J. Appl. Math. & Informatics Vol. **27**(2009), No. 5 - 6, pp. 1177 - 1194
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STABILITY OF A MIXED QUADRATIC AND ADDITIVE FUNCTIONAL EQUATION IN QUASI–BANACH SPACES

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ABSTRACT. In this paper we establish the general solution of the functional equation

$$f(2x+y)+f(x-2y)=2f(x+y)+2f(x-y)+f(-x)+f(-y)$$

and investigate the Hyers–Ulam–Rassias stability of this equation in quasi-Banach spaces. The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

AMS Mathematics Subject Classification :Primary 39B72, 46B03, 47Jxx.
Key words and phrases : Hyers–Ulam–Rassias stability, quadratic function, additive function, quasi-Banach space, p-Banach space.

1. Introduction

In 1940, S.M. Ulam [45] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta(\epsilon)>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with

$$d(h(x),H(x))<\epsilon$$

for all $x\in G_1$?

Received March 23, 2008. Accepted April 16, 2008. *Corresponding author.

In 1941, D. H. Hyers [12] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that there exists a unique additive mapping $L : E \to E'$ satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Th.M. Rassias [36] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1. (Th.M. Rassias) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $L : E \to E'$ which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{2}$$

for all $x \in E$. If $p < 0$ then inequality (1) holds for $x, y \neq 0$ and (2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The above inequality has provided a lot of influence in the development of what is now known as a *generalized Hyers–Ulam–Rassias stability* of functional equations. J.M. Rassias [35] followed the innovative approach of Th.M. Rassias' theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. Găvruta [9] provided a further generalization of Th.M. Rassias' theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam–Rassias stability to a number of functional equations and mappings (see [3], [8], [11], [13], [15]–[18], [23]–[34], [37], [38]). We also refer the readers to the books [1], [6], [14], [39]–[42].

Quadratic functional equation was used to characterize inner product spaces [1,2,19]. Several other functional equations were also to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{3}$$

is related to a symmetric bi-additive function [1,22]. It is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (3) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ for all x (see [1,22]). The biadditive function B is given by

$$B(x,y) = \frac{1}{4}\Big(f(x+y) - f(x-y)\Big). \tag{4}$$

A Hyers–Ulam stability problem for the quadratic functional equation (3) was proved by Skof for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space (see [44]). Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In the paper [7], Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (3). Grabiec [10] has generalized these results mentioned above. Jun and Lee [21] proved the Hyers–Ulam–Rassias stability of the pexiderized quadratic equation (3). K. Jun and H. Kim [20], have obtained the generalized Hyers-Ulam stability for a mixed type of cubic and additive functional equation.

In this paper, we deal with the next functional equation deriving from quadratic and additive functions:

$$f(2x+y) + f(x-2y) = 2f(x+y) + 2f(x-y) + f(-x) + f(-y) \tag{5}$$

It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of the functional equation (5). The main purpose of this paper is to establish the general solution of Eq. (5) and investigate the Hyers–Ulam–Rassias stability for Eq. (5).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1. [4,43] Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(*i*) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(*ii*) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(*iii*) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|.\|)$ is called a *quasi-normed space* if $\|.\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|.\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|.\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space.*

By the Aoki–Rolewicz theorem [43] (see also [4]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

2. Solutions of Eq. (5)

Throughout this section, X and Y will be real vector spaces. Before proceeding the proof of Theorem 2 which is the main result in this section, we shall need the following two lemmas.

Lemma 1. *If an even function* $f : X \to Y$ *satisfies* (5) *for all* $x, y \in X$, *then* f *is quadratic.*

Proof. Note that, in view of the evenness of f, we have $f(-x) = f(x)$ for all $x \in X$. Putting $x = y = 0$ in (5), we get $f(0) = 0$. Setting $y = 0$ in (5), we obtain that $f(2x) = 4f(x)$ for all $x \in X$. Replacing x and y by $x + y$ and $x - y$ in (5), respectively, we get by the evenness of f,

$$f(3x + y) + f(x - 3y) = f(x + y) + f(x - y) + 8f(y) + 8f(y) \tag{6}$$

for all $x, y \in X$. Replacing x and y by y and x in (6), respectively, we get

$$f(x + 3y) + f(3x - y) = f(x + y) + f(x - y) + 8f(x) + 8f(y) \tag{7}$$

for all $x, y \in X$. Adding (6) to (7), we get

$$\begin{aligned} f(3x + y)+&f(3x - y) + f(x + 3y) + f(x - 3y) \\ &= 2f(x + y) + 2f(x - y) + 16f(x) + 16f(y) \end{aligned} \tag{8}$$

for all $x, y \in X$. If we replace y by $x + y$ in (5), we have

$$f(3x + y) + f(x + 2y) = 2f(2x + y) + f(x + y) + f(x) + 2f(y) \tag{9}$$

for all $x, y \in X$. Replacing x and y by y and x in (9), respectively, we get

$$f(x + 3y) + f(2x + y) = 2f(x + 2y) + f(x + y) + 2f(x) + f(y) \tag{10}$$

for all $x, y \in X$. Adding (9) to (10), we get

$$\begin{aligned} f(3x + y) + f(x + 3y) &= f(2x + y) + f(x + 2y) \\ &\quad + 2f(x + y) + 3f(x) + 3f(y) \end{aligned} \tag{11}$$

for all $x, y \in X$. Replacing y by $-y$ in (11) and using the evenness of f, we get

$$\begin{aligned} f(3x - y) + f(x - 3y) &= 2f(2x - y) + f(x - 2y) \\ &\quad + 2f(x - y) + 3f(x) + 3f(y) \end{aligned} \tag{12}$$

for all $x, y \in X$. Adding (11) to (12), we obtain that

$$\begin{aligned} f(3x + y)+&f(3x - y) + f(x + 3y) + f(x - 3y) \\ &= [f(2x + y) + f(x - 2y)] + [f(2x - y) + f(x + 2y)] \\ &\quad + 2f(x + y) + 2f(x - y) + 6f(x) + 6f(y) \end{aligned} \tag{13}$$

for all $x, y \in X$. Since f is an even function, then by Replacing x and y by y and x in (5), respectively, we get that

$$f(2x + y) + f(x - 2y) = f(2x - y) + f(x + 2y)$$

for all $x, y \in X$. Therefore we obtain from (5) and (13) that

$$\begin{aligned} f(3x+y)+&f(3x-y)+f(x+3y)+f(x-3y) \\ &= 6f(x+y)+6f(x-y)+8f(x)+8f(y) \end{aligned} \tag{14}$$

for all $x, y \in X$. So we obtain from (8) and (14) that

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. Therefore the function$f : X \to Y$ is quadratic. □

Lemma 2. *If an odd function $f : X \to Y$ satisfies (5) for all $x, y \in X$, then f is additive.*

Proof. Note that, in view of the oddness of f, we have $f(-x) = -f(x)$ for all $x \in X$. Therefore $f(0) = 0$ and (5) implies the following equation

$$f(2x+y)+f(x-2y)=2f(x+y)+2f(x-y)-f(x)-f(y) \tag{15}$$

for all $x, y \in X$. Letting $y = 0$ in (15), we get that

$$f(2x)=2f(x) \tag{16}$$

for all $x \in X$. Replacing x and y by y and $-x$ in (15), respectively, and using the oddness of f, we get

$$f(2x+y)-f(x-2y)=2f(x+y)-2f(x-y)+f(x)-f(y) \tag{17}$$

for all $x, y \in X$. Adding (15) to (17), we obtain that

$$f(2x+y)=2f(x+y)-f(y) \tag{18}$$

for all $x, y \in X$. Replacing y by $2y$ in (18) and using (16), we get

$$f(x+y)=f(x+2y)-f(y)$$

for all $x, y \in X$. Replacing x and y by y and x in the last equation, respectively, we obtain

$$f(x+y)=f(2x+y)-f(x) \tag{19}$$

for all $x, y \in X$. Hence it follows from (18) and (19) that $f(x+y) = f(x)+f(y)$ for all $x, y \in X$. So the mapping $f : X \to Y$ is additive. □

Now we are ready to find out the general solution of (5).

Theorem 2. *A function $f : X \to Y$ satisfies (5) for all $x, y \in X$ if and only if there exist a symmetric bi-additive function $B : X \times X \to Y$ and an additive function $A : X \to Y$ such that $f(x) = B(x, x) + A(x)$ for all $x \in X$.*

Proof. If there exist a symmetric bi-additive function $B : X \times X \to Y$ and an additive function $A : X \to Y$ such that $f(x) = B(x,x) + A(x)$ for all $x \in X$, it is easy to show that

$$\begin{aligned} f(2x+y) + f(x-2y) &= 5B(x,x) + 5B(y,y) + 3A(x) - A(y) \\ &= 2f(x+y) + 2f(x-y) + f(-x) + f(-y) \end{aligned}$$

for all $x, y \in X$. Therefore the function $f : X \to Y$ satisfies (5).

Conversely, we decompose f into the even part and the odd part by putting

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. It is easy to show that the functions f_e and f_o satisfy (5). Hence by Lemma 1 and Lemma 2 we achieve that the functions f_e and f_o are quadratic and additive, respectively. Therefore there exists a symmetric bi-additive function $B : X \times X \to Y$ such that $f_e(x) = B(x,x)$ for all $x \in X$ (see [1]). So $f(x) = B(x,x) + A(x)$ for all $x \in X$, where $A(x) = f_o(x)$ for all $x \in X$. □

3. Hyers–Ulam–Rassias stability of Eq. (5)

Throughout this section, assume that X is a quasi-normed space with quasi-norm $\|.\|_X$ and that Y is a p-Banach space with p-norm $\|.\|_Y$. Let K be the modulus of concavity of $\|.\|_Y$.

In this section, using an idea of Găvruta [9] we prove the stability of Eq. (5) in the spirit of Hyers, Ulam and Rassias. For convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$Df(x,y) := f(2x+y) + f(x-2y) - 2f(x+y) - 2f(x-y) - f(-x) - f(-y)$$

for all $x, y \in X$.

We will use the following lemma in this section.

Lemma 3. [26] *Let $0 \le p \le 1$ and let $x_1, x_2, \dots, x_n$ be non-negative real numbers. Then*

$$\Big(\sum_{i=1}^{n} x_i\Big)^p \le \sum_{i=1}^{n} x_i^p. \tag{20}$$

Theorem 3. *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} 4^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0, \tag{21}$$

$$\widetilde{\varphi_e}(x) := \sum_{i=1}^{\infty} 4^{ip} \varphi^p\Big(\frac{x}{2^i}, 0\Big) < \infty \tag{22}$$

for all $x, y \in X$. Suppose that an even function $f : X \to Y$ satisfies the inequality

$$\|Df(x,y)\|_Y \le \varphi(x,y) \tag{23}$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}\Big) \tag{24}$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic function satisfying

$$\|f(x) - Q(x)\|_Y \le \frac{1}{4}[\widetilde{\varphi_e}(x)]^{\frac{1}{p}} \tag{25}$$

for all $x \in X$.

Proof. It follows from (22) that $\varphi(0,0) = 0$. So (23) implies that $f(0) = 0$. Letting $y = 0$ in (23), we get

$$\|f(2x) - 4f(x)\|_Y \le \varphi(x, 0) \tag{26}$$

for all $x \in X$. If we replace x in (26) by $\frac{x}{2^{n+1}}$ and multiply both sides of (26) by 4^n, then we have

$$\Big\|4^{n+1} f\Big(\frac{x}{2^{n+1}}\Big) - 4^n f\Big(\frac{x}{2^n}\Big)\Big\|_Y \le 4^n \varphi\Big(\frac{x}{2^{n+1}}, 0\Big) \tag{27}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space, we have

$$\begin{aligned}\Big\|4^{n+1} f\Big(\frac{x}{2^{n+1}}\Big) - 4^m f\Big(\frac{x}{2^m}\Big)\Big\|_Y^p &\le \sum_{i=m}^{n} \Big\|4^{i+1} f\Big(\frac{x}{2^{i+1}}\Big) - 4^i f\Big(\frac{x}{2^i}\Big)\Big\|_Y^p \\ &\le \sum_{i=m}^{n} 4^{ip} \varphi^p\Big(\frac{x}{2^{i+1}}, 0\Big)\end{aligned} \tag{28}$$

for all $x \in X$ and all non-negative integers m and n with $n \ge m$. Therefore we conclude from (22) and (28) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges for all $x \in X$. So one can define the mapping $Q : X \to Y$ by (24) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (28), we get

$$\|f(x) - Q(x)\|_Y^p \le \sum_{i=0}^{\infty} 4^{ip} \varphi^p\Big(\frac{x}{2^{i+1}}, 0\Big) = \frac{1}{4^p} \sum_{i=1}^{\infty} 4^{ip} \varphi^p\Big(\frac{x}{2^i}, 0\Big) \tag{29}$$

for all $x \in X$. Therefore we obtain (25). Now, we show that Q is quadratic. It follows from (21), (23) and (24),

$$\begin{aligned}\|DQ(x,y)\|_Y &= \lim_{n\to\infty} 4^n \Big\|Df\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big)\Big\|_Y \\ &\le \lim_{n\to\infty} 4^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0\end{aligned}$$

for all $x, y \in X$. Therefore the mapping $Q : X \to Y$ satisfies (5). Since f is even, then Q is even. So by Lemma 1 we get that the mapping $Q : X \to Y$ is quadratic.

To prove the uniqueness of Q, let $T : X \to Y$ be another quadratic mapping satisfying (25). Since

$$\lim_{n\to\infty} 4^{np}\sum_{i=1}^{\infty} 4^{ip}\varphi^p\Big(\frac{x}{2^{n+i}},0\Big) = \lim_{n\to\infty}\sum_{i=n+1}^{\infty} 4^{ip}\varphi^p\Big(\frac{x}{2^i},0\Big) = 0$$

for all $x \in X$, then it follows from (25) that

$$\|Q(x) - T(x)\|_Y^p = \lim_{n\to\infty} 4^{np}\Big\|f\Big(\frac{x}{2^n}\Big) - T\Big(\frac{x}{2^n}\Big)\Big\|_Y^p$$
$$\leq \frac{1}{4^p}\lim_{n\to\infty} 4^{np}\widetilde{\varphi_e}\Big(\frac{x}{2^n}\Big) = 0$$

for all $x \in X$. So $Q = T$. □

Theorem 4. *Let $\Phi : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty}\frac{1}{4^n}\Phi(2^n x, 2^n y) = 0, \tag{30}$$

$$\widetilde{\Phi_e}(x) := \sum_{i=0}^{\infty}\frac{1}{4^{ip}}\Phi^p(2^i x, 0) < \infty \tag{31}$$

for all $x, y \in X$. Suppose that an even function $f : X \to Y$ satisfies the inequality

$$\|Df(x,y)\|_Y \leq \Phi(x,y) \tag{32}$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x) \tag{33}$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic function satisfying

$$\Big\|f(x) - Q(x) - \frac{1}{3}f(0)\Big\|_Y \leq \frac{1}{4}[\widetilde{\Phi_e}(x)]^{\frac{1}{p}} \tag{34}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (32), we get

$$\|f(2x) - 4f(x) - f(0)\|_Y \leq \Phi(x,0) \tag{35}$$

for all $x \in X$. If we replace x in (35) by $2^n x$ and divide both sides of (35) to 4^{n+1}, we get that

$$\Big\|\frac{1}{4^{n+1}}f(2^{n+1}x) - \frac{1}{4^n}f(2^n x) - \frac{1}{4^{n+1}}f(0)\Big\|_Y \leq \frac{1}{4^{n+1}}\Phi(2^n x, 0) \tag{36}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space,

$$\Big\|\frac{1}{4^{n+1}}f(2^{n+1}x) - \frac{1}{4^m}f(2^m x) - \sum_{i=m}^{n}\frac{1}{4^{i+1}}f(0)\Big\|_Y^p$$

$$\begin{aligned} &\leq \sum_{i=m}^{n} \left\| \frac{1}{4^{i+1}} f(2^{i+1}x) - \frac{1}{4^i} f(2^i x) - \frac{1}{4^{i+1}} f(0) \right\|_Y^p \\ &\leq \frac{1}{4^p} \sum_{i=m}^{n} \frac{1}{4^{ip}} \Phi^p(2^i x, 0) \end{aligned} \tag{37}$$

for all $x \in X$ and all non-negative integers m and n with $n \geq m$. Since $\sum_{i=0}^{\infty} \frac{1}{4^i}$ converges, then it follows from (31) and (37) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges for all $x \in X$. So one can define the mapping $Q : X \to Y$ by (33) for all $x \in X$. The rest of the proof is similar to the proof of Theorem 3. □

Corollary 1. *Let θ be non-negative real number. Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\|_Y \leq \theta \tag{38}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfies

$$\|f(x) - Q(x)\|_Y \leq K\theta\Big[\frac{1}{(4^p - 1)^{\frac{1}{p}}} + \frac{1}{12}\Big]$$

for all $x \in X$, where K is the modulus of concavity of $\|.\|_Y$.

Proof. Let $\Phi(x, y) := \theta$ for all $x, y \in X$. It follows from (38) that $\|f(0)\|_Y \leq \theta/4$. So the result follows by Theorem 4. □

Corollary 2. *Let θ, r, s be non-negative real numbers such that $r, s > 2$ or $0 < r, s < 2$. Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\|_Y \leq \theta(\|x\|_X^r + \|y\|_X^s) \tag{39}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfies

$$\|f(x) - Q(x)\|_Y \leq \frac{\theta}{|2^{rp} - 4^p|^{\frac{1}{p}}} \|x\|_X^r$$

for all $x \in X$.

Proof. It follows from (39) that $f(0) = 0$. Hence the result follows by Theorems 3 and 4. □

Corollary 3. *Let θ, r be non-negative real numbers such that $r \in (0, 2) \cup (2, \infty)$. Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\|_Y \leq \theta \|x\|_X^r \tag{40}$$

for all $x, y \in X$. Then the function $f : X \to Y$ is quadratic.

Proof. It follows from (40) that $f(0)=0$. Letting $x=0$ in (40), we get that $f(2y)=4f(y)$ for all $y\in X$. By induction we infer that $f(2^n y)=4^n f(y)$ for all $y\in X$ and all $n\in\mathbb{Z}$. So the result follows by Theorems 3 and 4. □

Corollary 4. *Let θ, s be non-negative real numbers such that $s\in(0,2)\cup(2,\infty)$. Suppose that an even function $f:X\to Y$ satisfies the inequality*

$$\|Df(x,y)\|_Y \le \theta\|y\|_X^s \tag{41}$$

for all $x,y\in X$. Then the function $f:X\to Y$ is quadratic.

Corollary 5. *Let θ, r, s be non-negative real numbers such that $r+s\in(0,2)\cup(2,\infty)$. Suppose that an even function $f:X\to Y$ satisfies the inequality*

$$\|Df(x,y)\|_Y \le \theta\|x\|_X^r\|y\|_X^s \tag{42}$$

for all $x,y\in X$. Then the function $f:X\to Y$ is quadratic.

Proof. Similar to the proof of Corollary 3, we get $f(2^n x)=4^n f(x)$ for all $x\in X$ and all $n\in\mathbb{Z}$. So the result follows by Theorems 3 and 4. □

Theorem 5. *Let $\varphi: X\times X\to[0,\infty)$ be a function such that*

$$\lim_{n\to\infty} 2^n\varphi\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)=0, \tag{43}$$

$$\widetilde{\varphi_o}(x) := \sum_{i=1}^{\infty} 2^{ip}\varphi^p\Big(\frac{x}{2^i},0\Big)<\infty \tag{44}$$

for all $x,y\in X$. Suppose that an odd function $f:X\to Y$ satisfies the inequality (23) for all $x,y\in X$. Then the limit

$$A(x) := \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big) \tag{45}$$

exists for all $x\in X$ and $A:X\to Y$ is a unique additive function satisfying

$$\|f(x)-A(x)\|_Y \le \frac{1}{2}[\widetilde{\varphi_o}(x)]^{\frac{1}{p}} \tag{46}$$

for all $x\in X$.

Proof. It is clear that $f(0)=0$. Letting $y=0$ in (23), we have

$$\|f(2x)-2f(x)\|_Y \le \varphi(x,0) \tag{47}$$

for all $x\in X$. If we replace x in (47) by $\frac{x}{2^{n+1}}$ and multiply both sides of (47) by 2^n, we get

$$\Big\|2^{n+1}f\Big(\frac{x}{2^{n+1}}\Big)-2^n f\Big(\frac{x}{2^n}\Big)\Big\|_Y \le 2^n\varphi\Big(\frac{x}{2^{n+1}},0\Big) \tag{48}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space,

$$\left\|2^{n+1} f\Big(\frac{x}{2^{n+1}}\Big) - 2^m f\Big(\frac{x}{2^m}\Big)\right\|_Y^p \le \sum_{i=m}^{n} \left\|2^{i+1} f\Big(\frac{x}{2^{i+1}}\Big) - 2^i f\Big(\frac{x}{2^i}\Big)\right\|_Y^p \le \sum_{i=m}^{n} 2^{ip} \varphi^p\Big(\frac{x}{2^{i+1}}, 0\Big) \tag{49}$$

for all $x \in X$ and all non-negative integers m and n with $n \ge m$. Therefore we conclude from (44) and (49) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by (45) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (49), we get (46). Now, we show that A is additive. It follows from (23), (43) and (45),

$$\|DA(x,y)\|_Y = \lim_{n\to\infty} 2^n \left\|Df\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big)\right\|_Y \le \lim_{n\to\infty} 2^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0$$

for all $x, y \in X$. Therefore the mapping $A : X \to Y$ satisfies (5). Since f is an odd function, then (45) implies that the mapping $A : X \to Y$ is odd. Therefore by Lemma 2 we get that the mapping $A : X \to Y$ is additive.

To prove the uniqueness of A, let $T : X \to Y$ be another additive mapping satisfying (46). Since

$$\lim_{n\to\infty} 2^{np} \sum_{i=1}^{\infty} 2^{ip} \varphi^p\Big(\frac{x}{2^{n+i}}, 0\Big) = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 2^{ip} \varphi^p\Big(\frac{x}{2^i}, 0\Big) = 0$$

for all $x \in X$, then it follows from (45) and (46) that

$$\|A(x) - T(x)\|_Y^p = \lim_{n\to\infty} 2^{np} \left\|f\Big(\frac{x}{2^n}\Big) - T\Big(\frac{x}{2^n}\Big)\right\|_Y^p \le \frac{1}{2^p} \lim_{n\to\infty} 2^{np} \widetilde{\varphi_o}\Big(\frac{x}{2^n}, 0\Big) = 0$$

for all $x \in X$. So $A = T$. □

Theorem 6. *Let $\Phi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{2^n} \Phi(2^n x, 2^n y) = 0, \tag{50}$$

$$\widetilde{\Phi_o}(x) := \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \Phi^p(2^i x, 0) < \infty \tag{51}$$

for all $x, y \in X$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (32) for all $x, y \in X$. Then the limit

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \tag{52}$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfying

$$\|f(x) - A(x)\|_Y \leq \frac{1}{2}[\widetilde{\Phi_o}(x)]^{\frac{1}{p}} \tag{53}$$

for all $x \in X$.

Proof. It is clear that $f(0) = 0$. Letting $y = 0$ in (32), we have

$$\|f(2x) - 2f(x)\|_Y \leq \Phi(x, 0) \tag{54}$$

for all $x \in X$. If we replace x in (54) by $2^n x$ and divide both sides of (54) to 2^{n+1}, then we have

$$\left\|\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^n}f(2^n x)\right\|_Y \leq \frac{1}{2^{n+1}}\Phi(2^n x, 0) \tag{55}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space,

$$\begin{aligned}\left\|\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^m}f(2^m x)\right\|_Y^p &\leq \sum_{i=m}^{n}\left\|\frac{1}{2^{i+1}}f(2^{i+1}x) - \frac{1}{2^i}f(2^i x)\right\|_Y^p \\ &\leq \frac{1}{2^p}\sum_{i=m}^{n}\frac{1}{2^{ip}}\Phi^p(2^i x, 0)\end{aligned} \tag{56}$$

for all $x \in X$ and all non-negative integers m and n with $n \geq m$. It follows from (51) and (56) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by (52) for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 5. □

Corollary 6. *Let θ be non-negative real number. Suppose that an odd function $f : X \to Y$ satisfies the inequality (38) for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ satisfies*

$$\|f(x) - A(x)\|_Y \leq \frac{\theta}{(2^p - 1)^{\frac{1}{p}}}$$

for all $x \in X$.

Corollary 7. *Let θ, r, s be non-negative real numbers such that $r, s > 1$ or $0 < r, s < 1$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (39) for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ satisfies*

$$\|f(x) - A(x)\|_Y \leq \frac{\theta}{|2^{rp} - 2^p|^{\frac{1}{p}}}\|x\|_X^r$$

for all $x \in X$.

Proof. Since f is an odd function, then $f(0) = 0$. Hence the result follows by Theorems 5 and 6. □

Corollary 8. *Let θ, r be non-negative real numbers such that $r \in (0,1) \cup (1,\infty)$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (40) for all $x, y \in X$. Then the function $f : X \to Y$ is additive.*

Proof. Since $f(0) = 0$, letting $x = 0$ in (40), we get that $f(2y) = 2f(y)$ for all $y \in X$. By induction we infer that $f(2^n y) = 2^n f(y)$ for all $y \in X$ and all $n \in \mathbb{Z}$. So the result follows by Theorems 5 and 6. □

Corollary 9. *Let θ, s be non-negative real numbers such that $s \in (0,1) \cup (1,\infty)$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (41) for all $x, y \in X$. Then the function $f : X \to Y$ is additive.*

Corollary 10. *Let θ, r, s be non-negative real numbers such that $r + s \in (0,1) \cup (1,\infty)$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (42) for all $x, y \in X$. Then the function $f : X \to Y$ is additive.*

Proof. Similar to the proof of Corollary 8, we get $f(2^n x) = 2^n f(x)$ for all $x \in X$ and all $n \in \mathbb{Z}$. So the result follows by Theorems 5 and 6. □

We now prove our main theorems in this section.

Theorem 7. *Let $\varphi : X \times X \to [0,\infty)$ be a function satisfies (21) and (22) for all $x, y \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality (23) for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (5) and*

$$\|f(x) - Q(x) - A(x)\|_Y \le \frac{K^2}{8}\Big\{ [\widetilde{\varphi_e}(x) + \widetilde{\varphi_e}(-x)]^{\frac{1}{p}} + 2[\widetilde{\varphi_o}(x) + \widetilde{\varphi_o}(-x)]^{\frac{1}{p}} \Big\} \tag{57}$$

for all $x \in X$, where $\widetilde{\varphi_e}(x)$ and $\widetilde{\varphi_o}(x)$ has been defined in (22) and (44), respectively, for all $x \in X$.

Proof. It follows from (22) that $\varphi(0,0) = 0$. So (23) implies that $f(0) = 0$. Let $f_e(x) = \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$ and

$$\|Df_e(x,y)\|_Y \le \frac{K}{2}\Big[\varphi(x,y) + \varphi(-x,-y)\Big]$$

for all $x, y \in X$. Let

$$\psi(x,y) = \frac{K}{2}\Big[\varphi(x,y) + \varphi(-x,-y)\Big]$$

for all $x, y \in X$. So

$$\lim_{n\to\infty} 4^n \psi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0$$

for all $x, y \in X$. Since

$$\psi^p(x,y) \le \frac{K^p}{2^p}\Big[\varphi^p(x,y) + \varphi^p(-x,-y)\Big] \qquad \text{(by Lemma 3)}$$

for all $x, y \in X$, then

$$\widetilde{\psi_e}(x) := \sum_{i=1}^{\infty} 4^{ip}\psi^p\Big(\frac{x}{2^i}, 0\Big) < \infty$$

for all $x \in X$. So in view of Theorem 3, there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\|f_e(x) - Q(x)\|_Y \leq \frac{1}{4}[\widetilde{\psi_e}(x)]^{\frac{1}{p}} \leq \frac{K}{8}\Big[\widetilde{\varphi_e}(x) + \widetilde{\varphi_e}(-x)\Big]^{\frac{1}{p}} \tag{58}$$

for all $x \in X$.

Now, let $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$ and

$$\|Df_o(x, y)\|_Y \leq \psi(x, y)$$

for all $x, y \in X$. In view of Theorem 5, there exists a unique additive function $A : X \to Y$ satisfying

$$\|f_o(x) - A(x)\|_Y \leq \frac{1}{2}[\widetilde{\psi_o}(x)]^{\frac{1}{p}} \tag{59}$$

for all $x \in X$, where

$$\widetilde{\psi_o}(x) := \sum_{i=1}^{\infty} 2^{ip}\psi^p\Big(\frac{x}{2^i}, 0\Big). \tag{60}$$

Since

$$\widetilde{\psi_o}(x) \leq \frac{K^p}{2^p}\Big[\widetilde{\varphi_o}(x) + \widetilde{\varphi_o}(-x)\Big]$$

for all $x \in X$, it follows from (59) that

$$\|f_o(x) - A(x)\|_Y \leq \frac{K}{4}\Big[\widetilde{\varphi_o}(x) + \widetilde{\varphi_o}(-x)\Big]^{\frac{1}{p}} \tag{61}$$

for all $x \in X$. Hence (57) follows from (58) and (61). □

Theorem 8. *Let $\Phi : X \times X \to [0, \infty)$ be a function satisfies (50) and (51) for all $x, y \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality (32) for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (5) and*

$$\Big\|f(x) - Q(x) - A(x) - \frac{1}{3}f(0)\Big\|_Y \leq \frac{K^2}{8}\Big\{[\widetilde{\Phi_e}(x) + \widetilde{\Phi_e}(-x)]^{\frac{1}{p}} + 2[\widetilde{\Phi_o}(x) + \widetilde{\Phi_o}(-x)]^{\frac{1}{p}}\Big\} \tag{62}$$

for all $x \in X$, where $\widetilde{\Phi_e}(x)$ and $\widetilde{\Phi_o}(x)$ has been defined in (31) and (51), respectively, for all $x \in X$.

Proof. Similar to the proof of Theorem 7, the result follows from Theorems 4 and (6). □

Corollary 11. *Let θ be non-negative real number. Suppose that a function $f : X \to Y$ satisfies the inequality (38) for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (5) and*

$$\|f(x) - A(x) - Q(x)\|_Y \leq K\theta\Big[\frac{K}{(4^p - 1)^{\frac{1}{p}}} + \frac{1}{(2^p - 1)^{\frac{1}{p}}} + \frac{K}{12}\Big]$$

for all $x \in X$.

Proof. We decompose f into the even part (f_e) and the odd part (f_o). It is clear that

$$\|Df_e(x, y)\| \leq K\theta, \qquad \|Df_o(x, y)\| \leq K\theta$$

for all $x, y \in X$. Hence the result follows by applying Corollaries 1 and 6. □

Corollary 12. *Let θ, r, s be non-negative real numbers such that $r, s > 2$ or $0 < r, s < 1$. Suppose that a function $f : X \to Y$ satisfies the inequality (39) for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (5) and*

$$\|f(x) - A(x) - Q(x)\|_Y \leq K^2\theta\Big[\frac{1}{|2^{rp} - 4^p|^{\frac{1}{p}}} + \frac{1}{|2^{rp} - 2^p|^{\frac{1}{p}}}\Big]\|x\|_X^r$$

for all $x \in X$.

Proof. By decomposing f into the even part (f_e) and the odd part (f_o), we get

$$\|Df_e(x, y)\| \leq K\theta(\|x\|_X^r + \|y\|_X^s), \qquad \|Df_o(x, y)\| \leq K\theta(\|x\|_X^r + \|y\|_X^s)$$

for all $x, y \in X$. Hence the result follows by applying Corollaries 2 and 7. □

Corollary 13. *Let θ, r be non-negative real numbers such that $r \in (0, 1) \cup (2, \infty)$. Suppose that a function $f : X \to Y$ satisfies the inequality (40) for all $x, y \in X$. Then the function $f_e : X \to Y$ is quadratic and the function $f_o : X \to Y$ is additive.*

Proof. Since

$$\|Df_e(x, y)\| \leq K\theta\|x\|_X^r, \qquad \|Df_o(x, y)\| \leq K\theta\|x\|_X^r$$

for all $x, y \in X$, then the result follows by applying Corollaries 3 and 8. □

Corollary 14. *Let θ, s be non-negative real numbers such that $s \in (0, 1) \cup (2, \infty)$. Suppose that a function $f : X \to Y$ satisfies the inequality (41) for all $x, y \in X$. Then the function $f_e : X \to Y$ is quadratic and the function $f_o : X \to Y$ is additive.*

Corollary 15. *Let θ, r, s be non-negative real numbers such that $r+s \in (0,1) \cup (2,\infty)$. Suppose that a function $f : X \to Y$ satisfies the inequality (42) for all $x, y \in X$. Then the function $f_e : X \to Y$ is quadratic and the function $f_o : X \to Y$ is additive.*

Proof. Since

$$\|Df_e(x,y)\| \le K\theta \|x\|_X^r \|y\|_X^s, \qquad \|Df_o(x,y)\| \le K\theta \|x\|_X^r \|y\|_X^s$$

for all $x, y \in X$, then the result follows by applying Corollaries 5 and 10. □

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