J. Appl. Math. & Informatics Vol. **27***(2009), No. 5 - 6, pp. 1145 - 1156*
Website: http://www.kcam.biz

THE EXISTENCE AND CONTROLLABILITY OF SOLUTIONS FOR THE NEUTRAL FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAY TERMS

YOUNG CHEL AHN, HAE WON KIM AND DONG GUN PARK*

ABSTRACT. In this paper, we consider the existence, uniqueness of the solutions and controllability for the neutral functional integro-differential equations with delay terms by Sadovskii's fixed point theorem.

AMS Mathematics Subject Classification: 34G20,45L05
Key words and Phrases : Nonlinear impulsive evolution, strongly continuous semigroup, contraction mapping, fixed point theorem, cosine family, regular.

1. Introduction

Controllability of nonlinear systems in abstract spaces has been studied by Chukwu and Lenhart[6]. Quinn and Carmichael[14] have shown that the controllability problem in Banach space can be converted into a fixed pointed problem for a single-valued mapping. Kwun et al.[11] investigated the controllability and approximate controllability of delay Vloterra systems by using a fixed point theorem. Byszewski and Akca[4] studied the existence of solution of semi-linear functional-differential evolution nonlocal problem, where $-A$is the infinitesimal generator of a compact strongly continuous semigroup. Recently, Fu and Ezzinbi[8], by using fractional power of operators and Sadovskii's fixed point theorem, studied the existence of mild and strong solutions of semi-linear neutral functional differential evolution equations with nonlocal conditions.

In this paper, by investigating [8], we study the existence, uniqueness, magnitude, continuousness and controllability of solutions for neutral functional

Received September 26, 2008. Accepted February 3, 2009. *Corresponding author
This paper was supported by Dong-A university Research Foundation.

integro-differential equations in a phase space with nonlocal conditions as following:

$$\begin{cases} \dfrac{d}{dt}[x(t) - g(t, x_t)] + Ax(t) \\ \qquad = f(t, x_t, \displaystyle\int_0^t k(t, s, x_s)ds) + Cu(t), \quad t \in [0, b], \\ x(t) = \phi(t), x_0 = \phi \quad t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where X is a Banach space with norm $\|\cdot\|$ and $-A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators. A phase space $B : (-\infty, 0] \to X$ is a linear space of functions, $g : [0, b] \times B \to X$ and $k : [0, b] \times [0, b] \times B \to X$ are continuous functions, $f : [0, b] \times B \times X \to X$ is Hölder continuous. C is a bounded linear operator of another Banach space into U. control function $u(\cdot)$ is given in $L^2([0, b] : U)$

2. Preliminaries

Let $0 \in \rho$, then it is possible to define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \qquad x \in D(A^\alpha),$$

defines a norm on $D(A^\alpha)$. Hereafter we denote by X_α the Banach space, and $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha \leq 1$ and the imbedding is compact whenever the resolvent operator of A is compact. For semigroup $\{T(t)_{t \geq 0}\}$, the following properties will be used:

(1) there is a $M > 1$ such that $\|T(t)\| \leq M$, for all $t \in J = [0, b]$.

(2) For any $b > 0$, there exists a positive constant C_α such that

$$\|A^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, \ 0 < t \leq b.$$

To study the system (1.1), we assume that the histories $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta), t > 0, \theta \in (-\infty, 0]$ belong to some abstract phase space B, which is defined axiomatically. In this paper, we will employ an axiomatic definition of the phase space B introduced by Hale and Kato[7]. Thus B will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_B$. We will assume that B satisfies the following axioms:

(A) If $x : (-\infty, \sigma + b) \to X, b > 0$, is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in B$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:

(i) $x_t \in B$,

(ii) $\|x(t)\|_X \leq H\|x_t\|_B$,

(iii) $\|x_t\|_B \leq K(t - \sigma) \sup \Big\{\|x(s)\|_X : \sigma \leq s \leq t\Big\} + M(t - \sigma)\|x_\sigma\|_B$, where $H \geq 0$ is a constant, $K : [0,+\infty) \to [0,+\infty)$ is continuous function and $M : [0,+\infty) \to [0,+\infty)$ is locally bounded, and H, K, M are independent of $x(t)$.

(B) B is complete.

Now we give the basic assumptions on the system (1.1).

(H1) If $g : [0,b] \times B \to X$ is continuous function and $\beta \in (0,1)$ then

(i) for $0 \leqslant s_1, s_2 \leqslant b$, $\phi_1, \phi_2 \in B$, there exists $L > 0$ such that

$$\left\|A^\beta g(s_1,\phi_1) - A^\beta g(s_2,\phi_2)\right\| \leq L(|s_1 - s_2| + \|\phi_1 - \phi_2\|_B),$$

(ii) for $t \in J, , \phi \in B$ there exists $L_1 > 0$ such that

$$\|A^\beta g(t,\phi)\| \leq L_1\Big(\|\phi\|_B + 1\Big)t \in J. \tag{2.1}$$

(H2) $f(t,0,0) \equiv 0$ and $f : [0,b] \times B \times X \to X$ is Hölder continuous, that is, there exist $L_2 > 0$ and $0 < \sigma_1, \sigma_2, \sigma_3 < 1$ such that

$$\left\|f(t,x_1,y_1) - f(s,x_2,y_2)\right\|_X \leq L_2\Big(|t-s|^{\sigma_1} + \|x_1 - x_2\|_B^{\sigma_2} + \|y_1 - y_2\|^{\sigma_3}\Big).$$

(H3) $k(t,s,0) \equiv 0$ and $k : [0,b] \times [0,b] \times B \to X$ is continuous function and there exists $L_3 > 0$ such that

$$\|k(t,s,x_1) - k(t,s,x_2)\|_X \leq L_3\|x_1 - x_2\|_B \ .$$

3. Existence and uniqueness of mild solution

In this chapter, we prove the existence, uniqueness, magnitude and continuousness of mild solution for system (1.1). System (1.1) can be expressed to integral equations as following:

$$\begin{cases} x(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \displaystyle\int_0^t AT(t-s)g(s,x_s)ds \\ \quad + \displaystyle\int_0^t T(t-s)\left[Cu(s) + f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\right] ds, \quad t \in [0,b], \\ x(t) = \phi(t), x_0 = \phi \quad t \in (-\infty, 0], \end{cases} \tag{3.1}$$

where $\phi \in B, u \in L^2([0,b] : U)$.

Now for system (1.1), it is natural to define the mild solution as following:

Definition 3.1. We say that a continuous function $x(\cdot) : [-\infty, b] \to X$ is a mild solution of Cauchy problem (1.1) if for each $0 \leq t \leq b$, the function $AT(t-s)g(s,x_s), s \in [0,t)$ is integrable and integral equation (3.1) is verified:

Theorem 3.2. *Let $\phi \in B$. If the assumptions (H1)-(H3) are satisfied, a phase space B satisfies conditions (A), (B) and for $u \in L^2([0,b] : U)$, $\widehat{L} = \max\{L, L_1\}$,*

$$L_0 := M_0\widehat{L}(M+1) + \frac{1}{\beta}C_{1-\beta}b^\beta\widehat{L} + bML_2k_0(2 + bL_3) \leqslant \frac{k-P}{k+1} \tag{3.2}$$

holds, then Caushy problem (1.1) has a unique mild solution. Here $M_0 = \|A^{-\beta}\|$ *and* $P = \|T(t)\phi(0)\| + M\|C\|\|u\|_{L^2(0,b:u)}\sqrt{b}$.

Proof. For arbitrary $0 < t < b$, define a mapping Φ as following:

$$(\Phi x)(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)ds$$
$$+ \int_0^t T(t-s)\left[Cu(s) + f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\right] ds, \quad t \in J.$$

and for positive number $k > 0$, define a nonempty closed bounded set

$$B_k = \Big\{x \in B : \|x_t\| \leqslant k, 0 \leqslant t \leqslant b\Big\}.$$

Then by equation (2.1) and (H1), for $0 < \beta \le 1$, since

$$\| AT(t-s)g(t,x_t)\| \leqslant \|A^{1-\beta}T(t-s)A^\beta g(t,x_t)\| \leqslant \frac{C_{1-\beta}}{(t-s)^{1-\beta}} L_1(k+1),$$

$$\begin{aligned}&\|(\Phi x)(t)\|\\ &\leqslant \|T(t)\phi(0)\| + (M+1)M_0 L_1(k+1) + \frac{1}{\beta}C_{1-\beta}b^\beta L_1(k+1)\\ &\quad + M\|C\|\|u\|_{L^2(0,b:u)}\sqrt{b} + bML_2(2+bL_3)(k+1)\\ &\leqslant k\end{aligned}$$

holds and so $\|(\Phi x)\|_{C([0,b]:X)} = \sup_{0\leqslant t\leqslant b} \|(\Phi x)(t)\| \leqslant k$ follows. Thus $(\Phi x)(t) \in B_k$, that is, Φ is a mapping from $B_k$into B_k. And since

$$\begin{aligned}Q &= \left\|\int_0^t T(t-s)\left[f\left(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau\right) - f\left(s,y_s,\int_0^s k(s,\tau,y_\tau)d\tau\right)\right] ds\right\|\\ &\le bML_2\Big[\|x_s - y_s\|_B^{\sigma_2} + \|bL_3(x_\tau - y_\tau)\|_B^{\sigma_3}\Big],\end{aligned}$$

by (H2), there exist $k_2, k_3 > 0$ such that

$$\|x_s - y_s\|^{\sigma_2} \leqslant k_2\|x_s - y_s\|, \qquad \|x_s - y_s\|^{\sigma_3} \leqslant k_3\|x_s - y_s\|.$$

Let $k_0 = \max\Big\{k_2, k_3\Big\}$, then we have

$Q \le bML_2(k_2 + k_3(bL_3)^{\sigma_3})\|x_t - y_t\| \le bML_2k_0(2+bL_3)\|x_t - y_t\|$,

and so

$\|(\Phi x)(t) - (\Phi y)(t)\| \leqslant L_0 \sup_{0\leqslant t\leqslant b} \|x(t) - y(t)\|$.

Since $L_0 < 1$, Φ is contraction mapping and so we get fixed point $x(t) \in B_k$. Therefore equation (3.1) is mild solution of Cauchy problem (1.1). □

From the result of theorem 3.1, we can define solution mapping

$$W : L^2([0,b]:U) \to C([0,b]:X)$$

represented by

$$(Wu)(t) = x(\phi : u) \in C([0,b]:X)$$

Theorem 3.3. *Let $\phi \in B$ and $u(\cdot) \in L^2([0,b]:U)$. Then under the assumptions (H1)-(H3), solution mapping $(Wu)(t) = x(\phi : u)$ satisfies*

$$\|x(\phi:u)\|_{C([0,b]:X)} \leqslant D(M\|\phi\|_{C(0,b:X)} + M\|C\|\|u\|_{L^2(0,b:U)}\sqrt{b}), \quad 0 \leqslant t \leqslant b,$$

where D depends on $M,\ \beta,\ L_1,\ L_2,\ L_3,\ k$ and b.

Proof. By assumption,

$$\begin{aligned}
&\|x(\phi:u)(t)\| \\
&\leqslant \left\|T(t)\Big[\phi(0) - g(0,\phi)\Big]\right\| + \|g(t,x_t)\| + \left\|\int_0^t AT(t-s)g(s,x_s)ds\right\| \\
&\quad + \left\|\int_0^t T(t-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)ds\right\| + \left\|\int_0^t T(t-s)Cu(s)ds\right\| \\
&\leqslant M\|\phi\| + MC\|u\|_{L^2(0,b:U)}\sqrt{b} \\
&\quad + \Big\{MM_0L_1(k+1) + M_0L_1(k+1) + \frac{1}{\beta}b^\beta C_{1-\beta}L_1(k+1) \\
&\quad + bML_2k_0(2+bL_3)\Big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x(\phi:u)\|_{C(0,b:X)} &= \sup_{0\leqslant t\leqslant b}\|x(\phi:u)(t)\| \\
&\leqslant D(M\|\phi\|_{C([0,b]:x)} + MC\|u\|_{L^2(0,b:U)}\sqrt{b}).
\end{aligned}$$

□

Theorem 3.4. *Let $\phi \in B$ and $u(\cdot) \in L^2([0,b]:U)$. Then by the assumptions (H1)-(H3), solution (3.1) can be extended to the interval $[0,2b]$ and satisfies*

$$\|x(\phi:u)\|_{C(0,2b:X)} \leqslant D_1(M\|x(\phi:u)\|_{C(0,2b:X)} + MC\|u\|_{L^2(0,2b:U)}\sqrt{2b}),$$

where D_1 depends on $M,\ \beta,\ L_1,\ L_2,\ L_3,\ k$ and b.

Proof. For $t_1,\ t_2 > 0$ if $x(\phi:u)(t)$ is a solution of equation (3.1) at the interval $[0, t_1+t_2]$, then in case of $t \in [t_1, t_1+t_2]$,

$$\begin{aligned}
&x(\phi:u)(t) \\
&= T(t-t_1)[x(\phi:u)(t_1) - g(0,\phi)] + g(t,x_t) + \int_{t_1}^t AT(t-s)g(s,x_s)ds \\
&\quad + \int_{t_1}^t T(t-s)f(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau)ds + \int_{t_1}^t T(t-s)Cu(s)ds
\end{aligned}$$

holds and in case of $t \in [0, t_2]$,

$$\begin{aligned}
&x(\phi:u)(t_1+t) \\
&= T(t)[x(\phi:u)(t_1) - g(0,\phi)] + g(t,x_t) + \int_0^t AT(t-s)g(s+t_1, x_{s+t_1})ds
\end{aligned}$$

$$+\int_0^t T(t-s)(f(s+t_1, x_{s+t_1}, \int_0^{s+t_1} k(s+t_1,\tau,x_\tau)d\tau)+Cu(s+t_1))ds$$

holds. This means that $t \to x(\phi : u)(t_1+t)$ is the solution of equation (3.1) with initial data $x(\phi : u)(t_1)$ in $[0, t_2]$. Inversely, let $x^*(\phi : u)(t)$ be the solution of equation (3.1) in $[0, t_1]$ and $\widehat{x}^*(\phi : u)(t)$ be the solution of equation (3.1) with initial data $\widehat{x}^*(\phi : u)(t_1)$ in $[0, b]$. If

$$x(\phi : u)(t) = \begin{cases} x^*(\phi : u)(t), & 0 \leqslant t \leqslant t_1 \\ \widehat{x}^*(\phi : u)(t-t_1), & t_1 \leqslant t \leqslant t_1+b, \end{cases}$$

then $x(\phi : u)(t)$ is the solution of (3.1) in $[0, t_1+b]$. Thus let $t_1 = b$ then $x(\phi : u)(t)$ is the solution of (3.1) in $[0, 2b]$. By the assumptions,

$$\begin{aligned}
&\|x(\phi : u)(t+b)\| \\
&\leqslant \|T(t)[x(\phi : u)(b) - g(0,\phi)]\| + \|g(t,x_t)\| + \left\|\int_0^t AT(t-s)g(s+b,x_{s+b})ds\right\| \\
&\quad + \left\|\int_0^t T(t-s)f(s+b,x_{s+b},\int_0^{s+b} k(s+b,\tau,x_\tau)d\tau)ds\right\| \\
&\quad + \left\|\int_0^t T(t-s)Cu(s+b)ds\right\| \\
&\leqslant M\|x(\phi : u)(b)\| + M_0L_1(M+1)(k+1) + MC\|u\|_{L^2(0,2b:U)}\sqrt{2b} \\
&\quad + \frac{1}{\beta}b^\beta C_{1-\beta}L_1(k+1) + bML_2(k^{\sigma_2} + (bL_3k)^{\sigma_3})
\end{aligned}$$

holds. And so

$$\begin{aligned}
\|x(\phi : u)\|_{C(0,2b:X)} &= \sup_{0\leqslant t\leqslant b} \|x(\phi : u)(t+b)\| \\
&\leqslant D_1\Big(M\|x(\phi : u)(b)\|_{C(0,2b:X)} + MC\|u\|_{L^2(0,b:U)}\sqrt{2b}\Big).
\end{aligned}$$

□

Theorem 3.5. *Suppose $u_n \to u$ as $n \to \infty$ on U. Then, for each b, $x(\phi : u_n)(t)$ converges to $x(\phi : u)(t)$ on $C([0,b] : X)$ as $n \to \infty$.*

Proof. By theorem 3.2,

$$\|x(\phi : u)(t) - x(\phi : u_n)(t)\| \leqslant MC\|u-u_n\|_{L^2(0,b:U)}\sqrt{b},$$

holds and since $u_n \to u$ as $n \to \infty$, "$MC\|u-u_n\|_{L^2(0,b:U)}\sqrt{b} \longrightarrow 0$ follows. Therefore, $x(\phi : u_n)(t)$ converges to $x(\phi : u)(t)$ on $C([0,b] : X)$ as $n \to \infty$. □

4. Nonlocal controllability

In this chapter, we investigate controllability for neutral functional integro-differential equation (1.1) by using Sasovskii's fixed point theorem. That is, we find reachable conditions from initial value x_0 to target $x(x_0 : u)(b) = x^1$ after time b and consider how will be control function $u \in L^2([0,b] : U)$ represented.

Now suppose followings :

(H4) For $b \in J$, define a linear operator $W : U \to X$ as following :

$$Wu = \int_0^b T(b-s)Cu(s)ds.$$

Then $\widetilde{W}^{-1}$ exists on $L^2([0,b] : U)/\ker W$ and control function for arbitrary function $x(\cdot)$ is

$$u(t) = \widetilde{W}^{-1}\Big[x^1 - T(b)\{\phi - g(0,\phi)\} - g(b,x_b) - \int_0^b AT(b-s)g(s,x_s)ds$$
$$- \int_0^b T(b-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)ds\Big](t).$$

And by using above control function, define operator P as following:

$$\begin{aligned}(Px)(t) &= T(t)[\phi - g(0,\phi)] + g(t,x_t)\\ &+ \int_0^t AT(t-s)g(s,x_s)ds + \int_0^t T(t-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)ds\\ &+ \int_0^t T(t-s)C\widetilde{W}^{-1}\Big[x^1 - T(b)\{\phi(0) - g(0,\phi)\}\\ &- g(b,x_b) - \int_0^b AT(t-s)g(s,x_s)ds\\ &- \int_0^b T(t-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\Big](s)ds.\end{aligned}$$

Theorem 4.1. *If the assumptions (H1)-(H4) are satisfied and for $\phi \in B$, $u \in L^2([0,b] : U)$,*

$$L_1\left(MM_0 + M_0 + \frac{1}{\beta}b^\beta C_{1-\beta}\right)\left(1 + M\|C\|\|\widetilde{W}^{-1}\|b\right) < 1 \tag{4.1}$$

is satisfied, then Cauchy problem (1.1) is controllable on $[0,b]$.

Proof. For positive integer K, let

$$B_K = \Big\{x \in X : \|x_t\|_B \leqslant K, \quad 0 \leqslant t \leqslant b\Big\},$$

then B_K is clear bounded, closed convex set on X.

By (H1) and (2.1),

$$\begin{aligned}\|AT(t-s)g(s,x_s)\| &= \left\|A^{1-\beta}T(t-s)A^{\beta}g(s,x_s)\right\| \\ &\leqslant \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1(K+1)\end{aligned}$$

holds and from the Bocher's theorem[14], $AT(t-s)g(s,x_s)$ is integrable at $[0,b]$. Thus P is well defined on B_K. Now we will verify that there exists appropriate positive integer K such that $PB_K \subseteq B_K$. For positive integer K, if there exists a function $x_K(\cdot) \in B_K$ such that $Px_K \notin B_K$, then since $\|Px_K(t)\| > K$ for $t \in [0,b]$, from

$$\begin{aligned}K &< \|P(x_K)(t)\| \\ &= \Big\|T(t)\Big[\phi(0)-g(0,\phi)\Big]+g(t,x_t) \\ &\quad + \int_0^t AT(t-s)g(s,x_s)ds + \int_0^t T(t-s)f\left(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau\right)ds \\ &\quad + \int_0^t T(t-s)C\widetilde{W}^{-1}\Big[x^1 - T(b)\{\phi-g(0,\phi)\} \\ &\qquad - g(b,x_b) - \int_0^b AT(t-s)g(s,x_s)ds \\ &\qquad - \int_0^b T(t-s)f\left(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau\right)\Big](s)ds\Big\| \\ &\leqslant M\Big\{\|\phi\|+M_0L_1(K+1)\Big\}+M_0L_1(K+1) \\ &\quad + \frac{1}{\beta}b^{\beta}C_{1-\beta}L_1(K+1)+bML_2\Big(K^{\sigma_2}+(bL_3K)^{\sigma_3}\Big) \\ &\quad + MC\|\widetilde{W}^{-1}\|\Big[\|x^1\|+M\Big(\|\phi\|+M_0L_1(K+1)\Big) \\ &\quad + M_0L_1(K+1)+\frac{1}{\beta}b^{\beta}C_{1-\beta}L_1(K+1)+bML_2(K^{\sigma_2}+(bL_3K)^{\sigma_3})\Big]b,\end{aligned}$$

dividing on both sides by K and taking the lower limit as $K \to \infty$, we get

$$\begin{aligned}1 &\leqslant MM_0L_1+M_0L_1+\frac{1}{\beta}b^{\beta}C_{1-\beta}L_1 \\ &\quad + M\|C\|\|\widetilde{W}^{-1}\|b\Big\{MM_0L_1+M_0L_1+\frac{1}{\beta}b^{\beta}C_{1-\beta}L_1\Big\} \\ &= L_1\Big(MM_0+M_0+\frac{1}{\beta}b^{\beta}C_{1-\beta}\Big)\Big(1+M\|C\|\|\widetilde{W}^{-1}\|b\Big)\end{aligned}$$

However, this contradicts (4.1). Hence for positive integer K, $PB_K \subseteq B_K$.

Next we will show that operator P has a fixed point on B_K, which implies Equation (1.1) has a mild solution. To this end, we decompose P as $P = P_1 + P_2$, where the operators P_1, P_2 are defined on B_K, by

$$(P_1x)(t) = T(t)\Big[\phi(0) - g(0,\phi)\Big] + g(t,x_t)$$
$$+ \int_0^t AT(t-s)g(s,x_s)ds + \int_0^t T(t-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)ds,$$
$$(P_2x)(t) = \int_0^t T(t-s)C\widetilde{W}^{-1}\Big[x^1 - T(b)\Big[\phi(0) - g(0,\phi)\Big]$$
$$- g(b,x_b) - \int_0^b AT(b-s)g(s,x_s)ds$$
$$- \int_0^b T(b-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\Big](s)ds$$

for $0 \leqslant t \leqslant b$, and we will verify that P_1 is a contraction while P_2 is a compact operator. To prove P_1 is a contraction, we take $x,\ y \in B_K$. Then for each $t \in [0,b]$ and (H1) and equation (3.2), we have

$$\|(P_1x)(t) - (P_1y)(t)\| \leqslant L_0 \sup_{0\leqslant t\leqslant b} \|x(t) - y(t)\|.$$

Thus

$$\|P_1x - P_1y\|_C = \sup_{0\leqslant t\leqslant b} \Big\|(P_1x)(t) - (P_1y)(t)\Big\| \leqslant L_0\|x-y\|_C$$

and so by $L_0 < 1$, we see that P_1 is a contraction.

To prove P_2 is a compact operator, firstly we prove that P_2 is continuous on B_K. Let sequence $\{x_n\} \subseteq B_K$ with $\{x_n\} \to x$ in B_K. As $n \to \infty$, since

$$g(s, x_{ns}) \longrightarrow g(s,x_s),$$

$$f\left(s, x_{ns}, \int_0^s k(s,\tau,x_{n\tau})d\tau\right) \longrightarrow f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right).$$

And so if $n \to \infty$ then we see

$$\|P_2x_n - P_2x\|_{C([0,b]:x)} = \sup_{0\leqslant t\leqslant b} \Big\|(P_2x_n)(t) - (P_2x)(t)\Big\| \longrightarrow 0.$$

That is, P_2 is continuous. Next we prove that $\{P_2x : x \in B_K\}$ is a family of equicontinuous functions. To prove this we fix $t_1 > 0$ and let $t_2 > t_1$ and $\epsilon > 0$ be enough small. Then we have

$$\Big\|(P_2x)(t_2) - (P_2x)(t_1)\Big\|$$
$$= \Big\| \int_0^{t_2} T(t_2-s)C\widetilde{W}^{-1}\Big[x^1 - T(b)\{\phi - g(0,\phi)\} - g(b,x_b)$$

$$- \int_0^b AT(b-s)g(s,x_s)ds - \int_0^b T(b-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\Big](s)ds$$
$$- \int_0^{t_1} T(t_1-s)C\widetilde{W}^{-1}\Big[x^1 - T(b)\{\phi - g(0,\phi)\} - g(b,x_b)$$
$$- \int_0^b AT(b-s)g(s,x_s)ds - \int_0^b T(b-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\Big](s)ds\Big\|,$$

and let

$$Y = C\widetilde{W}^{-1}\Big[x^1 - T(b)\{\phi - g(0,\phi)\} - g(b,x_b) - \int_0^b AT(b-s)g(s,x_s)ds - \int_0^b T(b-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\Big],$$

$$\Big\|(P_2x)(t_2) - (P_2x)(t_1)\Big\| \leqslant \int_0^{t_1-\varepsilon} \|T(t_2-s) - T(t_1-s)\|\|Y(s)\|ds + \int_{t_1-\varepsilon}^{t_1} \|T(t_2-s) - T(t_1-s)\|\|Y(s)\|ds + \int_{t_1}^{t_2} \|T(t_2-s)\|\|Y(s)\|ds$$

holds. And we see that $\|(P_2x)(t_2) - (P_2x)(t_1)\|$ tends to zero independently of $x \in B_k$ $T(t)$ as $t_2 - t_1 \to 0$ since the compactness of $T(t)(t > 0)$ implies the continuity of $T(t)(t > 0)$ in t. Similarly, by using the compactness of the set $g(B_K)$, we can prove that the functions $P_2x, x \in B_K$ are equicontinuous at $t = 0$. Hence P_2 maps B_K into a family of equicontinuous functions. Finally, we verify that $V(t) = \Big\{(P_2x)(t) : x \in B_K\Big\}$ is relatively compact in X. Since

$$\begin{aligned} V(0) &= (P_2x)(0) \\ &= \int_0^0 T(0-s)C\widetilde{W}^{-1}\Big[x^1 - T(b)\{\phi - g(0,\phi)\} \\ &\quad - g(b,x_b) - \int_0^b AT(b-s)g(s,x_s)ds \\ &\quad - \int_0^b T(b-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\Big](s)ds \\ &= 0, \end{aligned}$$

$V(0)$ is relatively compact in X. Let $0 \leqslant t \leqslant b$ be fixed and $0 < \epsilon < t$, and also for $x \in B_K$, we define

$$
\begin{aligned}
(P_{2,\epsilon}x)(t) = \int_0^{t-\epsilon} T(t-s)C\widetilde{W}^{-1}\Big[x^1 - T(b)\{\phi - g(0,\phi)\}\\
- g(b, x_b) - \int_0^b AT(b-s)g(s,x_s)ds \\
- \int_0^b T(b-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\Big](s)ds.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
(P_{2,\epsilon}x)(t) = T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)C\widetilde{W}^{-1}\Big[x^1 - T(b)\{\phi - g(0,\phi)\}\\
- g(b, x_b) - \int_0^b AT(b-s)g(s,x_s)ds \\
- \int_0^b T(b-s)f\left(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\right)\Big](s)ds.
\end{aligned}
$$

and from the compactness of $T(\epsilon)$, we obtain $V_\epsilon(t) = \Big\{(P_{2,\epsilon}x)(t) : x \in B_K\Big\}$ is relatively compact in X for every $0 < \epsilon < t$. Moreover, for every $x \in B_K$, we have

$$
\begin{aligned}
&\Big\|(P_2x)(t) - (P_{2,\epsilon}x)(t)\Big\| \\
&\leqslant \int_{t-\epsilon}^t M\|C\|\|\widetilde{W}^{-1}\|\Big[\|x^1\| + M\Big\{\|\phi\| + M_0L_1(K+1)\Big\} \\
&\qquad + M_0L_1(K+1) + \frac{1}{\beta}b^\beta C_{1-\beta}L_1(K+1) \\
&\qquad + bML_2(K^{\sigma_2} + (bL_3K)^{\sigma_3})\Big](s)ds.
\end{aligned}
$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t)$. Hence the set $V(t)$ is also relatively compact in X. Thus by Arzela-Ascoli's theorem, P_2 is compact operator. By this time, we show that $P = P_1 + P_2$ is a condensing map on B_K, and by the fixed point of Sadovskii, there exists a fixed point $x(\cdot)$ for P on B_K. Therefore, we can see that the nonlocal Cauchy problem (1.1) has a mild solution with $x_0 = \phi$ and $x(b) = x^1$. □

References

1. L. Byszewski, *Theorems about the existence and uniqueness of a solution of a semilinear evolution nonlocal Cauchy problem,* J. Math. Anal. Appl., 162, 496-505, (1991).
2. L. Byszewski, *Existence, uniqueness and asymptotic stability of solutions of abstract nonlocal Cauchy problems,* Dynamics Systems Appl., 5, 595-606, (1996).
3. L. Byszewski, *Application of properties of the right-hand sides of evolution equation to an investigation of nonlocal evolution problem,* Nonlinear Anal., 33, 413-426, (1998).

4. L. Byszewski, H. Akca, *Existence of solutions of a semi-linear functional differential evolution nonlocal problem,* Nonlinear Anal., 34, 65-72, (1998).
5. L. Byszewski, V. Lakshimikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in Banach space,* Appl. Anal., 40, 11-19, (1990).
6. E.N. Chukwu, S.M. Lenhart, *Controllability questions for nonlinear systems in abstract spaces, Journal of Optimization theory and Applications,* 68, 437-462, (1991).
7. X.L. Fu, *Controllability of neutral functional differential systems in abstract space,* Appl.Math and Comp, 141, 281-296, (2003).
8. X. Fu, K. Ezzinbi, *Existence of solutions for neutral functional differential evolution equations with nonlocal conditions*, Nonlinear Anal., 54, 215-227, (2003).
9. E. Hernandez, H.R. Henriquez, *Existence results for partial neutral functional differential equations with unbounded delay,* J. Math. Anal. Appl., 221, 452-475, (1998).
10. Y. Lin, H. Liu, *Semilinear integrodifferential equation with nonlocal Cauchy problem,* Nonlinear Anal., 26, 1023-1033, (1996).
11. Y.C. Kwun, J.Y. Park and J.W. Ryu, *Approximate controllability and controllability for delay Voltera systems,* bull. Korean Math. Soc., 28, 131-145, (1991).
12. S.K. Ntouyas, P.Ch. Tsamatos, *Global existence for semilinear evolution equations with nonlocal condition,* J. Math. Anal. Appl., 210, 679-687, (1997).
13. A. Pazy, *Semigroups of Linear Operators and Applications to Patial Differential Equations,* Springer, New York, (1983).
14. M.D. Quinn, N. Carmichael, *An approach to nonlinear control problems using fixed point methods, degree theory and pseudo-inverses,* Numer. Funct. Anal. Opt., 7, 197-219, (1984-1985).
15. B.N. Sadovskii, *On a fixed point principle, Funct.* Anal. Appl., 1, 74-76, (1967).

Young Chel Ahn
e-mail : math0623@@hanmail.net

Hae Won Kim
e-mail : uto83@@hanmail.net

Dong Gun Park
Department of Mathematics, Dong-A University, Pusan 604-714, South Korea
e-mail: dgpark@@dau.ac.kr