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MULTIPLICITY OF POSITIVE SOLUTIONS FOR MULTIPOINT BOUNDARY VALUE PROBLEMS WITH ONE-DIMENSIONAL P-LAPLACIAN

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ABSTRACT. In this paper, we consider the multipoint boundary value problem for the one-dimensional p-Laplacian

$$(\phi_p(u'))'(t)+q(t)f(t,u(t),u'(t))=0,\ \ t\in(0,1),$$

subject to the boundary conditions:

$$u(0)=\sum_{i=1}^{n-2}\alpha_i u(\xi_i),\ \ u(1)=\sum_{i=1}^{n-2}\beta_i u(\xi_i),$$

where $\phi_p(s)=|s|^{n-2}s$, $p>1$, $\xi_i\in(0,1)$ with $0<\xi_1<\xi_2<\cdots<\xi_{n-2}<1$ and α_i, $\beta_i\in[0,1)$, $0<\Sigma_{i=1}^{n-2}\alpha_i,\Sigma_{i=1}^{n-2}\beta_i<1$. Using a fixed point theorem due to Bai and Ge, we study the existence of at least three positive solutions to the above boundary value problem. The important point is that the nonlinear term f explicitly involves a first-order derivative.

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1. Introduction

In this paper, we study the existence of multiple positive solutions to the boundary value problem (BVP for short) for the one-dimensional p-Laplacian

$$(\phi_p(u'))'(t)+q(t)f(t,u(t),u'(t))=0,\ \ t\in(0,1), \tag{1.1}$$

$$u(0)=\sum_{i=1}^{n-2}\alpha_i u(\xi_i),\ \ u(1)=\sum_{i=1}^{n-2}\beta_i u(\xi_i), \tag{1.2}$$

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where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$ and α_i, β_i, f satisfy

(H_1) α_i, $\beta_i \in [0,\infty)$ satisfy $0 < \Sigma_{i=1}^{n-2}\alpha_i$, $\Sigma_{i=1}^{n-2}\beta_i < 1$;

(H_2) $f \in C([0,1] \times [0,+\infty) \times R, (0,+\infty))$;

(H_3) $q(t) \in L^1[0,1]$ is nonnegative on $(0,1)$ and q(t) is not identically zero on any subinterval of $(0,1)$. Furthermore $q(t)$ satisfies $0 < \int_0^1 q(t)dt < \infty$.

In this paper, a positive of solution of (1.1) and (1.2) means a solution u satisfying $u(t) > 0$, $0 < t < 1$.

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev in [1]. Since then there has been much current attention focused on the study of nonlinear multipoint boundary value problems, see [2-7].

Recently, it was found that the non-linear term f does not depend on the first-order derivative for the following p-Laplacian differential equation:

$$(\phi_p(u'))'(t) + q(t)f(t,u(t)) = 0, \quad t \in (0,1), \tag{1.3}$$

$$u(0) = \sum_{i=1}^{n-2}\alpha_i u(\xi_i), \quad u(1) = \sum_{i=1}^{n-2}\beta_i u(\xi_i). \tag{1.4}$$

Ma et al. in [8] proved the existence of positive solutions of (1.3) and (1.4), via monotone iterative technique.

More recently, using a fixed point theorem due to Avery and Peterson, Wang and Ge in [9] provide sufficient conditions for the existence of multiple positive solutions to Eq. (1.1) subject to one of the following boundary conditions:

$$u(0) = \sum_{i=1}^{n-2}\alpha_i u(\xi_i), \quad u'(1) = \sum_{i=1}^{n-2}\beta_i u(\xi_i),$$

$$u'(0) = \sum_{i=1}^{n-2}\alpha_i u(\xi_i), \quad u(1) = \sum_{i=1}^{n-2}\beta_i u(\xi_i),$$

and Feng and Ge in [10] provide sufficient conditions for the existence of multiple positive solutions to Eq. (1.1) subject to the following boundary condition:

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{n-2}\beta_i u(\xi_i).$$

By applying monotone iterative techniques, Sun, Qu and Ge in [11] provides two approximating the solutions for the multipoint BVP (1.1) and (1.2).

Our purpose in this paper is to show the existence of at least three positive solutions to the multipoint BVP (1.1) and (1.2) by using fixed point theorem due to Bai and Ge.

2. Preliminaries

For the convenience of readers, we provide some background material from the theory of cones in Banach spaces. We also state in this section the fixed point theorem due to Bai and Ge.

Definition 2.1. Let E be a real Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone provided that
(i) $au \in P$ for all $u \in P$ and all $a \geq 0$ and
(ii) $u, -u \in P$ implies $u = 0$.
Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. The map ψ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\psi : P \to [0, +\infty)$ is continuous and

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map α is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Definition 2.3. Let $r > a > 0$ and $L > 0$ be constants, ψ a nonnegative continuous concave functional and α, β nonnegative continuous convex functionals on the cone P. Define the following convex sets:

$P(\alpha, r; \beta, L) = \{u \in P | \alpha(u) < r, \beta(u) < L\}$,
$\overline{P}(\alpha, r; \beta, L) = \{u \in P | \alpha(u) \leq r, \beta(u) \leq L\}$,
$P(\alpha, r; \beta, L; \psi, a) = \{u \in P | \alpha(u) < r, \beta(u) < L, \psi(u) > a\}$,
$\overline{P}(\alpha, r; \beta, L; \psi, a) = \{u \in P | \alpha(u) \leq r, \beta(u) \leq L, \psi(u) \geq a\}$.

The following assumptions as regards the nonnegative continuous convex functionals α, β are used:
$(B1)$ there exists $M > 0$ such that $\|x\| \leq M \max\{\alpha(x), \beta(x)\}$, for all $x \in P$;
$(B2)$ $P(\alpha, r; \beta, L) \neq \emptyset$, for any $r > 0$ and $L > 0$.

To prove our results, we need the following fixed point theorem due to Bai and Ge in [12].

Theorem 2.1. *Let E be a Banach space, $P \subset E$ be a cone and $r_2 \geq d > b > r_1 > 0$, $L_2 \geq L_1 > 0$. Assume that α and β are nonnegative continuous convex functionals satisfying $(B1)$ and $(B2)$, ψ is nonnegative continuous concave functional on P such that $\psi(y) \leq \alpha(y)$ for all $y \in \overline{P}(\alpha, r_2; \beta, L_2)$ and $T : \overline{P}(\alpha, r_2; \beta, L_2) \to (\alpha, r_2; \beta, L_2)$ is a completely continuous operator. Suppose that*

(S_1) $\{y \in \overline{P}(\alpha, d; \beta, L_2; \psi, b) | \psi(y) > b\} \neq \emptyset$, $\psi(Ty) > b$ for $y \in \overline{P}(\alpha, d; \beta, L_2; \psi, b)$,

(S_2) $\alpha(Ty) < r_1$, $\beta(Ty) < L_1$ for all $y \in \overline{P}(\alpha, r_1; \beta, L_1)$ and

(S_3) *$\psi(Ty) > b$ for all $y \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b)$ with $\alpha(Ty) > d$.*
Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P}(\alpha, r_2; \beta, L_2)$ with

$$y_1 \in P(\alpha, r_1; \beta, L_1),\ \ y_2 \in \{\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) | \psi(y) > b\},$$
$$y_3 \in P(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_2)).$$

3. Related lemmas

We consider the Banach space $E = (C^1[0,1], \|\cdot\|)$ with the maximum norm

$$\|x\| = \max\{\max_{t\in[0,1]} |x(t)|, \max_{t\in[0,1]} |x'(t)|\}.$$

Define the cone $P \in E$ by

$$P = \{x \in E | x(t) \geq 0, x \text{ is concave on } [0,1]\}.$$

It follows from (H_3) that there exists a natural number $k > \max\{\frac{1}{\xi_1}, \frac{2}{1-\xi_{n-2}}\}$ such that $0 < \int_{\frac{1}{k}}^{1-\frac{1}{k}} q(t)dt < \infty$. Let the nonnegative continuous concave functional ψ, the nonnegative continuous convex functionals α, β, be defined on the cone P by

$$\alpha(x) = \max_{0\leq t\leq 1} |x(t)|,\ \beta(x) = \max_{0\leq t\leq 1} |x'(t)|, \psi(x) = \min_{\frac{1}{k}\leq t\leq 1-\frac{1}{k}} |x(t)| \text{ for } x \in P,$$

respectively. Then on the cone P, α and β are convex functionals satisfy $(B1)$ and $(B2)$, while ψ is a concave functional with $\frac{1}{k}\alpha(x) \leq \psi(x) \leq \alpha(x)$, for $x \in P$.

For any $x \in C^{1+}[0,1] = \{x \in C^1[0,1] : x(t) \geq 0\}$, we consider the problem,

$$(\phi_p(u'))'(t) + q(t)f(t, x(t), x'(t)) = 0,\ \ t \in (0,1), \tag{3.1}$$

$$u(0) = \sum_{i=1}^{n-2} \alpha_i u(\xi_i),\ \ u(1) = \sum_{i=1}^{n-2} \beta_i u(\xi_i). \tag{3.2}$$

Lemma 3.1. *Let $(H_1) - (H_2)$ hold. Then for any $x \in C^{1+}[0,1]$, the problem (3.1) and (3.2) has a unique solution $u(t)$ give by*

$$u(t) = \frac{\sum_{i=1}^{n-2} \alpha_i}{1-\sum_{i=1}^{n-2} \alpha_i} \int_0^{\xi_i} \phi_q(A_x - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau)ds$$
$$+ \int_0^t \phi_q(A_x - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau)ds,$$

where A_x satisfies

$$\begin{aligned}
&\frac{1}{1-\sum_{i=1}^{n-2} \alpha_i} \sum_{i=1}^{n-2} \alpha_i \int_0^{\xi_i} \phi_q(A_x - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau)ds \\
&+ \frac{1}{1-\sum_{i=1}^{n-2} \beta_i} \int_0^1 \phi_q(A_x - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau)ds \\
&- \frac{1}{1-\sum_{i=1}^{n-2} \beta_i} \sum_{i=1}^{n-2} \beta_i \int_0^{\xi_i} \phi_q(A_x - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau)ds = 0.
\end{aligned} \tag{3.3}$$

Then there exists a $A_x \in (0, \int_0^1 q(s)f(s, x(s), x'(s))ds)$ satisfying (3.3). This implies that there is a unique $\sigma_x \in (0,1)$, such that $A_x = \int_0^{\sigma_x} q(s)f(s, x(s), x'(s))ds$.

Proof. For any $f(t,x(t),x'(t))>0$, define

$$\begin{aligned}H_x(c) &= \frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\sum_{i=1}^{n-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(c-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &+\frac{1}{1-\sum_{i=1}^{n-2}\beta_i}\int_0^1\phi_q\Big(c-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &-\frac{1}{1-\sum_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_0^{\xi_i}\phi_q\Big(c-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &=\frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\sum_{i=1}^{n-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(c-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &+\frac{1}{1-\sum_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1\phi_q\Big(c-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &+\int_0^1\phi_q\Big(c-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds.\end{aligned}$$

Then, $H_x:R\to R$ is continuous and strictly increasing.

$$H_x(0)<0,\ H_x\Big(\int_0^1 q(s)f(s,x(s),x'(s))\Big)ds>0,$$

implies the existence of a unique $c\in(0,\int_0^1 q(s)f(s,x(s),x'(s))ds)$, such that $H_x(c)=0$. Then the existence of $\sigma_x\in(0,1)$ is obvious. □

Lemma 3.2. *Let $(H_1)-(H_2)$ hold. Then for any $x\in C^{1+}[0,1]=\{x\in C^1[0,1]: x(t)\ge 0\}$, the solution of BVP (3.1) and (3.2) can also be expressed:*

$$\begin{aligned}u(t) &= -\frac{\sum_{i=1}^{n-2}\beta_i}{1-\sum_{i=1}^{n-2}\beta_i}\int_{\xi_i}^1\phi_q\Big(B_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &-\int_t^1\phi_q\Big(B_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds,\end{aligned}\tag{3.4}$$

where B_x satisfies

$$\begin{aligned}&-\frac{1}{1-\sum_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1\phi_q\Big(B_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &-\frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\int_0^1\phi_q\Big(B_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &+\frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\sum_{i=1}^{n-2}\alpha_i\int_{\xi_i}^1\phi_q\Big(B_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds=0.\end{aligned}\tag{3.5}$$

Then there exists a $B_x\in(0,\int_0^1 q(s)f(s,x(s),x'(s))ds)$ satisfying (3.5). This implies that there is a unique $\tau_x\in(0,1)$, such that $B_x=\int_0^{\tau_x}q(s)f(s,x(s),x'(s))ds$.

Proof. The proof is similar to Lemma 3.1. we omit it here. □

Lemma 3.3. *If $x\in C^{1+}[0,1]$, then the unique solution $u(t)$ of the BVP (3.1) and (3.2) has the following properties:*

(*i*) *the graph of $u(t)$ is concave;*

(*ii*) *$u(t)\ge 0$;*

(*iii*) *there exists a unique $t_0\in(0,1)$ such that $u(t_0)=\max_{0\le t\le 1}u(t)$, moreover $u'(t_0)=0$;*

(*iv*) *$\sigma_x=\tau_x=t_0$.*

Proof. Suppose that $u(t)$ is the solution of (3.1) and (3.2). Then

(i) From the fact that $(\phi_p(u'))' = -q(t)f(t,x(t),x(t')) \le 0$, we know that $(\phi_p(u'))$ is nonincreasing. It follows that $u'(t)$ is also nonincreasing. Thus, we know the graph of $u(t)$ is concave down on (0,1).

(ii) Suppose $u(0) = \min\{u(0), u(1)\}$, property (i) implies $u(\xi_i) \ge u(0)$. Thus from $u(0) = \sum_{i=1}^{n-2} \alpha_i u(\xi_i) \ge \sum_{i=1}^{n-2} \alpha_i u(0)$, we have $u(0) \ge 0$. So, $u(t) \ge 0$, $t \in [0,1]$.

(iii) From $u(0) - \sum_{i=1}^{n-2} \alpha_i u(\xi_i) = 0 \le (1 - \sum_{i=1}^{n-2} \alpha_i)u(0)$, we have $\exists \xi_i$, st. $u(\xi_i) \ge u(0)$. similarly, from $u(1) - \sum_{i=1}^{n-2} \beta_i u(\xi_i) = 0 \le (1 - \sum_{i=1}^{n-2} \beta_i)u(1)$, we have $\exists \xi_i$, st. $u(\xi_i) \ge u(1)$. So there is a $t_0 \in (0,1)$, such that $u(t_0) = \max_{0\le t\le 1} u(t)$, i.e., $u'(t_0) = 0$. If there exist t_1, $t_2 \in (0,1)$, $t_1 < t_2$, such that $u'(t_1) = 0 = u'(t_2)$, then

$$0 = \phi_p(u'(t_2)) - \phi_p(u'(t_1)) = -\int_{t_1}^{t_2} q(s)f(s,x(s),x'(s))ds < 0,$$

this is a contradiction.

(iv) From Lemmas 3.1 and 3.2 , we have $u'(t) = \phi_q(\int_t^{\sigma_x} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)$ and $u'(t) = \phi_q$ $(\int_t^{\tau_x} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)$, so $u'(\sigma_x) = u'(\tau_x) = u'(t_0) = 0$. Therefore $\sigma_x = \tau_x = t_0$. □

Lemma 3.4. *Let $(H_1) - (H_2)$ hold. For any $u \in P$, define the operator,*

$$(Tu)(t) = \begin{cases} \frac{\sum_{i=1}^{n-2}\alpha_i}{1-\sum_{i=1}^{n-2}\alpha_i} \int_0^{\xi_i} \phi_q(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds \\ +\int_0^t \phi_q(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds, \ 0 \le t \le \sigma_u, \\ \frac{\sum_{i=1}^{n-2}\beta_i}{1-\sum_{i=1}^{n-2}\beta_i} \int_{\xi_i}^1 \phi_q(\int_{\sigma_u^s} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds \\ +\int_t^1 \phi_q(\int_{\sigma_u}^s q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds, \ \sigma_u \le t \le 1. \end{cases}$$

Then $T: P \to P$ is completely continuous.

4. Existence of triple positive solutions to (1.1) and (1.3).

We are now to apply the fixed point theorem due to Bai and Ge to the operator T in order to get sufficient conditions for the existence of at least three positive solutions to BVP (1.1) and (1.2).

Now for convenience we introduce the following notations. Let

$$t^* = \frac{\xi_{n-2}+1}{2}, \quad K = \phi_q(\int_0^1 q(\tau)d\tau),$$

$$M = \min\{\int_{\frac{1}{k}}^{t^*} \phi_q(\int_s^{t^*} q(\tau)d\tau)ds, \int_{t^*}^{1-\frac{1}{k}} \phi_q(\int_{t^*}^s q(\tau)d\tau)ds\},$$

$$N = \max_{0\le t\le 1}\{\int_0^{t^*} \phi_q(\int_s^{t^*} q(\tau)d\tau)ds + \frac{\sum_{i=1}^{n-2}\alpha_i}{1-\sum_{i=1}^{n-2}\alpha_i} \int_0^{\xi_i} \phi_q(\int_s^{\sigma_u} q(\tau)d\tau)ds,$$
$$\int_{t^*}^1 \phi_q(\int_{t^*}^s q(\tau)d\tau)ds + \frac{\sum_{i=1}^{n-2}\beta_i}{1-\sum_{i=1}^{n-2}\beta_i} \int_{\xi_i}^1 \phi_q(\int_{\sigma_u}^s q(\tau)d\tau)ds\}.$$

Theorem 4.1. *Assume* $(H_1)-(H_2)$ *hold and there exist constants* $r_2 \geq kb > b > r_1 > 0$, $L_2 \geq L_1 > 0$ *such that* $\phi_p(kb/M) \leq \min\{\phi_p(r_2/N), \phi_p(L_1/K)\}$. *If* f *satisfies the following conditions:*

(A_1) $f(t,u,v) \leq \min\{\phi_p(r_1/N), \phi_p(L_1/K)\}$, *for* $(t,u,v) \in [0,1] \times [0,r_1] \times [-L_1, L_1]$;

(A_2) $f(t,u,v) > \phi_p(kb/M)$ *for* $(t,u,v) \in [\frac{1}{k}, 1-\frac{1}{k}] \times [b, bk] \times [-L_2, L_2]$;

(A_3) $f(t,u,v) \leq \min\{\phi_p(r_2/N), \phi_p(L_2/K)\}$, *for* $(t,u,v) \in [0,1] \times [0,r_2] \times [-L_2, L_2]$,

then BVP (1.1) and (1.2) has at least three positive solutions u_1, u_2 *and* u_3 *such that*

$$\max_{0\leq t\leq 1} |u(t)| < r_1, \ \max_{0\leq t\leq 1} |u_1'(t)| < L_1,$$

$$b < \min_{\frac{1}{k}\leq t\leq 1-\frac{1}{k}} |u_2(t)| \leq \min_{0\leq t\leq 1} |u_2(t)| \leq r_2, \ \max_{0\leq t\leq 1} |u_2'(t)| < L_2,$$

$$\max_{0\leq t\leq 1} |u_3'(t)| \leq kb, \ \min_{\frac{1}{k}\leq t\leq 1-\frac{1}{k}} |u_3(t)| \leq b, \quad \max_{0\leq t\leq 1} |u_3'(t)| \leq L_2.$$

Proof. The BVP (1.1) and (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Tu$. Thus we set out to verify that the operator T satisfies Theorem2.1. Now the proof is divided into some steps.

Firstly, we show that

$$T : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2). \tag{4.1}$$

In fact, if $u \in \overline{P}(\alpha, r_2; \beta, L_2)$, then $\alpha(u) \leq r_2$, $\beta(u) \leq L_2$, then condition (A_3) implies $f(t, u(t), u'(t)) \leq \phi_p(r_2/N)$.

$$\begin{aligned}
\alpha(Tu) &= \max_{0\leq t\leq 1} |(Tu)(t)| = (Tu)(\sigma_u)\\
&= \Big[\frac{\sum_{i=1}^{n-2}\alpha_i}{1-\sum_{i=1}^{n-2}\alpha_i} \textstyle\int_0^{\xi_i} \phi_q(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\\
&\qquad + \textstyle\int_0^{\sigma_u} \phi_q(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\Big]\\
&= \Big[\frac{\sum_{i=1}^{n-2}\beta_i}{1-\sum_{i=1}^{n-2}\beta_i} \textstyle\int_{\xi_i}^{1} \phi_q(\int_{\sigma_u}^{s} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\\
&\qquad + \textstyle\int_{\sigma_u}^{1} \phi_q(\int_{\sigma_u}^{s} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\Big]\\
&\leq \max\Big\{\frac{\sum_{i=1}^{n-2}\alpha_i}{1-\sum_{i=1}^{n-2}\alpha_i} \textstyle\int_0^{\xi_i} \phi_q(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\\
&\qquad + \textstyle\int_0^{t^*} \phi_q(\int_s^{t^*} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds, \frac{\sum_{i=1}^{n-2}\beta_i}{1-\sum_{i=1}^{n-2}\beta_i} \int_{\xi_i}^{1} \phi_q(\int_{\sigma_u}^{s} q(\tau)\\
&\qquad f(\tau,u(\tau),u'(\tau))d\tau)ds + \textstyle\int_{t^*}^{1} \phi_q(\int_{t^*}^{s} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\Big\}\\
&\leq \max\Big\{ \textstyle\int_0^{t^*} \phi_q(\int_s^{t^*} q(\tau)\phi_p(r_2/N)d\tau)ds +\\
&\qquad \frac{\sum_{i=1}^{n-2}\alpha_i}{1-\sum_{i=1}^{n-2}\alpha_i} \textstyle\int_0^{\xi_i} \phi_q(\int_s^{\sigma_u} q(\tau)\phi_p(r_2/N)d\tau)ds,\\
&\textstyle\int_{t^*}^{1} \phi_q(\int_{t^*}^{s} q(\tau)\phi_p(r_2/N)d\tau)ds + \frac{\sum_{i=1}^{n-2}\beta_i}{1-\sum_{i=1}^{n-2}\beta_i} \int_{\xi_i}^{1} \phi_q(\int_{\sigma_u}^{s} q(\tau)\phi_p(r_2/N)d\tau)ds\Big\}
\end{aligned}$$

$= \frac{r_2}{N} \max\Big\{ \int_0^{t^*} \phi_q(\int_s^{t^*} q(\tau)d\tau)ds + \frac{\sum_{i=1}^{n-2}\alpha_i}{1-\sum_{i=1}^{n-2}\alpha_i} \int_0^{\xi_i} \phi_q(\int_s^{\sigma_u} q(\tau)d\tau)ds,$

$\int_{t^*}^1 \phi_q(\int_{t^*}^s q(\tau)d\tau)ds + \frac{\sum_{i=1}^{n-2}\beta_i}{1-\sum_{i=1}^{n-2}\beta_i} \int_{\xi_i}^1 \phi_q(\int_{\sigma_u}^s q(\tau)d\tau)ds\Big\}$

$= \frac{r_2}{N} N = r_2.$

On the other hand, for $u \in P$, there is $Tu \in P$, then Tu is concave on $[0,1]$, and $\max_{0\le t\le 1} |(Tu)'(t)| = \max\{(Tu)'(0), -(Tu)'(1)\}$, so

$$\begin{aligned}
\beta(Tu) &= \max_{0\le t\le 1} |(Tu)'(t)| \\
&= \max\{(Tu)'(0), -(Tu)'(1)\} \\
&= \max\Big\{\phi_q \int_0^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau, \phi_q \int_{\sigma_u}^1 q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big\} \\
&\le \max\Big\{\phi_q \int_0^1 q(\tau)\phi_p(L_2/K)d\tau, \phi_q \int_0^1 q(\tau)\phi_p(L_2/K)d\tau\Big\} \\
&= \frac{L_2}{K}(\int_0^1 q(\tau)d\tau) \\
&= \frac{L_2}{K} K = L_2.
\end{aligned}$$

Thus (4.1) holds.

In the same way, if $u \in \overline{P}(\alpha, r_1, \beta, L_1)$, then Assumption (A_1) yields $f(t,u,v) \le \min\{\phi_p(r_1/N), \phi_p(L_1/K)\}$, $0 \le t \le 1$. As in the argument above we can obtain that $T : \overline{P}(\alpha, r_1; \beta, L_1) \to \overline{P}(\alpha, r_1; \beta, L_1)$. Therefore condition (S_2) of Theorem 2.1 is satisfied.

Next we show that condition (S_1) in Theorem 2.1 holds. We take $u(t) = kb$, for $t \in [0,1]$. It is easy to see that $u(t) = kb \in \overline{P}(\alpha, kb, \beta, L_2, \psi, b)$ and $\psi(u) = kb > b$, hence $\{u \in \overline{P}(\alpha, kb, \beta, L_2, \psi, b) | \psi(u) > b\} \ne \emptyset$. Thus for $\{u \in P(\alpha, kb, \beta, L_2, \psi, b)$ there is $b \le u(t) \le kb$, for $1/k \le t \le 1 - 1/k$, hence by condition (A_2) of this theorem, one has $f(t,u(t),u'(t)) > \phi_p(kb/M)$, for $t \in [1/k, 1-1/k]$. Noting from Lemma 3.3 and combing the conditions on ψ and P, one arrives at

$$\begin{aligned}
\psi(Tu) &= \min_{\frac{1}{k}\le t\le 1-\frac{1}{k}} |(Tu)(t)| \\
&\ge \frac{1}{k} \max_{0\le t\le 1} |(Tu)(t)| = \frac{1}{k}(Tu)(\sigma_u) \\
&= \frac{1}{k}\Big[\frac{\sum_{i=1}^{n-2}\alpha_i}{1-\sum_{i=1}^{n-2}\alpha_i} \int_0^{\xi_i} \phi_q(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds \\
&\qquad + \int_0^{\sigma_u} \phi_q(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\Big] \\
&= \frac{1}{k}\Big[\frac{\sum_{i=1}^{n-2}\beta_i}{1-\sum_{i=1}^{n-2}\beta_i} \int_{\xi_i}^1 \phi_q(\int_{\sigma_u}^s q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds \\
&\qquad + \int_{\sigma_u}^1 \phi_q(\int_{\sigma_u}^s q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\Big] \\
&\ge \frac{1}{k}\min\{\int_0^{t^*} \phi_q(\int_s^{t^*} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds, \\
&\qquad \int_{t^*}^1 \phi_q(\int_{t^*}^s q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\} \\
&\ge \frac{1}{k}\min\{\int_{\frac{1}{k}}^{t^*} \phi_q(\int_s^{t^*} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds, \\
&\qquad \int_{t^*}^{1-\frac{1}{k}} \phi_q(\int_{t^*}^s q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\} \\
&> \frac{1}{k}\min\{\int_{\frac{1}{k}}^{t^*} \phi_q(\int_s^{t^*} q(\tau)\phi_p(kb/M)d\tau)ds, \int_{t^*}^{1-\frac{1}{k}} \phi_q(\int_{t^*}^s q(\tau)\phi_p(kb/M)d\tau)ds\}
\end{aligned}$$

$$= \frac{1}{k}\frac{kb}{M}\min\{\int_{\frac{1}{k}}^{t^*}\phi_q(\int_s^{t^*}q(\tau)d\tau)ds, \int_{t^*}^{1-\frac{1}{k}}\phi_q(\int_{t^*}^{s}q(\tau)d\tau)ds\}$$
$$= \frac{1}{k}\frac{kb}{kb}M = b.$$

Therefore we have $\psi(Tu) > b$, for all $u \in \overline{P} \in (\alpha, kb, \beta, L_2, \psi, b)$. Consequently, condition (S_1) in Theorem 2.1 is satisfied.

We finally prove (S_3) in Theorem 2.1 is also satisfied. Suppose that $u \in \overline{P} \in (\alpha, kb, \beta, L_2, \psi, b)$ with $\alpha(Tx) > kb$. Then , by the definition of ψ and $Tu \in P$ we have

$$\psi(Tu) = \min_{\frac{1}{k}\leq t\leq 1-\frac{1}{k}}|(Tu)(t)| \geq \frac{1}{k}\max_{0\leq t\leq 1}|(Tu)(t)| = \frac{1}{k}\alpha(Tu) > \frac{1}{k}kb = b.$$

Thus condition (S_3) of Theorem 2.1 is also satisfied. Therefore 2.1 leads to the conclusion that the BVP (1.1) and (1.2) has at least three positive solutions u_1,u_2 and u_3, in $\overline{P}(\alpha, r_2; \beta, L_2)$ with $y_1, y_2, y_3 \in \overline{P}(\alpha, r_2; \beta, L_2)$, with

$$u_1 \in P(\alpha, r_1; \beta, L_1),\ u_2 \in \{\overline{P}(\alpha, r_2; \beta, L_2; \psi, b)|\psi(y) > b\},$$

$$u_3 \in P(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_2)).$$

□

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