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INTERVAL OSCILLATION CRITERIA FOR A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

CUN-HUA ZHANG

ABSTRACT. This paper is concerned with the interval oscillation of the second order nonlinear ordinary differential equation $\left(r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)'+p(t)\left|y'(t)\right|^{\alpha-1}y'(t)+q(t)f(y(t))g(y'(t))=0$. By constructing a generalized Riccati transformation and using the method of averaging techniques, we establish some interval oscillation criteria when $f(y)$ is not differetiable but satisfies the condition $\frac{f(y)}{|y|^{\alpha-1}y}\geq\mu_0>0$ for $y\neq 0$.

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1. Introduction

This paper deals with the second order nonlinear differential equation defined on the half line $[t_0,\infty)$ of the form

$$\left(r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)'+p(t)\left|y'(t)\right|^{\alpha-1}y'(t)+q(t)f(y(t))g(y'(t))=0, \tag{1}$$

where $\alpha>0$ is a constant and the functions r, p, q, f and g satisfy the following conditions:

(H1): $r(t)\in C^1([t_0,\infty),(0,\infty))$ and $p(t)\in C([t_0,\infty),\mathbb{R})$;
(H2): $q(t)\in C([t_0,\infty),[0,\infty))$ and $q(t)\not\equiv 0$ on any interval $[T,\infty)$ for some $T\geq t_0$;
(H3): $f\in C(\mathbb{R},\mathbb{R})$ and $yf(y)>0$ for $y\neq 0$;
(H4): $g\in C(\mathbb{R},\mathbb{R})$ and $g(y)\geq K>0$ for $y\neq 0$.

By a solution of (1) we mean that a function $y(t)\in C^1([T_y,\infty),\mathbb{R})$ for $T_y\geq t_0$, which has the property $\left|y'(t)\right|^{\alpha-1}y'(t)\in C^1([T_y,\infty),\mathbb{R})$ and satisfies (1) for all $t\in[T_y,\infty)$. A nontrivial solution $y(t)$ of (1) is called oscillatory if it

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has arbitrary large zeros, otherwise it is called non-oscillatory. Eq.(1) is called oscillatory if all its solutions are oscillatory.

The oscillation of (1) has been investigated extensively by many authors (see, i.e. [2]-[7]). By using a generalized Riccati transformation and the averaging techniques, Li [4], Wang and Yang [6] investigated the oscillation of (1) when $p(t) \equiv 0$, $g(y) \equiv 1$ and $f(y) = |y|^{\alpha-1}y$, that is, the following second order half-linear differential equation

$$\left(r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + q(t)|y(t)|^{\alpha-1}y(t) = 0, \tag{2}$$

and obtained some interval oscillation criteria. When $\alpha = 1$ and $g(y) \equiv 1$, (1) is reduced to the following second order nonlinear ordinary differential equation

$$(r(t)y'(t))' + p(t)y'(t) + q(t)f(y(t)) = 0. \tag{3}$$

Under the condition that $f'(y) \geq \mu > 0$ or $f(y)/y \geq \mu > 0$ for $y \neq 0$, by using the methods similar to those in [4] and [6], Li and Agarwal [3] investigated the oscillation of Eq.(3) and established some interval oscillation criteria. Subsequently, Li [5] considered the following nonlinear differential equation

$$(r(t)y'(t))' + p(t)y'(t) + q(t)f(y(t))g(y'(t)) = 0, \tag{4}$$

and obtained some interval oscillation criteria under the assumption that $f'(y) \geq \mu > 0$ or $f(y)/y \geq \mu > 0$ and $g(y) \geq K > 0$ for $y \neq 0$.

However, when $\alpha \neq 1$, the results obtained by Li [5] can not be applied. Motivated by the ideas of Kong [2], Li and Agarwal [3] and Li [5], Yan, Chu and Zhang [7] investigated the interval oscillation of Eq.(1) under the condition that

(H): $f(y)$ is differentiable and $\frac{f'(y)}{|f(y)|^{(\alpha-1)/\alpha}} \geq \mu > 0$ for $y \neq 0$.

Unfortunately, Yan, Chu and Zhang [7] did not consider the case when $f(y)$ is not differentiable. Therefore, as a continuation, in the present paper, by using a generalized Riccati transformation, the averaging techniques and an inequality due to Hardy, Littlewood and Polya [1], we investigate the oscillation of Eq.(1) when $f(y)$ satisfies the following condition:

(H5): $\frac{f(y)}{|y|^{\alpha-1}y} \geq \mu_0 > 0$ for $y \neq 0$, where μ_0 is a constant.

The remainder of this paper is organized as follows. In the next section, we introduce the concept concerning averaging functions and establish an inequality to be useful in our discussion by means of a well-known inequality due to Hardy, Littlewood and Polya[1]. In Section 3, in terms of the inequality establishes in Section 2, we give some oscillation criteria of Eq.(1) under the condition (H1)-(H5).

2. Preliminaries

In order to prove our main result we use the following well-known inequality due to Hardy, Littlewood and Polya[1] and the concepts concerning averaging functions.

Lemma 1. *Let $p > 1$ and A, B be nonnegative real numbers, then*

$$pAB^{p-1} - A^p \le (p-1) B^p,$$

where equality holds if and only if $A = B$.

In the sequel we say that a function $H = H(t,s)$ belongs to a function class X, denoted by $H \in X$, if $H \in C(D,[0,\infty))$, where $D = \{(t,s) : -\infty < s \le t < \infty\}$, which satisfies

$$H(t,t) = 0, H(t,s) > 0, \text{ for } t > s, \tag{5}$$

and has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)} \quad \text{and} \quad \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}, \tag{6}$$

where h_1 and h_2 are nonnegative continuous function on D.

First, we establish an interesting lemma, which will be useful for establishing oscillation criteria of (1).

Lemma 2. *Let $A_0(t)$, $A_1(t) \in C([t_0,\infty),\mathbb{R})$, $A_2(t) \in C([t_0,\infty),(0,\infty))$ and $w(t) \in C^1([t_0,\infty),\mathbb{R})$. If there exists a interval $(a,b) \subset [t_0,\infty)$ such that*

$$w'(s) \le -A_0(s) + A_1(s) w(s) - A_2(s) |w(s)|^{(\alpha+1)/\alpha}, s \in (a,b), \tag{7}$$

then for every $H \in X$ and $c \in (a,b)$

$$\begin{aligned} &\frac{1}{H(c,a)} \int_a^c \left\{ H(s,a) A_0(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\phi_1^{\alpha+1}(s,a)}{(A_2(s))^\alpha H^{(\alpha-1)/2}(s,a)} \right\} ds \\ &+ \frac{1}{H(b,c)} \int_c^b \left\{ H(b,s) A_0(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\phi_2^{\alpha+1}(b,s)}{(A_2(s))^\alpha H^{(\alpha-1)/2}(b,s)} \right\} ds \\ &\le 0, \end{aligned} \tag{8}$$

where

$$\begin{cases} \phi_1(s,a) = \left| h_1(s,a) + A_1(s) \sqrt{H(s,a)} \right|, \\ \phi_2(b,s) = \left| h_2(b,s) - A_1(s) \sqrt{H(b,s)} \right|. \end{cases}$$

Proof. Multiplying (7) by $H(s,t)$ and integrating it with respect to s from t to c for $t \in (a,c]$, we have

$$\begin{aligned} \int_t^c H(s,t) A_0(s) ds \le & -\int_t^c H(s,t) w'(s) ds + \int_t^c H(s,t) A_1(s) w(s) ds \\ & - \int_t^c H(s,t) A_2(s) |w(s)|^{(\alpha+1)/\alpha} ds. \end{aligned} \tag{9}$$

In view of (5) and (6), we see that

$$\int_t^c H(s,t) w'(s) ds = H(c,t) w(c) - \int_t^c h_1(s,t) \sqrt{H(s,t)} w(s) ds. \tag{10}$$

Using (10) in (9) leads to

$$\begin{aligned}
&\int_t^c H(s,t)\, A_0(s)\, ds \\
\le\ & -H(c,t)\, w(c) + \int_t^c \left[h_1(s,t)\sqrt{H(s,t)} + H(s,t)\, A_1(s)\right] w(s)\, ds \\
& - \int_t^c H(s,t)\, A_2(s)\, |w(s)|^{(\alpha+1)/\alpha}\, ds \\
\le\ & -H(c,t)\, w(c) + \int_t^c \left|h_1(s,t)\sqrt{H(s,t)} + H(s,t)\, A_1(s)\right| |w(s)|\, ds \\
& - \int_t^c H(s,t)\, A_2(s)\, |w(s)|^{(\alpha+1)/\alpha}\, ds. \qquad (11)
\end{aligned}$$

Let

$$\begin{aligned}
A &= (H(s,t)\, A_2(s))^{\alpha/(\alpha+1)}\, |w(s)|, \quad p = \frac{\alpha+1}{\alpha}, \\
B &= \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \frac{\left|h_1(s,t) + A_1(s)\sqrt{H(s,t)}\right|^{\alpha}}{(A_2(s))^{\alpha^2/(\alpha+1)}\, H^{\alpha(\alpha-1)/(2(\alpha+1))}(s,t)}.
\end{aligned}$$

In view of Lemma 1, we obtain

$$\begin{aligned}
&\left|h_1(s,t)\sqrt{H(s,t)} + H(s,t)\, A_1(s)\right| |w(s)| - H(s,t)\, A_2(s)\, |w(s)|^{(\alpha+1)/\alpha} \\
\le\ & \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\left|h_1(s,t) + A_1(s)\sqrt{H(s,t)}\right|^{\alpha+1}}{(A_2(s))^{\alpha}\, H^{(\alpha-1)/2}(s,t)}.
\end{aligned}$$

Hence (11) implies

$$\begin{aligned}
&\int_t^c H(s,t)\, A_0(s)\, ds \\
\le\ & -H(c,t)\, w(c) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \int_t^c \frac{\left|h_1(s,t) + A_1(s)\sqrt{H(s,t)}\right|^{\alpha+1}}{(A_2(s))^{\alpha}\, H^{(\alpha-1)/2}(s,t)} ds. \qquad (12)
\end{aligned}$$

Letting $t \to a^+$ in (12) and then multiplying (12) by $1/H(c,a)$, one can get

$$\begin{aligned}
&\frac{1}{H(c,a)} \int_a^c H(s,a)\, A_0(s)\, ds \\
\le\ & -w(c) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{1}{H(c,a)} \int_a^c \frac{\left|h_1(s,a) + A_1(s)\sqrt{H(s,a)}\right|^{\alpha+1}}{(A_2(s))^{\alpha}\, H^{(\alpha-1)/2}(s,a)} ds. \qquad (13)
\end{aligned}$$

Next, we go back to (7). Similarly, we multiply (7) by $H(t,s)$ and then integrate it with respect to s from c to t for $t \in [c,b)$, then we get that

$$\int_c^t H(t,s)\,A_0(s)\,ds \le -\int_c^t H(t,s)\,w'(s)\,ds + \int_c^t H(t,s)\,A_1(s)\,w(s)\,ds - \int_c^t H(t,s)\,A_2(s)\,|w(s)|^{(\alpha+1)/\alpha}\,ds. \tag{14}$$

It follows from (5) and (6) that

$$\int_c^t H(t,s)\,w'(s)\,ds = -H(t,c)\,w(c) + \int_c^t h_2(s,t)\,\sqrt{H(s,t)}w(s)\,ds, \tag{15}$$

thus (14) becomes

$$\begin{aligned}&\int_c^t H(t,s)\,A_0(s)\,ds\\ \le\;& H(t,c)\,w(c) + \int_c^t \left[-h_2(t,s)\sqrt{H(t,s)} + A_1(s)\,H(t,s)\right] w(s)\,ds\\ &- \int_t^c H(t,s)\,A_2(s)\,|w(s)|^{(\alpha+1)/\alpha}\,ds\\ \le\;& H(t,c)\,w(c) + \int_c^t \left|h_2(t,s)\sqrt{H(t,s)} - A_1(s)\,H(t,s)\right| |w(s)|\,ds\\ &- \int_t^c H(t,s)\,A_2(s)\,|w(s)|^{(\alpha+1)/\alpha}\,ds.\end{aligned} \tag{16}$$

Let

$$A = (H(t,s)\,A_2(s))^{\alpha/(\alpha+1)}\,|w(s)|, \quad p = \frac{\alpha+1}{\alpha},$$

and

$$B = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \frac{\left|h_2(t,s) - A_1(s)\sqrt{H(t,s)}\right|^{\alpha}}{(A_2(s))^{\alpha^2/(\alpha+1)}\,H^{\alpha(\alpha-1)/(2(\alpha+1))}(t,s)}.$$

By Lemma 1, we also have

$$\begin{aligned}&\left|h_2(t,s)\sqrt{H(t,s)} - A_1(s)\,H(t,s)\right| |w(s)| - H(t,s)\,A_2(s)\,|w(s)|^{(\alpha+1)/\alpha}\\ \le\;& \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\left|h_2(t,s) - A_1(s)\sqrt{H(t,s)}\right|^{\alpha+1}}{(A_2(s))^{\alpha}\,H^{(\alpha-1)/2}(t,s)}.\end{aligned} \tag{17}$$

Hence, (16) implies

$$\begin{aligned}&\int_c^t H(t,s)\,A_0(s)\,ds\\ \le\;& H(t,c)\,w(c) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \int_c^t \frac{\left|h_2(t,s) - A_1(s)\sqrt{H(t,s)}\right|^{\alpha+1}}{(A_2(s))^{\alpha}\,H^{(\alpha-1)/2}(t,s)}\,ds.\end{aligned} \tag{18}$$

Letting $t \to b^-$ in (18) and then divide both sides of (18) by $H(b,c)$ to get

$$\begin{aligned} &\frac{1}{H(b,c)}\int_c^b H(b,s)\,A_0(s)\,ds \\ \leq\;& w(c)+\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{1}{H(b,c)}\int_c^b \frac{\left|h_2(b,s)-A_1(s)\sqrt{H(b,s)}\right|^{\alpha+1}}{(A_2(s))^\alpha H^{(\alpha-1)/2}(b,s)}ds. \quad (19)\end{aligned}$$

Adding the inequalities (13) and (19) we can obtain (8) and the proof is complete. □

3. Main results

In this section we consider the oscillation of Eq.(1) when the function $f(y)$ is not differetiable. In this case we always assume that (H5) holds. In the following discussions, we always use the notation β to denote $\frac{1}{(\alpha+1)^{\alpha+1}}$ for any $\alpha > 0$, that is,

$$\beta = \frac{1}{(\alpha+1)^{\alpha+1}}.$$

Theorem 1. *Assume that (H1)-(H5) hold and $(a,b) \subset [t_0,\infty)$. If there exist $c \in (a,b)$ and $H \in X$, $\rho \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\begin{aligned} &\frac{1}{H(c,a)}\int_a^c \left[H(s,a)\,K\mu_0\rho(s)\,q(s)-\beta\frac{\rho(s)\,r(s)\,\varphi_1^{\alpha+1}(s,a)}{H^{(\alpha-1)/2}(s,a)}\right]ds \\ &+\frac{1}{H(b,c)}\int_c^b \left[H(b,s)\,K\mu_0\rho(s)\,q(s)-\beta\frac{\rho(s)\,r(s)\,\varphi_2^{\alpha+1}(b,s)}{H^{(\alpha-1)/2}(b,s)}\right]ds \\ &> 0, \end{aligned} \quad (20)$$

then every solution of Eq.(1) has at least one zero in (a,b), where

$$\begin{aligned} \varphi_1(s,a) &= \left|h_1(s,a)+\left(\tfrac{\rho'(s)}{\rho(s)}-\tfrac{p(s)}{r(s)}\right)\sqrt{H(s,a)}\right|, \\ \varphi_2(b,s) &= \left|h_2(b,s)-\left(\tfrac{\rho'(s)}{\rho(s)}-\tfrac{p(s)}{r(s)}\right)\sqrt{H(b,s)}\right|. \end{aligned} \quad (21)$$

Proof. Suppose that there exists a solution $y(t)$ of Eq.(1) such that it has no zero in (a,b), that is, $y(t) \neq 0$ for $t \in (a,b)$. Now, we define

$$w(t) = \rho(t)\frac{r(t)\,|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\alpha-1}y(t)}, t \in (a,b). \quad (22)$$

From (1) and (22) we have

$$\begin{aligned} w'(t) \;=\;& -\rho(t)\,q(t)\frac{f(y(t))}{|y(t)|^{\alpha-1}y(t)}g(y'(t))+\left(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right)w(t) \\ &-\frac{\alpha}{(\rho(t)\,r(t))^{1/\alpha}}|w(t)|^{(\alpha+1)/\alpha}\;,\ \text{for } t \in (a,b). \end{aligned} \quad (23)$$

In view of (H4) and (H5), we obtain from (23) that

$$w'(t) \leq -K\mu_0\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right)w(t) - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|w(t)|^{(\alpha+1)/\alpha}. \quad (24)$$

Comparing the inequality (24) with (7), we identify that

$$A_0(t) = K\mu_0\rho(t)q(t), \quad A_1(t) = \frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}, \quad A_2(t) = \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}.$$

Appling Lemma 2 to (24) we see that inequality (20) fails to hold. So, we can conclude that every solution of Eq.(1) has at least one zero in (a, b). Thus the proof is complete. □

In Theorem 1, if $\rho(t) = \exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}ds\right)$ for $t \geq t_0$, then $\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} = 0$. Thus we have the following result.

Corollary 1. *Assume that (H1)-(H5) hold and $(a, b) \subset [t_0, \infty)$. If there exist $c \in (a, b)$ and $H \in X$ such that*

$$\frac{1}{H(c,a)}\int_a^c \left[H(s,a)K\mu_0 q(s) - \beta\frac{r(s)h_1^{\alpha+1}(s,a)}{H^{(\alpha-1)/2}(s,a)}\right]\exp\left(\int_{t_0}^s \frac{p(u)}{r(u)}du\right)ds$$
$$+\frac{1}{H(b,c)}\int_c^b \left[H(b,s)K\mu_0 p(s) - \beta\frac{r(s)h_2^{\alpha+1}(b,s)}{H^{(\alpha-1)/2}(b,s)}\right]\exp\left(\int_{t_0}^s \frac{p(u)}{r(u)}du\right)ds$$
$$> 0. \quad (25)$$

Then every solution of Eq.(1) has at least one zero in (a, b).

Theorem 2. *Assume that (H1)-(H5) hold. If, for every $T \geq t_0$, there exist $H \in X$, $\rho \in C^1([t_0, \infty), (0, \infty))$ and $a, b, c \in \mathbb{R}$ such that $T \leq a < c < b$ and (20) holds, then every solution of Eq.(1) is oscillatory.*

Proof. Pick up a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By the assumption, for every $i \in \mathbb{N}$, there are $a_i, b_i, c_i \in \mathbb{R}$ such that $T_i \leq a_i < c_i < b_i$ and (20) holds, where a, b, c are replaced by a_i, b_i, c_i, repectively. From Theorem 1, every solution $y(t)$ has at least one zero, $t_i \in (a_i, b_i)$. Noting that $t_i > a_i \geq T_i$, $i \in \mathbb{N}$, we see that every solution of Eq.(1) has arbitary large zeros. Thus, every solution of Eq.(1) is oscillatory. The proof is complete. □

Corollary 2. *Assume that (H1)-(H5) hold. If, for every $T \geq t_0$, there exist $H \in X$ and $a, b, c \in \mathbb{R}$ such that $T \leq a < c < b$ and (25) holds, then every solution of Eq.(1) is oscillatory.*

Theorem 3. *Assume that (H1)-(H5) hold. If for some $\rho \in C^1([t_0, \infty), (0, \infty))$, $H \in X$ and for each $l \geq t_0$ such that*

$$\limsup_{t\to\infty}\int_l^t \left[H(s,l)K\mu_0\rho(s)q(s) - \beta\frac{\rho(s)r(s)\varphi_1^{\alpha+1}(s,l)}{H^{(\alpha-1)/2}(s,l)}\right]ds > 0, \quad (26)$$

and

$$\limsup_{t\to\infty}\int_l^t\left[H(t,s)K\mu_0\rho(s)q(s)-\beta\frac{\rho(s)\,r(s)\,\varphi_1^{\alpha+1}(t,s)}{H^{(\alpha-1)/2}(t,s)}\right]ds>0, \tag{27}$$

where $\varphi_1(s,l)$ *and* $\varphi_2(t,s)$ *defined by (21), and* a *and* b *in (21) are replaced by* l *and* t*, respectively, then every solution of Eq.(1) is oscillatory.*

Proof. For any $T\geq t_0$, let $a=T$. In (26) we choose $l=a$. Then there exists $c>a$ such that

$$\int_a^c\left[H(s,a)K\rho(s)q(s)-\beta\frac{\rho(s)\,r(s)\,\varphi_1^{\alpha+1}(s,a)}{H^{(\alpha-1)/2}(s,a)}\right]ds>0. \tag{28}$$

In (27) we choose $l=c$. Then there exists $b>c$ such that

$$\int_c^b\left[H(b,s)K\rho(s)q(s)-\beta\frac{\rho(s)\,r(s)\,\varphi_2^{\alpha+1}(b,s)}{H^{(\alpha-1)/2}(b,s)}\right]ds>0. \tag{29}$$

Combining (28) and (29) we obtain (20). The conclusion thus comes from Theorem 1. The proof is complete. □

Corollary 3. *Assume that (H1)-(H5) hold. If for some* $H\in X$ *and each* $l\geq t_0$ *such that*

$$\limsup_{t\to\infty}\int_l^t\left[H(s,l)K\mu_0q(s)-\beta\frac{r(s)\,h_1^{\alpha+1}(s,l)}{H^{(\alpha-1)/2}(s,l)}\right]\cdot\exp\left(\int_{t_0}^s\frac{p(u)}{r(u)}du\right)ds>0, \tag{30}$$

and

$$\limsup_{t\to\infty}\int_l^t\left[H(t,s)K\mu_0q(s)-\beta\frac{r(s)\,h_2^{\alpha+1}(t,s)}{H^{(\alpha-1)/2}(t,s)}\right]\cdot\exp\left(\int_{t_0}^s\frac{p(u)}{r(u)}du\right)ds>0, \tag{31}$$

then every solution of Eq.(1) is oscillatory.

Theorem 4. *Assume that (H1)-(H5) hold. If, for every* $T\geq t_0$*, there exist* $H\in X_0$*,* $\rho\in C^1([t_0,\infty),(0,\infty))$ *and* $a,c\in\mathbb{R}$ *such that* $T\leq a<c$ *and*

$$\begin{aligned}&\int_a^c H(s-a)K\mu_0\left[\rho(s)q(s)+\rho(2c-s)q(2c-s)\right]ds\\ >\;&\beta\int_a^c\left[\frac{\rho(s)\,r(s)}{H^{(\alpha-1)/2}(s-a)}\psi_1^{\alpha+1}+\frac{\rho(2c-s)\,r(2c-s)}{H^{(\alpha-1)/2}(s-a)}\psi_2^{\alpha+1}(s,a)\right]ds,\end{aligned} \tag{32}$$

then every solution of Eq.(1) is oscillatory, where $\psi_1(s,a)$ *and* $\psi_2(s,a)$ *defined by (21).*

Proof. Let $b = 2c - a$. Then $H(b-c) = H(c-a) = H((b-a)/2)$, and for any $\eta \in L[a,b]$, we have

$$\int_c^b \eta(s)\, ds = \int_a^c \eta(2c-s)\, ds.$$

Hence

$$\int_c^b H(b-s)\, w(s)\, ds = \int_a^c H(s-a)\, w(2c-s)\, ds.$$

Thus that (32) holds implies that (21) holds for $H \in X_0$, $\rho \in C^1([t_0,\infty),(0,\infty))$ and therefore every solution of (1) is oscillatory by Theorem 1. The proof is complete. □

Corollary 4. *Assume that (H1)-(H5) hold. If, for every $T \geq t_0$, there exist $H \in X$ and $a, c \in \mathbb{R}$ such that $T \leq a < c$ and*

$$\begin{aligned}&\int_a^c H(s-a)\left(K\mu_0 q(s) - \beta \frac{r(s)\, h^{\alpha+1}(s-a)}{H^{(\alpha-1)/2}(s-a)}\right)\exp\left(\int_{t_0}^s \frac{p(u)}{r(u)}du\right) ds \\ &+ \int_a^c H(s-a)\left[K\mu_0 q(2c-s) - \beta\frac{r(2c-s)\, h^{\alpha+1}(s-a)}{H^{(\alpha-1)/2}(s-a)}\right] \\ &\cdot \exp\left(\int_{t_0}^{2c-s}\frac{p(u)}{r(u)}du\right) ds > 0. \end{aligned} \tag{33}$$

From above oscillation criteria, we can obtain different sufficient conditions for oscillation of all solutions of Eq.(1) by different choices of $H(t,s)$.

Let

$$H(t,s) = (t-s)^\lambda, \quad t \geq s \geq t_0,$$

where $\lambda > \alpha$ is a constant.

Corollary 5. *Assume that (H1)-(H5) hold. Then every solution of Eq. (1) is oscillatory provided that for each $l \geq t_0$ and for some $\lambda > \alpha$, there exists a function $\rho \in C^1([t_0,\infty),(0,\infty))$ such that the following two inequalities hold:*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\alpha}} \int_l^t (s-l)^\lambda\, \eta(s) ds > 0, \tag{34}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\alpha}} \int_l^t (t-s)^\lambda\, \zeta(s) ds > 0, \tag{35}$$

where

$$\eta(s) = K\mu_0\rho(s)q(s) - \beta\rho(s)\, r(s)\left|\frac{\lambda}{s-l} + \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)}\right|^{\alpha+1}$$

and

$$\zeta(s) = K\mu_0\rho(s)q(s) - \beta\rho(s)\, r(s)\left|\frac{\lambda}{t-s} - \frac{\rho'(s)}{\rho(s)} + \frac{p(s)}{r(s)}\right|^{\alpha+1}.$$

Corollary 6. *Assume that (H1)-(H5) hold. Then every solution of (1) is oscillatory provided that for each $l \geq t_0$ and for some $\lambda > \alpha$, the following two inequalities hold:*

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-\alpha}}\int_l^t (s-l)^\lambda\left[K\mu_0 q(s)-\beta r(s)\left(\frac{\lambda}{s-l}\right)^{\alpha+1}\right] \cdot\exp\left(\int_{t_0}^s \frac{p(u)}{r(u)}du\right)ds>0, \tag{36}$$

and

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-\alpha}}\int_l^t (t-s)^\lambda\left[K\mu_0 q(s)-\beta r(s)\left(\frac{\lambda}{t-s}\right)^{\alpha+1}\right] \cdot\exp\left(\int_{t_0}^s \frac{p(u)}{r(u)}du\right)ds>0. \tag{37}$$

Theorem 5. *Suppose that (H1)-(H5) hold, $\lim_{t\to\infty} R(t)=\infty$ and $\alpha\in\mathbb{N}$. Then every solution of equation (1) is oscillatory provided that for each $l > t_0$ and for some $\lambda > \alpha$, the following two inequalities hold:*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\left\{\int_l^t\left[K\mu_0 q(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{|p(s)|^{\alpha+1}}{r^\alpha(s)}\right](R(s)-R(l))^\lambda\right.\\&\left.-\frac{1}{(\alpha+1)^{\alpha+1}}\sum_{k=1}^{\alpha}\frac{\binom{\alpha+1}{k}\lambda^{\alpha+1-k}|p(s)|^k}{r^{(1+k(\alpha-1))/\alpha}(s)}(R(s)-R(l))^{\lambda+k-\alpha-1}\right\}ds\\&>\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\lambda^{\alpha+1}}{\lambda-\alpha}\end{aligned} \tag{38}$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\left\{\int_l^t\left[Kq(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{|p(s)|^{\alpha+1}}{r^\alpha(s)}\right](R(t)-R(s))^\lambda\right.\\&\left.-\frac{1}{(\alpha+1)^{\alpha+1}}\sum_{k=1}^{\alpha}\frac{\binom{\alpha+1}{k}\lambda^{\alpha+1-k}|p(s)|^k}{r^{(1+k(\alpha-1))/\alpha}(s)}(R(t)-R(s))^{\lambda+k-\alpha-1}\right\}ds\\&>\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\lambda^{\alpha+1}}{\lambda-\alpha},\end{aligned} \tag{39}$$

where $\binom{\alpha+1}{k}=\frac{(\alpha+1)!}{k!(\alpha+1-k)!}$, $k=1,2,\cdots,\alpha$.

Proof. In view of $H(t,s)=(R(t)-R(s))^\lambda$, for $t\geq t_0$, we have

$$h_1(t,s)=\frac{\lambda(R(t)-R(s))^{(\lambda-2)/2}}{r^{1/\alpha}(t)},\quad h_2(t,s)=\frac{\lambda(R(t)-R(s))^{(\lambda-2)/2}}{r^{1/\alpha}(s)}.$$

Let $\rho(t)\equiv 1$ for $t\geq t_0$, then $\rho'(t)\equiv 0$ and hence we have

$$\begin{aligned}\frac{\rho(s)\,r(s)\,\varphi_1^{\alpha+1}(s,l)}{H^{(\alpha-1)/2}(s,l)} &= r(s)\frac{\left|h_1(s,l)-\frac{p(s)}{r(s)}\sqrt{H(s,l)}\right|^{\alpha+1}}{H^{(\alpha-1)/2}(s,l)}\\ &= r(s)\frac{\left|\frac{\lambda[R(s)-R(l)]^{(\lambda-2)/2}}{r^{1/\alpha}(s)}-\frac{p(s)}{r(s)}\left(R(s)-R(l)\right)^{\lambda/2}\right|^{\alpha+1}}{[R(s)-R(l)]^{\lambda(\alpha-1)/2}}\\ &= r(s)\left|\frac{\lambda}{(R(s)-R(l))\,r^{1/\alpha}(s)}-\frac{p(s)}{r(s)}\right|^{\alpha+1}(R(s)-R(l))^{\lambda}\\ &\le r(s)\left(\frac{\lambda}{(R(s)-R(l))\,r^{1/\alpha}(s)}+\frac{|p(s)|}{r(s)}\right)^{\alpha+1}(R(s)-R(l))^{\lambda}. \quad (40)\end{aligned}$$

Thus from (40) one can see that

$$\begin{aligned}& H(s,l)Kq(s)-\beta\frac{r(s)\,\varphi_1^{\alpha+1}(s,l)}{H^{(\alpha-1)/2}(s,l)}\\ \ge\ & \left[Kq(s)-\beta r(s)\left(\frac{\lambda}{(R(s)-R(l))\,r^{1/\alpha}(s)}+\frac{|p(s)|}{r(s)}\right)^{\alpha+1}\right](R(s)-R(l))^{\lambda}\\ =\ & \left(Kq(s)-\beta\frac{|p(s)|^{\alpha+1}}{r^{\alpha}(s)}\right)(R(s)-R(l))^{\lambda}\\ & -\beta\sum_{k=1}^{\alpha}\frac{\binom{\alpha+1}{k}\lambda^{\alpha+1-k}\,|p(s)|^{k}}{r^{(1+k(\alpha-1))/\alpha}(s)}(R(s)-R(l))^{\lambda+k-\alpha-1}\\ & -\beta\frac{\lambda^{\alpha+1}}{r^{1/\alpha}(s)}(R(s)-R(l))^{\lambda-\alpha-1}.\end{aligned}$$

Noting that $\int_l^t \frac{1}{r^{1/\alpha}(s)}\left(R(s)-R(l)\right)^{\lambda-\alpha-1}ds=\frac{1}{\lambda-\alpha}\left(R(t)-R(l)\right)^{\lambda-\alpha}$. In view of $\lim_{t\to\infty}R(t)=\infty$, we can obtain

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_l^t\frac{1}{r^{1/\alpha}(s)}\left(R(s)-R(l)\right)^{\lambda-\alpha-1}ds=\frac{1}{\lambda-\alpha}. \quad (41)$$

From (38) and (41) we have that

$$\begin{aligned}& \limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\left(\int_l^t H(s,l)Kq(s)-\beta\frac{r(s)\,\varphi_1^{\alpha+1}(s,l)}{H^{(\alpha-1)/2}(s,l)}\right)ds\\ =\ & \limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_l^t\left[\left(Kq(s)-\beta\frac{|p(s)|^{\alpha+1}}{r^{\alpha}(s)}\right)(R(s)-R(l))^{\lambda}\right.\\ & \left.-\beta\sum_{k=1}^{\alpha}\frac{\binom{\alpha+1}{k}\lambda^{\alpha+1-k}\,|p(s)|^{k}}{r^{(1+k(\alpha-1))/\alpha}(s)}(R(s)-R(l))^{\lambda+k-\alpha-1}\right]ds\end{aligned}$$

$$
\begin{aligned}
&-\limsup_{t\to\infty}\frac{\beta\lambda^{\alpha+1}}{R^{\lambda-\alpha}(t)}\int_l^t \frac{1}{r^{1/\alpha}(s)}\left(R(s)-R(l)\right)^{\lambda-\alpha-1}ds\\
= &\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_l^t\left[\left(Kq(s)-\beta\frac{|p(s)|^{\alpha+1}}{r^{\alpha}(s)}\right)\left(R(s)-R(l)\right)^{\lambda}\right.\\
&\left.-\beta\sum_{k=1}^{\alpha}\frac{\binom{\alpha+1}{k}\lambda^{\alpha+1-k}|p(s)|^{k}}{r^{(1+k(\alpha-1))/\alpha}(s)}\left(R(s)-R(l)\right)^{\lambda+k-\alpha-1}\right]ds-\frac{\beta\lambda^{\alpha+1}}{\lambda-\alpha}\\
> &\ 0,
\end{aligned}
$$

that is,

$$
\limsup_{t\to\infty}\left(\frac{1}{R^{\lambda-\alpha}(t)}H(s,l)Kq(s)-\beta\frac{r(s)\varphi_1^{\alpha+1}(s,l)}{H^{(\alpha-1)/2}(s,l)}\right)ds>0.
$$

Similarly, we can obtain, by using the same method as the above and (39), that

$$
\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\left(\int_l^t H(t,s)Kq(s)-\beta\frac{r(s)\varphi_2^{\alpha+1}(t,s)}{H^{(\alpha-1)/2}(t,s)}\right)ds>0.
$$

By Theorem 3, every solution of Eq.(1) is oscillatory. Thus the proof is complete. □

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Cun-Hua Zhang received her BSc from Northwest Normal University in 1997 and MSc from Lanzhou University in 2006. Now she is a Lecturer at Lanzhou Jiaotong University. Her research interest is qualitative analysis for ordinary differential equations.

Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, P. R. China
e-mail: chzhang72@163.com