r(\mathcal{E}_h)$ satisfying

$$\begin{cases} (\Delta_t U_j, v) + A(U_{j,\theta}, v) + A(\Delta_t U_j, v) = (f(U_{j,\theta}), v), \quad \forall v \in D_r(\mathcal{E}_h),\\ U(\cdot, 0) = U_0, \end{cases} \tag{3.2}$$

where $U_0 \in D_r(\mathcal{E}_h)$ is an appropriate initial approximation to $u_0(x)$, for example $U_0(x) = \widetilde{u}(x)$ to be defined later. For $\theta = 0$, (3.2) yields the Crank-Nicolson discontinuous Galerkin approximations and for $\theta = 1$, (3.2) yields a backward Euler discontinuous Galerkin approximations.

For $\lambda > 0$, we let

$$A_\lambda(\phi, \psi) = A(\phi, \psi) + \lambda(\phi, \psi), \quad \forall \phi, \psi \in H^2(\mathcal{E}_h).$$

Then the following lemmas can be proved by simple calculations, together with Lemma 3.2 and the definition of $|\!|\!| \cdot |\!|\!|_1$.

Lemma 3.3. *For $\lambda > 0$, there exists a constant C independent of h satisfying*

$$|A_\lambda(\phi, \psi)| \le C |\!|\!|\phi|\!|\!|_1 |\!|\!|\psi|\!|\!|_1, \quad \forall \phi, \psi \in H^2(\mathcal{E}_h).$$

Lemma 3.4. *For a sufficiently large $\sigma > 0$ and $\lambda > 0$, there exists a positive constant β such that*

$$A_\lambda(v,v) \geq \beta |||v|||_1^2, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h).$$

By Lemma 3.3 and Lemma 3.4, if $\lambda > 0$ there exists $\widetilde{u} \in \mathcal{D}_r(\mathcal{E}_h)$ satisfying

$$A_\lambda(u - \widetilde{u}, v) = 0, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h).$$

Theorem 3.1. *For r, $s \geq 2$, there exists a constant C independent of h satisfying the following stastements:*

$$\begin{aligned}
&\text{(i)} && |||u - \widetilde{u}|||_1 \leq C\frac{h^{\mu-1}}{r^{s-2}}\|u\|_s,\\
&\text{(ii)} && \|u - \widetilde{u}\| \leq C\frac{h^{\mu}}{r^{s-2}}\|u\|_s,\\
&\text{(iiii)} && |||u_t - \widetilde{u}_t|||_1 \leq C\frac{h^{\mu-1}}{r^{s-2}}\|u_t\|_s,\\
&\text{(iv)} && \|u_t - \widetilde{u}_t\| \leq C\frac{h^{\mu}}{r^{s-2}}\|u_t\|_s.
\end{aligned}$$

Now we state the following lemma which is essential in the proof of the optimal convergence of the fully discrete approximations in $\ell^\infty(L^2)$ norm. The proof can be found in [8].

Lemma 3.5. *If $\rho_{j,\theta}$ satisfies*

$$\Delta_t\widetilde{u}_j - \widetilde{u}_t(t_{j,\theta}) = (\Delta t)\rho_{j,\theta},$$

then there exists a constant C independent of h such that
(*i*) *if* $0 < \theta \leq 1$, $\quad |||\rho_{j,\theta}|||_1 \leq C\|u_{tt}\|_{L^\infty(H^2)}$;
(*ii*) *if* $\theta = 0$, $\quad |||\rho_{j,\theta}|||_1 \leq C\Delta t\|u_{ttt}\|_{L^\infty(H^2)}$.

Proof. If $\theta \in (0,1]$, then there exist t_j^* and t_j^{**} such that

$$\rho_{j,\theta} = \frac{1+\theta}{2}\widetilde{u}_{tt}(t_j^*) + \frac{1}{2}\widetilde{u}_{tt}(t_j^{**}).$$

Therefore, by Theorem 3.1, we get

$$\begin{aligned}
|||\rho_{j,\theta}|||_1 &\leq C\|\widetilde{u}_{tt}\|_{L^\infty(|||\cdot|||_1)}\\
&\leq C\Big(\|\widetilde{u}_{tt} - u_{tt}\|_{L^\infty(|||\cdot|||_1)} + \|u_{tt}\|_{L^\infty(|||\cdot|||_1)}\Big)\\
&\leq C\|u_{tt}\|_{L^\infty(\|\cdot\|_2)}.
\end{aligned}$$

If $\theta = 0$, then there exist $t^{**}_{j,1}$ and $t^{**}_{j,2}$ such that

$$\rho_{j,\theta} = \Delta t \left\{ \frac{1}{6}\widetilde{u}_{ttt}(t^{**}_{j,1}) - \frac{1}{8}\widetilde{u}_{ttt}(t^{**}_{j,2}) \right\}.$$

By the similar argument as above, we get

$$|\!|\!|\rho_{j,\theta}|\!|\!|_1 \le C\Delta t \|u_{ttt}\|_{L^\infty(\|\cdot\|_2)}.$$

4. Optimal $\ell^\infty(L^2)$ error estimates

Now in order to prove the optimal convergence of $u - U$ in $\ell^\infty(L^2)$ norm, we denote $\eta^n = u(\cdot, t^n) - \widetilde{u}(\cdot, t^n)$ and $\zeta^n = \widetilde{u}(\cdot, t^n) - U^n$.

Theorem 4.1. *If Δt is sufficiently small, then there exist C^* and $\widehat{C}$ independent of h, Δt, and r such that for $\mu = \min(r+1, s)$, $r \ge 1$*

(i) if $0 < \theta \le 1$, then

$$\|U - u\|_{\ell^\infty(L^2)} \le C^* \frac{h^\mu}{r^{s-2}}\Big(\|u\|_{\ell^2(H^s)} + \|u_t\|_{\ell^2(H^s)}\Big) + \widehat{C}\Delta t \frac{h^\mu}{r^{s-2}}(\|u_{tt}\|_{L^\infty(H^2)});$$

(ii) if $\theta = 0$, then

$$\|U - u\|_{\ell^\infty(L^2)} \le C^* \frac{h^\mu}{r^{s-2}}\Big(\|u\|_{\ell^2(H^s)} + \|u_t\|_{\ell^2(H^s)}\Big) + \widehat{C}(\Delta t)^2 \frac{h^\mu}{r^{s-2}}(\|u_{ttt}\|_{L^\infty(H^2)}).$$

Proof. By subtracting (3.2) from (3.1), we get for $j = 0, 1, 2, \cdots, N-1$

$$\begin{aligned}&\left(\frac{\zeta_{j+1} - \zeta_j}{\Delta t}, v\right) + (u_t(t_{j,\theta}), v) - \left(\frac{\widetilde{u}_{j+1} - \widetilde{u}_j}{\Delta t}, v\right) + A(u_{j,\theta}, v) - A(U_{j,\theta}, v)\\ &+ A\Big(u_t(t_{j,\theta}), v\Big) - A(\Delta_t U_j, v) = \Big(f(u_{j,\theta}) - f(U_{j,\theta}), v\Big), \quad \forall v \in D_r(\mathcal{E}_h).\end{aligned} \tag{3.3}$$

From (3.3) and the definitions of η and ζ, we obtain

$$\begin{aligned}&\left(\frac{\zeta_{j+1} - \zeta_j}{\Delta t}, v\right) + A(\zeta_{j,\theta}, v) + A(\Delta_t \zeta_j, v)\\ &= -(u_t(t_{j,\theta}), v) + \left(\frac{\widetilde{u}_{j+1} - \widetilde{u}_j}{\Delta t}, v\right) - A(\eta_{j,\theta}, v) - A(u_t(t_{j,\theta}), v)\\ &\quad + A(\Delta_t \widetilde{u}_j, v) + \Big(f(u_{j,\theta}) - f(U_{j,\theta}), v\Big).\end{aligned} \tag{3.4}$$

We put $v = \zeta_{j,\theta} + \Delta_t\zeta_j$ in (3.4) to get

$$\begin{aligned}
&\left(\frac{\zeta_{j+1}-\zeta_j}{\Delta t}, \zeta_{j,\theta} + \Delta_t\zeta_j\right) + A(\zeta_{j,\theta}, \zeta_{j,\theta}) + A(\zeta_{j,\theta}, \Delta_t\zeta_j) + A(\Delta_t\zeta_j, \zeta_{j,\theta}) \\
&+ A(\Delta_t\zeta_j, \Delta_t\zeta_j) \\
= &- (u_t(t_{j,\theta}), \zeta_{j,\theta} + \Delta_t\zeta_j) + \left(\frac{\widetilde{u}_{j+1}-\widetilde{u}_j}{\Delta t}, \zeta_{j,\theta} + \Delta_t\zeta_j\right) \\
&- A(u_{j,\theta} - \widetilde{u}_{j,\theta}, \zeta_{j,\theta} + \Delta_t\zeta_j) - A(u_t(t_{j,\theta}), \zeta_{j,\theta} + \Delta_t\zeta_j) \\
&+ A(\Delta_t\widetilde{u}_j, \zeta_{j,\theta} + \Delta_t\zeta_j) + \Big(f(u_{j,\theta}) - f(U_{j,\theta}), \zeta_{j,\theta} + \Delta_t\zeta_j\Big).
\end{aligned} \tag{3.5}$$

By the definition of A, we get

$$\begin{aligned}
&(\Delta_t\zeta_j, \zeta_{j,\theta}) + \|\Delta_t\zeta_j\|^2 + A(\zeta_{j,\theta}, \zeta_{j,\theta}) + A(\Delta_t\zeta_j, \Delta_t\zeta_j) + \sum_{k=1}^{N_h}(\nabla\zeta_{j,\theta}, \nabla\Delta_t\zeta_j)_{E_k} \\
&- \sum_{k=1}^{P_h}\int_{e_k}\{\nabla\zeta_{j,\theta}\cdot n_k\}[\Delta_t\zeta_j] - \sum_{k=1}^{P_h}\int_{e_k}\{\nabla\Delta_t\zeta_j\cdot n_k\}[\zeta_{j,\theta}] + J^\sigma(\zeta_{j,\theta}, \Delta_t\zeta_j) \\
&+ \sum_{k=1}^{N_h}(\nabla\Delta_t\zeta_j, \nabla\zeta_{j,\theta})_{E_k} - \sum_{k=1}^{P_h}\int_{e_k}\{\nabla\Delta_t\zeta_j\cdot n_k\}[\zeta_{j,\theta}] \\
&- \sum_{k=1}^{P_h}\int_{e_k}\{\nabla\zeta_{j,\theta}\cdot n_k\}[\Delta_t\zeta_j] + J^\sigma(\Delta_t\zeta_j, \zeta_{j,\theta}) \\
= &- (u_t(t_{j,\theta}), \zeta_{j,\theta} + \Delta_t\zeta_j) + \left(\frac{\widetilde{u}_{j+1}-\widetilde{u}_j}{\Delta t}, \zeta_{j,\theta} + \Delta_t\zeta_j\right) \\
&- A(u_{j,\theta} - \widetilde{u}_{j,\theta}, \zeta_{j,\theta} + \Delta_t\zeta_j) - A(u_t(t_{j,\theta}), \zeta_{j,\theta} + \Delta_t\zeta_j) + A(\Delta_t\widetilde{u}_j, \zeta_{j,\theta} + \Delta_t\zeta_j) \\
&+ \Big(f(u_{j,\theta}) - f(U_{j,\theta}), \zeta_{j,\theta} + \Delta_t\zeta_j\Big).
\end{aligned}$$

From the equation above, we obtain

$$\begin{aligned}
&(\Delta_t\zeta_j, \zeta_{j,\theta}) + \|\Delta_t\zeta_j\|^2 + A_\lambda(\zeta_{j,\theta}, \zeta_{j,\theta}) + A_\lambda(\Delta_t\zeta_j, \Delta_t\zeta_j) \\
&+ 2\sum_{k=1}^{N_h}(\nabla\zeta_{j,\theta}, \nabla\Delta_t\zeta_j)_{E_k} - 2\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\zeta_{j,\theta}\cdot n_k\}[\Delta_t\zeta_j] \\
&- 2\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\Delta_t\zeta_j\cdot n_k\}[\zeta_{j,\theta}] + 2J^\sigma(\zeta_{j,\theta}, \Delta_t\zeta_j) \\
= &- (u_t(t_{j,\theta}), \zeta_{j,\theta} + \Delta_t\zeta_j) + (\Delta_t\widetilde{u}_j, \zeta_{j,\theta} + \Delta_t\zeta_j) - A(u_{j,\theta} - \widetilde{u}_{j,\theta}, \zeta_{j,\theta} + \Delta_t\zeta_j) \\
&- A(u_t(t_{j,\theta}), \zeta_{j,\theta} + \Delta_t\zeta_j) + A(\Delta_t\widetilde{u}_j, \zeta_{j,\theta} + \Delta_t\zeta_j) \\
&+ (f(u_{j,\theta}) - f(U_{j,\theta}), \zeta_{j,\theta} + \Delta_t\zeta_j) + \lambda(\zeta_{j,\theta}, \zeta_{j,\theta}) + \lambda(\Delta_t\zeta_j, \Delta_t\zeta_j).
\end{aligned} \tag{3.6}$$

By simple calculations, we get

$$
\begin{aligned}
(\Delta_t\zeta_j, \zeta_{j,\theta}) &= \left(\frac{\zeta_{j+1}-\zeta_j}{\Delta t}, \frac{1+\theta}{2}\zeta_{j+1} + \frac{1-\theta}{2}\zeta_j\right) \\
&= \frac{1}{2\Delta t}\big[(\zeta_{j+1}, (1+\theta)\zeta_{j+1}) + (\zeta_{j+1}, (1-\theta)\zeta_j) - (\zeta_j, (1+\theta)\zeta_{j+1}) \\
&\quad - (\zeta_j, (1-\theta)\zeta_j)\big] \\
&= \frac{1}{2\Delta t}\left[(1+\theta)\|\zeta_{j+1}\|^2 - 2\theta(\zeta_j, \zeta_{j+1}) - (1-\theta)\|\zeta_j\|^2\right] \\
&\geq \frac{1}{2\Delta t}\left[(1+\theta)\|\zeta_{j+1}\|^2 - \theta\|\zeta_j\|^2 - \theta\|\zeta_{j+1}\|^2 - (1-\theta)\|\zeta_j\|^2\right] \\
&= \frac{1}{2\Delta t}\left[\|\zeta_{j+1}\|^2 - \|\zeta_j\|^2\right].
\end{aligned}
\tag{3.7}
$$

The coercivity of A_λ in Lemma 3.4 implies that

$$
A_\lambda(\zeta_{j,\theta}, \zeta_{j,\theta}) + A_\lambda(\Delta_t\zeta_j, \Delta_t\zeta_j) \geq C(|||\zeta_{j,\theta}|||_1^2 + |||\Delta_t\zeta_j|||_1^2). \tag{3.8}
$$

By simple calculations, we get the following two inequalities

$$
\begin{aligned}
\sum_{k=1}^{N_h}(\nabla\zeta_{j,\theta}, \nabla\Delta_t\zeta_j)_{E_k} &= \sum_{k=1}^{N_h}\left(\frac{\nabla\zeta_{j+1}-\nabla\zeta_j}{\Delta t}, \frac{1+\theta}{2}\nabla\zeta_{j+1} + \frac{1-\theta}{2}\nabla\zeta_j\right)_{E_k} \\
&\geq \frac{1}{2\Delta t}(|||\nabla\zeta_{j+1}|||_0^2 - |||\nabla\zeta_j|||_0^2)
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
J^\sigma(\zeta_{j,\theta}, \Delta_t\zeta_j) &= \sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|}\int_{e_k}[\zeta_{j,\theta}][\Delta_t\zeta_j]ds \\
&= \sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|}\int_{e_k}\left[\frac{1+\theta}{2}\zeta_{j+1} + \frac{1-\theta}{2}\zeta_j\right]\left[\frac{\zeta_{j+1}-\zeta_j}{\Delta t}\right]ds \\
&\geq \frac{1}{2\Delta t}\left[J^\sigma(\zeta_{j+1}, \zeta_{j+1}) - J^\sigma(\zeta_j, \zeta_j)\right].
\end{aligned}
\tag{3.10}
$$

Substituting the inequalites (3.7)-(3.10) into (3.6), we get

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left[\|\zeta_{j+1}\|^2 - \|\zeta_j\|^2\right] + C(|||\xi_{j,\theta}|||_1^2 + |||\Delta_t\zeta_j|||_1^2) + \|\Delta_t\zeta_j\|^2 \\
&+ \frac{1}{2\Delta t}\left[|||\nabla\zeta_{j+1}|||_0^2 - |||\nabla\zeta_j|||_0^2\right] + \frac{2}{2\Delta t}\left[J^\sigma(\zeta_{j+1}, \zeta_{j+1}) - J^\sigma(\zeta_j, \zeta_j)\right]
\end{aligned}
$$

$$\begin{aligned}
\le & -(u_t(t_{j,\theta}),\zeta_{j,\theta}+\Delta_t\zeta_j)+(\Delta_t\widetilde{u}_j,\zeta_{j,\theta}+\Delta_t\zeta_j)-A\Big(u_{j,\theta}-\widetilde{u}_{j,\theta},\zeta_{j,\theta}+\Delta_t\zeta_j\Big)\\
& -A\Big(u_t(t_{j,\theta}),\zeta_{j,\theta}+\Delta_t\zeta_j\Big)+A(\Delta_t\widetilde{u}_j,\zeta_{j,\theta}+\Delta_t\zeta_j)\\
& +\Big(f(u_{j,\theta})-f(U_{j,\theta}),\zeta_{j,\theta}+\Delta_t\zeta_j\Big)+\lambda(\zeta_{j,\theta},\zeta_{j,\theta})+\lambda(\Delta_t\zeta_j,\Delta_t\zeta_j)\\
& +2\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\zeta_{j,\theta}\cdot n_k\}[\Delta_t\zeta_j]+2\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\Delta_t\zeta_j\cdot n_k\}[\zeta_{j,\theta}].
\end{aligned} \tag{3.11}$$

Notice that

$$\begin{aligned}
-u_t(t_{j,\theta})+\Delta_t\widetilde{u}_j &= -\eta_t(t_{j,\theta})-\widetilde{u}_t(t_{j,\theta})+\Delta_t\widetilde{u}_j\\
&= -\eta_t(t_{j,\theta})+\Delta t\rho_{j,\theta}.
\end{aligned} \tag{3.12}$$

From (3.11), (3.12), and the definition of η, we obtain

$$\begin{aligned}
& \frac{1}{2\Delta t}\big[\|\zeta_{j+1}\|^2-\|\zeta_j\|^2\big]+C(|||\zeta_{j,\theta}|||_1^2+|||\Delta_t\zeta_j|||_1^2)+\|\Delta_t\zeta_j\|^2\\
& +\frac{1}{2\Delta t}\big[|||\nabla\zeta_{j+1}|||^2-|||\nabla\zeta_j|||^2\big]+\frac{2}{2\Delta t}\big[J^\sigma(\zeta_{j+1},\zeta_{j+1})-J^\sigma(\zeta_j,\zeta_j)\big]\\
\le & -\Big(\eta_t(t_{j,\theta}),\zeta_{j,\theta}+\Delta_t\zeta_j\Big)+(\Delta t\rho_{j,\theta},\zeta_{j,\theta}+\Delta_t\zeta_j)-A(\eta_{j,\theta},\zeta_{j,\theta}+\Delta_t\zeta_j)\\
& -A\Big(\eta_t(t_{j,\theta}),\zeta_{j,\theta}+\Delta_t\zeta_j\Big)+A(\Delta t\rho_{j,\theta},\zeta_{j,\theta}+\Delta_t\zeta_j)\\
& +\Big(f(u_{j,\theta})-f(U_{j,\theta}),\zeta_{j,\theta}+\Delta_t\zeta_j\Big)+\lambda(\zeta_{j,\theta},\zeta_{j,\theta})+\lambda(\Delta_t\zeta_j,\Delta_t\zeta_j)\\
& +2\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\zeta_{j,\theta}\cdot n_k\}[\Delta_t\zeta_j]+2\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\Delta_t\zeta_j\cdot n_k\}[\zeta_{j,\theta}]
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
= & -\Big(\eta_t(t_{j,\theta}),\zeta_{j,\theta}+\Delta_t\zeta_j\Big)+(\Delta t\rho_{j,\theta},\zeta_{j,\theta}+\Delta_t\zeta_j)+\lambda(\eta_{j,\theta},\zeta_{j,\theta}+\Delta_t\zeta_j)\\
& +\lambda\Big(\eta_t(t_{j,\theta}),\zeta_{j,\theta}+\Delta_t\zeta_j\Big)+A(\Delta t\rho_{j,\theta},\zeta_{j,\theta}+\Delta_t\zeta_j)\\
& +\Big(f(u_{j,\theta})-f(U_{j,\theta}),\zeta_{j,\theta}+\Delta_t\zeta_j\Big)+\lambda(\zeta_{j,\theta},\zeta_{j,\theta})+\lambda(\Delta_t\zeta_j,\Delta_t\zeta_j)\\
& +2\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\zeta_{j,\theta}\cdot n_k\}[\Delta_t\zeta_j]+2\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\Delta_t\zeta_j\cdot n_k\}[\zeta_{j,\theta}]\\
= & \sum_{i=1}^{10} I_i.
\end{aligned} \tag{3.14}$$

Now for sufficiently small $\varepsilon > 0$, we obtain the following estimates of I_i:

$$
\begin{aligned}
|I_1| &\le \|\eta_t(t_{j,\theta})\|\Big(\|\zeta_{j,\theta}\| + \|\Delta_t\zeta_j\|\Big),\\
&\le C\|\eta_t(t_{j,\theta})\|^2 + \varepsilon\Big(\|\zeta_{j,\theta}\|^2 + \|\Delta_t\zeta_j\|^2\Big),\\
|I_2| &\le \|\Delta t\rho_{j,\theta}\|\|\Big(\|\zeta_{j,\theta}\| + \|\Delta_t\zeta_j\|\Big)\\
&\le C(\Delta t)^2\|\rho_{j,\theta}\|^2 + \varepsilon\Big(\|\zeta_{j,\theta}\|^2 + \|\Delta_t\zeta_j\|^2\Big),\\
|I_3| &\le C\|\eta_{j,\theta}\|^2 + \varepsilon\Big(\|\zeta_{j,\theta}\|^2 + \|\Delta_t\zeta_j\|^2\Big),\\
|I_4| &\le C\|\eta_t(t_{j,\theta})\|^2 + \varepsilon\Big(\|\zeta_{j,\theta}\|^2 + \|\Delta_t\zeta_j\|^2\Big),\\
|I_5| &\le C(\Delta t)^2|\!|\!|\rho_{j,\theta}|\!|\!|_1^2 + \varepsilon\Big(|\!|\!|\zeta_{j,\theta}|\!|\!|_1^2 + |\!|\!|\Delta_t\zeta_j|\!|\!|_1^2\Big),\\
|I_6| &\le K(\|\eta_{j,\theta}\| + \|\zeta_{j,\theta}\|)\Big(\|\zeta_{j,\theta}\| + \|\Delta_t\zeta_j\|\Big),\\
&\le C\Big(\|\eta_{j,\theta}\|^2 + \|\zeta_{j,\theta}\|^2\Big) + \varepsilon\|\Delta_t\zeta_j\|^2,\\
|I_7| &\le \lambda\|\zeta_{j,\theta}\|^2,\\
|I_8| &\le \lambda\|\Delta_t\zeta_j\|^2,\\
|I_9| &\le C|\!|\!|\nabla\zeta_{j,\theta}|\!|\!|_0^2 + \varepsilon|\!|\!|\Delta_t\zeta_j|\!|\!|_1^2,\\
|I_{10}| &\le CJ^\sigma(\zeta_{j,\theta},\zeta_{j,\theta}) + \varepsilon|\!|\!|\Delta_t\zeta_j|\!|\!|_0^2,
\end{aligned}
$$

for each $1 \le i \le 10$. Substituting the estimation of I_i $1 \le i \le 10$ into (3.13), we get

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left[\|\zeta_{j+1}\|^2 - \|\zeta_j\|^2\right] + C(|\!|\!|\zeta_{j,\theta}|\!|\!|_1^2 + |\!|\!|\Delta_t\zeta_j|\!|\!|_1^2) + \|\Delta_t\zeta_j\|^2\\
&\quad + \frac{1}{2\Delta t}\left[|\!|\!|\nabla\zeta_{j+1}|\!|\!|_0^2 - |\!|\!|\nabla\zeta_j|\!|\!|_0^2\right] + \frac{2}{2\Delta t}\left[J^\sigma(\zeta_{j+1},\zeta_{j+1}) - J^\sigma(\zeta_j,\zeta_j)\right]\\
&\le C\Big[\|\eta_t(t_{j,\theta})\|^2 + (\Delta t)^2|\!|\!|\rho_{j,\theta}|\!|\!|_1^2 + \|\eta_{j,\theta}\|^2 + \|\zeta_{j,\theta}\|^2 + |\!|\!|\nabla\zeta_{j,\theta}|\!|\!|_0^2 + J^\sigma(\zeta_{j,\theta},\zeta_{j,\theta})\Big]\\
&\le C\Big[\|\eta_{j,\theta}\|^2 + \|\eta_t(t_{j,\theta})\|^2 + (\Delta t)^2\|\rho_{j,\theta}\|^2 + \|\zeta_j\|^2 + \|\zeta_{j+1}\|^2 + |\!|\!|\nabla\zeta_j|\!|\!|_0^2\\
&\quad + |\!|\!|\nabla\zeta_{j+1}|\!|\!|_0^2 + J^\sigma(\zeta_{j,\theta},\zeta_{j,\theta})\Big],
\end{aligned}
\tag{3.15}
$$

for sufficiently small $\lambda > 0$. From (3.14), we have

$$
\begin{aligned}
&\|\zeta_{j+1}\|^2 - \|\zeta_j\|^2 + C(\Delta t)\Big(|\!|\!|\zeta_{j,\theta}|\!|\!|_1^2 + |\!|\!|\Delta_t\zeta_j|\!|\!|_1^2\Big) + 2(\Delta t)\|\Delta_t\zeta_j\|_1^2\\
&\quad + \Big(|\!|\!|\nabla\zeta_{j+1}|\!|\!|_0^2 - |\!|\!|\nabla\zeta_j|\!|\!|_0^2\Big) + (a+b)\Big(J^\sigma(\zeta_{j+1},\zeta_{j+1}) - J^\sigma(\zeta_j,\zeta_j)\Big)
\end{aligned}
$$

$$\le C(\Delta t)\Big[\|\eta_{j,\theta}\|^2 + \|\eta_t(t_{j,\theta})\|^2 + (\Delta t)^2 |\!|\!|\rho_{j,\theta}|\!|\!|_1^2 + \|\zeta_j\|^2 + \|\zeta_{j+1}\|^2 + |\!|\!|\nabla\zeta_j|\!|\!|_0^2 + |\!|\!|\nabla\zeta_{j+1}|\!|\!|_0^2 + J^\sigma(\zeta_{j+1},\zeta_{j+1}) + J^\sigma(\zeta_j,\zeta_j)\Big]. \tag{3.16}$$

Summing (3.15) from $j = 0$ to $j = N-1$, we obtain

$$\|\zeta_N\|^2 + 2|\!|\!|\nabla\zeta_N|\!|\!|_0^2 + 2J^\sigma(\zeta_N,\zeta_N) + C(\Delta t)\sum_{j=0}^{N-1}\{|\!|\!|\zeta_{j,\theta}|\!|\!|_1^2 + |\!|\!|\Delta_t\zeta_j|\!|\!|_1^2\}$$

$$\le \|\zeta_0\|^2 + 2|\!|\!|\nabla\zeta_0|\!|\!|_0^2 + 2J^\sigma(\zeta_0,\zeta_0) + C(\Delta t)\sum_{j=1}^{N}[\|\zeta_j\|^2 + |\!|\!|\nabla\zeta_j|\!|\!|_0^2 + J^\sigma(\zeta_j,\zeta_j)]$$

$$+ C(\Delta t)\sum_{j=0}^{N}[\|\eta_j\|^2 + \|\eta_t(t_{j,\theta})\|^2 + (\Delta t)^2|\!|\!|\rho_{j,\theta}|\!|\!|_1^2].$$

If Δt is sufficiently small, then by the discrete version of Gronwall's lemma we obtain

$$\|\zeta_N\|^2 + |\!|\!|\nabla\zeta_N|\!|\!|_0^2 + J^\sigma(\zeta_N,\zeta_N) + C(\Delta t)\sum_{j=0}^{N-1}\{|\!|\!|\zeta_{j,\theta}|\!|\!|_1^2 + |\!|\!|\Delta_t\zeta_j|\!|\!|_1^2\}$$

$$\le C\Big\{\|\zeta_0\|^2 + 2|\!|\!|\nabla\zeta_0|\!|\!|_0^2 + 2J^\sigma(\zeta_0,\zeta_0) + C(\Delta t)\sum_{j=0}^{N}\Big[\|\eta_j\|^2 + \|\eta_t(t_{j,\theta})\|^2 + (\Delta t)^2|\!|\!|\rho_{j,\theta}|\!|\!|_1^2\Big]\Big\}.$$

Choosing an appropriate initial approximation $U^0(x)$ and usinging Lemma 3.5, we get the following results.

(i) If $0 < \theta \le 1$, then

$$\|\zeta_N\|^2 + |\!|\!|\nabla\zeta_N|\!|\!|_0^2 + J^\sigma(\zeta_N,\zeta_N) + C(\Delta t)\sum_{j=0}^{N-1}\{|\!|\!|\zeta_{j,\theta}|\!|\!|_1^2 + |\!|\!|\Delta_t\zeta_j|\!|\!|_1^2\}$$

$$\le C(\Delta t)\frac{h^{2\mu}}{r^{2(s-2)}}\sum_{j=0}^{N}[\|u_j\|_{H^s}^2 + \|u_t(t_{j,\theta})\|_{H^s}^2 + (\Delta t)^2|\!|\!|u_{tt}|\!|\!|_{L^\infty((t_j,t_{j+1}),H^2)}^2].$$

(ii) If $\theta = 0$, then

$$\|\zeta_N\|^2 + |\!|\!|\nabla\zeta_N|\!|\!|_0^2 + J^\sigma(\zeta_N,\zeta_N) + C(\Delta t)\sum_{j=0}^{N-1}\{|\!|\!|\zeta_{j,\theta}|\!|\!|_1^2 + |\!|\!|\Delta_t\zeta_j|\!|\!|_1^2\}$$

$$\le C(\Delta t)\frac{h^{2\mu}}{r^{2(s-2)}}\sum_{j=1}^{N}[\|u_j\|_{H^s}^2 + \|u_t(t_{j,\theta})\|_{H^s}^2] + (\Delta t)^4|\!|\!|u_{ttt}|\!|\!|_{L^\infty((t_j,t_{j+1}),H^2)}.$$

This implies the results of Theorem 4.1.

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