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NEIGHBORHOOD CONDITION AND FRACTIONAL f-FACTORS IN GRAPHS

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ABSTRACT. Let G be a graph with vertex set $V(G)$ and let f be a nonnegative integer-valued function defined on $V(G)$. A spanning subgraph F of G is called a fractional f-factor if $d_G^h(x) = f(x)$ for all $x \in V(G)$, where $d_G^h(x) = \sum_{e \in E_x} h(e)$ is the fractional degree of $x \in V(F)$ with $E_x = \{e : e = xy \in E(G)\}$. In this paper it is proved that if $\delta(G) \geq \frac{b^2(k-1)}{a}$, $n > \frac{(a+b)(k(a+b)-2)}{a}$ and $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \geq \frac{bn}{a+b}$ for any independent subset $\{x_1, x_2, ..., x_k\}$ of $V(G)$, then G has a fractional f-factor . Where $k \geq 2$ be a positive integer not larger than the independence number of G, a and b are integers such that $1 \leq a \leq f(x) \leq b$ for every $x \in V(G)$. Furthermore, we show that the result is best possible in some sense.

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1. Introduction

In our daily life many problems on optimization and network design, e.g., coding design, building blocks, the file transfer problems on computer networks, scheduling problems and so on, are related to the factors, factorizations and fractional factors [1]. In particular, a wide variety of systems can be described by complex networks. Such systems include: the cell, where we model the chemicals by nodes and their interactions by edges; the world wide web, which is a virtual network of web pages connected by hyper-links; and the food chain webs, the networks by which human diseases spread, the human collaboration networks etc. It is well-known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes,

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respectively. Henceforth we use the term *graph* instead of *network*. If the links in the network have different roles, then the edges of the graph can be with weights in $[0, 1]$. So we should consider the fractional graph theory.

The graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $d_G(x)$ the degree of a vertex x in G and by $N_G(x)$ the set of vertices adjacent to x in G. We use $N_G[x]$ to denote $N_G(x) \cup \{x\}$ and $\delta(G)$ to denote the minimum degree of G. For a subset $S \subseteq V(G)$, we denote by $N_G(S)$ the union of $N_G(x)$ for every $x \in S$, by $G[S]$ the subgraph of G induced by S, by $G-S$ the subgraph obtained from G by deleting the vertices in S together with the edges incident to the vertices in S.

Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for any $x \in V(G)$. A fractional (g, f)-indicator function is a function h that assigns to each edge of a graph G a number $h(e)$ in $[0, 1]$ so that for each vertex x we have $g(x) \leq d_G^h(x) \leq f(x)$, where $d_G^h(x) = \sum_{e \in E_x} h(e)$ is the fractional degree of $x \in G$ with $E_x = \{e : e = xy \in E(G)\}$. If $g(x) = f(x)$ for every $x \in V(G)$, a fractional (g, f)-indicator function is called a fractional f-indicator function. Let h be a fractional f-indicator function of a graph G. Set $E_h = \{e : e \in E(G) \text{ and } h(e) \neq 0\}$. If G_h is a spanning subgraph of G such that $E(G_h) = E_h$, then G_h is called a fractional f-factor of G. The other terminologies and notations can be found in [3].

Many authors have investigated graph factors [7,8], factorizations [12,14], and fractional (g, f)-factors [4,9,13]. There is a necessary and sufficient condition for a graph to have a fractional f-factor which was given by Guizhen Liu and Lanju Zhang.

Theorem 1. [9] *Let G be a graph. Then G has a fractional f-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - f(T) \geq 0,$$

where $T = \{x \in V(G) \setminus S \text{ and } d_{G-S}(x) \leq f(x)\}$.

In [6], Yanjun Li and Maocheng Cai gave the following result for a graph to have an $[a, b]$-factor.

Theorem 2. [6] *Let G be a graph of order n, and let a and b be integers such that $1 \leq a < b$. Then G has an $[a, b]$-factor if $\delta(G) \geq a$, $n \geq 2a + b + \frac{a^2-a}{b}$ and*

$$\max\{d_G(x), d_G(y)\} \geq \frac{an}{a+b}$$

for any two nonadjacent vertices x and y in G.

In [10], Haruhide Matsuda gave a sufficient condition in terms of neighborhood union for the existence of $[a, b]$-factors.

Theorem 3. [10] *Let a and b be integers such that $1 \le a < b$, and let G be a graph of order n with $n \ge \frac{2(a+b)(a+b-1)}{b}$, and $\delta(G) \ge a$. If*

$$|N_G(x) \cup N_G(y)| \ge \frac{an}{a+b}$$

for any two nonadjacent vertices x and y of G , then G has an $[a, b]$-factor.

In [5], Jianxiang Li proved the following theorem, which is an extension of Theorem 3.

Theorem 4. [5] *Let a and b be integers such that $1 \le a < b$, and let G be a graph of order n with $n \ge \frac{(a+b)(k(a+b)-2)}{b}$. If $\delta(G) \ge (k-1)a$, and*

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \ge \frac{an}{a+b}$$

for any independent subset $\{x_1, x_2, ...x_k\}$ of $V(G)$, where $k \ge 2$, then G has an $[a, b]$-factor.

It is easy to see that Theorem 3 is a special case of Theorem 4 for $k = 2$.

In this paper, we give a neighborhood condition for a graph to have a fractional f-factor. Our main result is an extension of Theorem 4.

Theorem 5. *Let G be a graph of order n, and let a and b be integers with $1 \le a \le b$, and let $f(x)$ be a nonnegative integer-valued functions defined on $V(G)$ such that $a \le f(x) \le b$ for each $x \in V(G)$. Let k be a positive integer not larger than the independence number of G. Then G has a fractional f-factor if $\delta(G) \ge \frac{b^2(k-1)}{a}$, $n > \frac{(a+b)(k(a+b)-2)}{a}$ and*

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \ge \frac{bn}{a+b} \tag{1}$$

for any independent subset $\{x_1, x_2, ...x_k\}$ of $V(G)$, where $k \ge 2$.

In Section 3, we shall show that the condition (1) in Theorem 5 is best possible in some sense. Furthermore, we present a conjecture on the existence of f-factor.

2. Proof of Theorem 5

Proof of Theorem 5. Let G be a graph satisfying the hypothesis of Theorem 5, we prove the theorem by contradiction. Suppose that G has no fractional f-factors. Then, according to Theorem 1, there exist some subset $S \subseteq V(G)$ such that

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - f(T) \le -1, \tag{2}$$

where $T = \{x \in V(G) \setminus S \text{ and } d_{G-S}(x) \le f(x)\}$. We choose such subsets S and T which satisfy $|T|$ is as small as possible. We first prove the following claims.

Claim 1. $d_{G-S}(x) \le f(x) - 1 \le b - 1$ for all $x \in T$.

Proof. If $d_{G-S}(x) \ge f(x)$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2). This contradicts the choice of S and T. Therefore,

$$d_{G-S}(x) \le f(x) - 1 \le b - 1$$

for all $x \in T$ holds. □

Claim 2. $|T| \geq a+1$.

Proof. If $|T| \leq a$, then by (2) and since $|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) \geq \frac{b^2(k-1)}{a} \geq b(k-1)$ (since $a \leq b$) for all $x \in T$, we obtain

$$\begin{aligned}
-1 &\geq \delta_G(S,T) = f(S) + d_{G-S}(T) - f(T) \\
&\geq a|S| + d_{G-S}(T) - b|T| \\
&\geq |T||S| + d_{G-S}(T) - b|T| \\
&= \textstyle\sum_{x\in T}(|S| + d_{G-S}(x) - b) \\
&\geq \textstyle\sum_{x\in T}(b(k-1) - b) \geq 0,
\end{aligned}$$

which is a contradiction. Thus $|T| \geq a+1$. □

Since $T \neq \phi$, in the following we shall construct a sequence $x_1, x_2, ...x_\pi$ of vertices of T. Let

$$h_1 = \min\{d_{G-S}(x) | x \in T\}$$

and choose $x_1 \in T$ to be a vertex such that $d_{G-S}(x_1) = h_1$. By Claim1, we have $h_1 \leq f(x) - 1 \leq b-1$.

If $j \geq 2$ and $T \setminus (\bigcup_{i=1}^{j-1} N_T[x_i]) \neq \phi$, define

$$h_j = \min\left\{ d_{G-S}(x) | x \in T \setminus \left(\bigcup_{i=1}^{j-1} N_T[x_i] \right) \right\}$$

and choose $x_j \in T \setminus (\bigcup_{i=1}^{j-1} N_T[x_i])$ to be a vertex such that $d_{G-S}(x_j) = h_j$. This defines a nondecreasing sequence of integers $0 \leq h_1 \leq h_2 \leq \cdots \leq h_\pi \leq f(x) - 1 \leq b-1$(by Claim 1) and a sequence of independent vertices $x_1, x_2, ..., x_\pi$ in T with $d_{G-S}(x_i) = h_i$ $(1 \leq i \leq \pi)$ and $T \setminus (\bigcup_{i=1}^{\pi} N_T[x_i]) = \phi$.

Claim 3. $|T| \geq (k-1)b+1$.

Proof. Suppose that $|T| \leq (k-1)b$. Since $|S| + h_1 \geq d_G(x_1) \geq \delta(G) \geq \frac{b^2(k-1)}{a}$, by (2) and $0 \leq h_1 \leq b-1$, it follows that

$$\begin{aligned}
-1 &\geq f(S) + d_{G-S}(T) - f(T) \\
&\geq a|S| + h_1|T| - b|T| \\
&= a|S| + (h_1 - b)|T| \\
&\geq a(\tfrac{b^2(k-1)}{a} - h_1) + (h_1 - b)(k-1)b \\
&= b^2(k-1) + h_1(b(k-1) - a) - b^2(k-1) \\
&\geq 0.
\end{aligned}$$

This is a contradiction. Thus we have $|T| \geq (k-1)b+1$. □

Since $d_{G-S}(x) \leq b-1$(by Claim 1) and $|T| \geq (k-1)b+1$(by Claim 3), we have $\pi \geq k$ and we can take the independent subset $\{x_1, x_2, ..., x_k\} \subseteq T$.

By the assumption of the theorem we can get the following inequalities :

$$\frac{bn}{a+b} \leq |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \leq |S| + \sum_{i=1}^{k} h_i.$$

It follows that

$$|S| \geq \frac{bn}{a+b} - \sum_{i=1}^{k} h_i. \tag{3}$$

Since $n - |S| - |T| \geq 0$ and $b - h_k \geq 1$, we have $(n - |S| - |T|)(b - h_k) \geq 0$. Note that

$$|N_T[x_i]| - |N_T[x_i] \cap (\bigcup_{j=1}^{i-1} N_T[x_j])| \geq 1, i = 2, 3, ..., k-1$$

and

$$\left|\bigcup_{j=1}^{i} N_T[x_j]\right| \leq \sum_{j=1}^{i} |N_T[x_j]| \leq \sum_{j=1}^{i} (d_{G-S}(x_j) + 1) = \sum_{j=1}^{i} (h_j + 1), i = 1, 2, ..., k.$$

Therefore we have

$$\begin{aligned}
& (n - |S| - |T|)(b - h_k) \\
\geq\ & f(S) + d_{G-S}(T) - f(T) + 1 \\
\geq\ & a|S| + d_{G-S}(T) - b|T| + 1 \\
\geq\ & a|S| + h_1|N_T[x_1]| + h_2(|N_T[x_2]| - |N_T[x_2] \cap N_T[x_1]|) + \cdots \\
& + h_{k-1}\left(|N_T[x_{k-1}]| - \left|N_T[x_{k-1}] \cap (\bigcup_{i=1}^{k-2} N_T[x_i])\right|\right) \\
& + h_k\left(|T| - \left|\bigcup_{i=1}^{k-1} N_T[x_i]\right|\right) - b|T| + 1 \\
\geq\ & a|S| + (h_1 - h_k)|N_T[x_1]| + \sum_{i=2}^{k-1} h_i + (h_k - b)|T| - h_k \sum_{i=2}^{k-1} |N_T[x_i]| + 1 \\
=\ & a|S| + (h_1 - h_k)(h_1 + 1) + \sum_{i=2}^{k-1} h_i + (h_k - b)|T| - h_k \sum_{i=2}^{k-1} (h_i + 1) + 1 \\
=\ & a|S| + h_1^2 + \sum_{i=1}^{k-1} h_i + (h_k - b)|T| - h_k \sum_{i=1}^{k-1} (h_i + 1) + 1.
\end{aligned}$$

Thus it follows that

$$0 \leq n(b - h_k) - (a + b - h_k)|S| + h_k \sum_{i=1}^{k-1} h_i - \sum_{i=1}^{k-1} h_i + h_k(k-1) - h_1^2 - 1. \tag{4}$$

By (3) and (4), $h_1 \leq h_2 \leq \cdots \leq h_k \leq b - 1$, and $n > \frac{(a+b)(k(a+b)-2))}{a}$, we have

$$0 \leq n(b - h_k) - (a + b - h_k)(\frac{bn}{a+b} - \sum_{i=1}^{k} h_i) + h_k \sum_{i=1}^{k-1} h_i$$

$$
\begin{aligned}
& -\sum_{i=1}^{k-1} h_i + h_k(k-1) - h_1^2 - 1 \\
= \; & -\frac{an}{a+b}h_k + (a+b)\sum_{i=1}^{k} h_i - h_k \sum_{i=1}^{k} h_i + h_k \sum_{i=1}^{k-1} h_i - \sum_{i=1}^{k-1} h_i \\
& + h_k(k-1) - h_1^2 - 1 \\
= \; & -\frac{an}{a+b}h_k + ((a+b-1)h_1 - h_1^2) + (a+b-1)\sum_{i=2}^{k-1} h_i \\
& + h_k(a+b+k-1) - h_k^2 - 1 \\
\leq \; & -\frac{an}{a+b}h_k + (a+b-1)h_k + (a+b-1)\sum_{i=2}^{k-1} h_k \\
& + h_k(a+b+k-1) - h_k^2 - 1 \\
= \; & -\frac{an}{a+b}h_k + k(a+b)h_k - h_k^2 - 1.
\end{aligned}
$$

If $h_k > 0$, then $0 < 2h_k - h_k^2 - 1 \leq 0$ (since $n > \frac{(a+b)(k(a+b)-2))}{a}$), that is a contradiction. If $h_k = 0$, then $0 \leq -1$, a contradiction. So we conclude that G has a fractional f-factor. Completing the proof of Theorem 5. □

3. Remarks

Remark 1. By the following example we can demonstrate that the condition (1) in Theorem 5 is best possible in the following sense. The condition (1) in Theorem 5 can not be replaced by $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \geq \frac{bn}{a+b} - 1$. We let $G_1 = K_{bt}$ be a complete graph and $G_2 = (at+1)K_1$ be $at+1$ independent vertices. Then let $G = G_1 + G_2$ be the join of G_1 and G_2(that is, $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$), where t is a sufficiently large positive integer(For some k, we choose $t > \frac{k(a+b)-2}{a} - \frac{1}{a+b}$, thus the conditions $\delta(G) \geq \frac{b^2(k-1)}{a}$ and $n > \frac{(a+b)(k(a+b)-2)}{a}$ suffice). Then it follows that $n = |G_1| + |G_2| = (a+b)t + 1$ and

$$
\frac{bn}{a+b} > |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| = bt > \frac{bn}{a+b} - 1
$$

for any subset $\{x_1, x_2, ... x_k\}$ of G_2. We take $S = V(G_1)$ and $f(x) = a$ for $x \in V(G_1)$; $T = V(G_2)$ and $f(x) = b$ for $x \in V(G_2)$. It is easy to see G has no fractional f-factors because $\delta_G(S,T) = f(S) - f(T) = a|S| - b|T| = abt - b(at+1) = -b < 0$.

Remark 2. We can see that the minimum degree bound in Theorem 5 $\delta(G) \geq \frac{b^2(k-1)}{a}$ is best possible when $b = a$ and $k = 2$.

Finally we present the following conjecture.

Conjecture. *Let G be a connected graph of order n, and let a and b be integers with $1 \le a \le b$, and let $f(x)$ be a nonnegative integer-valued functions defined on $V(G)$ such that $a \le f(x) \le b$ for each $x \in V(G)$. If $f(V(G))$ is even, $\delta(G) \ge \frac{b^2(k-1)}{a}$, $n > \frac{(a+b)(k(a+b)-2)}{a}$ and $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \ge \frac{bn}{a+b}$ for any independent subset $\{x_1, x_2, ... x_k\}$ of $V(G)$, where $k \ge 2$ be a positive integer not larger than the independence number of G, then G has an f-factor*

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