SOLUTION OF RICCATI TYPES MATRIX DIFFERENTIAL EQUATIONS USING MATRIX DIFFERENTIAL TRANSFORM METHOD

REZA ABAZARI

ABSTRACT. In this work, we successfully extended dimensional differential transform method (DTM), by presenting and proving some new theorems, to solve the non-linear matrix differential Riccati equations(first and second kind of Riccati matrix differential equations). This technique provides a sequence of matrix functions which converges to the exact solution of the problem. Examples show that the method is effective.

AMS Mathematics Subject Classification : 15A30; 15A24; 15A90; 39B42.
Key words and phrases : Matrix differential Riccati equations; differential transformation method; continuous matrix solutions.

1. Introduction

Consider the first kind time-varying matrix differential Riccati equations(1-MDREs)

$$X'(t) = A_{21}(t) + A_{22}(t)X(t) - X(t)A_{11}(t) - X(t)A_{12}(t)X(t),$$
$$X(t_0) = X_0, \tag{1}$$

where $A_{11}(t) \in R^{n \times n}, A_{22}(t) \in R^{m \times m}, A_{12}(t) \in R^{n \times m}, A_{21}(t) \in R^{m \times n}, X(t) \in R^{m \times n}$, and second kind time-varying matrix differential Riccati equations(2-MDREs)

$$[X(t) + B(t)]X'(t) = B_{21}(t) + B_{22}(t)X(t) - X(t)B_{11}(t) - X(t)B_{12}(t)X(t),$$
$$X(t_0) = X_0, \tag{2}$$

where $B_{11}(t) \in R^{n \times n}, B_{22}(t) \in R^{n \times n}, B_{12}(t) \in R^{n \times n}, B_{21}(t) \in R^{n \times n}, X(t) \in R^{n \times n}$ and $B(t) \in R^{n \times n}$.

MDREs play a fundamental role in control theory, for example, optimal control [1], filtering and estimation, decoupling Boundary values problems [2,3], and

order reduction, etc. In the past a number of unconventional numerical methods that are suited only for time-invariant MDREs have been designed, but despite their special structure, no unconventional methods that are suited for time-varying MDREs have been constructed, except (carefully) re-designed conventional linear multistep methods and Runge-Kutta methods. Implicit conventional methods which are preferred to explicit ones for stiff systems require solving nonlinear systems of equations (of possibly much higher dimensions than the original problem itself for Runge-Kutta methods) which not only pose implementation difficulties but also may be expensive because they require solving non-linear matrix equations which may be costly.

Many authors studied Eq.(1) and similar of these equation, by different numerical method, such as L. Dieci, et al.[2], Kenney, et, al.[4], Chiu, et, al.[5,6]. In [7,8], the Cubic Matrix Splines method employed to approximate the solution of special case of Matrix differential equation.

The differential transform method is a semi-numerical-analytic-technique that formalizes the Taylor series in a totally different manner. It was first introduced by Zhou in a study about electrical circuits [9]. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylors series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for linear and nonlinear matrix differential equations. It is possible to solve system of differential equations [10], differential-algebraic equations [11], partial differential equations [12], fractional differential equations [13], pantograph equations [14], integro-differential equations [15] by using this method.

The purpose of this research is to extended the differential transformation method(DTM) to obtained the solution of Eqs. (1) and (2).

2. Basic definitions

The basic definitions of matrix differential transformation are introduced as follows:

Definition 1. If $u(t) \in R^{n \times n}$ can be expressed by Taylor's series about fixed point t_i, then $u(t)$ can be represented as

$$u(t) = \sum_{k=0}^{\infty} \frac{u^{(k)}(t_i)}{k!}(t - t_i)^k. \tag{3}$$

If $u_n(t)$ is be the n-partial sums of a Taylor's series (2), then

$$u_n(t) = \sum_{k=0}^{n} \frac{u^{(k)}(t_i)}{k!}(t - t_i)^k + R_n(t). \tag{4}$$

where $u_n(t)$ is called the n-th Taylor polynomial for $u(t)$ about t_i and $R_n(t)$ is remainder term.

If $U(k)$ is defined as

$$U(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}u(t)\right]_{t=t_i}, \qquad \text{where} \quad k = 0, 1, .., \infty \tag{5}$$

then Eq (3) reduce to

$$u(t) = \sum_{k=0}^{\infty} U(k)(t - t_i)^k. \tag{6}$$

and the n-partial sums of a Taylor's series (6) reduce to

$$u_n(t) = \sum_{k=0}^{n} U(k)(t - t_i)^k + R_n(t). \tag{7}$$

The $U(k)$ defined in Eq (5), is called the matrix differential transform of matrix function $u(t)$. For simplicity assume that $t_0 = 0$, then solution (6) reduce to

$$u(t) = \sum_{k=0}^{n} U(k)t^k + R_{n+1}(t). \tag{8}$$

From the above definitions, it can be found that the concept of the one - dimensional matrix differential transform is derived from the Taylor series expansion.

Now we state the fundamental theorem of this paper. Assume that the matrixes $W(k), U(k)$, and $V(k)$, in $R^{n \times n}$, are the differential transform versions of the matrix functions $w(t), u(t)$, and $v(t)$, in $R^{n \times n}$, respectively, then we have:

Theorem 1. *If $w(t) = c_1 u(t) \pm c_2 v(t)$, where $c_1, c_2 \in \mathbb{R}$, then*
$$W(k) = c_1 U(k) \pm c_2 V(k).$$

Theorem 2. *If $w(t) = \frac{d^m}{dt^m}u(t)$, then $W(k) = \frac{(k+m)!}{k!}U(k + m)$.*

Proof. From definition 1, we get
$$\frac{d^k}{dt^k}w(t) = \frac{d^k}{dt^k}\left[\frac{d^m}{dt^m}u(t)\right] = \frac{d^{k+m}}{dt^{k+m}}u(t).$$
Therefore
$$\left[\frac{d^k}{dt^k}w(t)\right]_{t=t_i} = \left[\frac{d^{k+m}}{dt^{k+m}}u(t)\right]_{t=t_i} = (k + m)!U(k + m),$$
then from (5), we have $W(k) = \frac{(k+m)!}{k!}U(k + m)$. $\qquad \square$

Remark 1. In this paper, the natation $\otimes$ is applied to denoted the multiplicative notation of differential transform version of matrixes functions.

Theorem 3. *If $w(t) = u(t)v(t)$, then $W(k) = \sum_{l=0}^{k} U(l)V(k - l)$.*

Proof. By using the Lienitis formula, we get
$$\frac{d^k}{dt^k}w(t) = \frac{d^k}{dt^k}\left[u(t)v(t)\right] = \sum_{l=0}^{k}\binom{k}{l}\frac{d^l}{dt^l}u(t)\frac{d^{k-l}}{dt^{k-l}}v(t).$$

Therefore

$$\left[\frac{d^k}{dt^k}w(t)\right]_{t=t_i} = \sum_{l=0}^{k} \binom{k}{l} l!(k-l)!U(l)V(k-l),$$

then from (5), we have

$$W(k) = U(k) \otimes V(k) = \sum_{l=0}^{k} U(l)V(k-l).$$

$\square$

Remark 2. Matrix multiplication is a noncommutative operation, i.e. it is possible for $u(t)v(t) \neq v(t)u(t)$, even when both products exist and have the same shape. Therefor for $w(t) = v(t)u(t)$, we have $W(k) = \sum_{l=0}^{k} V(l)U(k-l)$.

Theorem 4. *If* $w(t) = \frac{d^m}{dt^m}u(t)\frac{d^n}{dt^n}v(t)$, *then*

$$W(k) = \sum_{l=0}^{k} \frac{(l+m)!(k-l+n)!}{l!(k-l)!}U(l+m)V(k-l+n).$$

Proof. Analogously to previous Theorems, we get

$$\frac{d^k}{dt^k}w(t) = \frac{d^k}{dt^k}\left[\frac{d^m}{dt^m}u(t)\frac{d^n}{dt^n}v(t)\right] = \sum_{l=0}^{k} \binom{k}{l} \frac{d^{m+l}}{dt^{m+l}}u(t)\frac{d^{n+k-l}}{dt^{n+k-l}}v(t).$$

Therefore $\left[\frac{d^k}{dt^k}w(t)\right]_{t=t_i} = \sum_{l=0}^{k} \binom{k}{l} (l+m)!(k-l+n)!U(l+m)V(k-l+n)$.

Then from (5), we have $W(k) = \sum_{l=0}^{k} \frac{(l+m)!(k-l+n)!}{l!(k-l)!}U(l+m)V(k-l+n)$. $\square$

Remark 3. From Remark. 2, if $w(t) = \frac{d^m}{dt^m}v(t)\frac{d^n}{dt^n}u(t)$, then

$$W(k) = \sum_{l=0}^{k} \frac{(l+m)!(k-l+n)!}{l!(k-l)!}V(l+m)U(k-l+n).$$

Theorem 5. *Assume that* $W(k), U(k,)$ *and* $V(k)$, *are the differential transform version of matrix functions* $w(t), u(t)$, *and* $v(t)$, *respectively, then*
(A) If $w(t) = u(t)\frac{d^n}{dt^n}v(t)$, *then* $W(k) = \sum_{l=0}^{k} \frac{(k-l+n)!}{(k-l)!}U(l)V(k-l+n)$.
(B) If $w(t) = \frac{d^m}{dt^m}u(t)v(t)$, *then* $W(k) = \sum_{l=0}^{k} \frac{(l+m)!}{l!}U(l+m)V(k-l)$.

Proof. It is obvious from Theorem 4. $\square$

3. Convergence analysis

In this section, we show that the presented matrix differential transformation method is convergence.

Theorem 6. *(Matrix form of Taylor series) Let the matrix function* $u : R^{n \times n} \to R^{n \times n}$, *and its first* n *derived functions be relatively continuous and finite on an interval* T *and differentiable on* $T - Q$, *(Q countable). Let* $t_0, t \in T$. *Then formulas (5) and (6) hold, with*

$$R_n(x) = \frac{1}{n!} \int_{t_0}^x u^{(n+1)}(t)(x-t)^n dt, \tag{9}$$

$$\|R_n(x)\|_\infty \le \frac{|x - t_0|^{n+1}}{(n+1)!} \|u^{(n+1)}(t)\|_\infty. \tag{10}$$

where $\|.\|_\infty$ *is infinity matrix norm.*

Proof. By definition 1, we get

$$R_n(x) = u(x) - u(t_0) - \sum_{k=1}^n u^{(k)}(t_0) \frac{(x - t_0)^k}{k!}.$$

We use the right side as a pattern to define a function $h : R^{n \times n} \to R^{n \times n}$. This time, we keep x fixed (say, $x = a \in T$) and replace t_0 by a variable t. Thus we set

$$h(t) = u(a) - u(t) - \frac{u'(t)}{1!}(a-t) - \dots - \frac{u^{(n)}(t)}{n!}(a-t)^n, \quad t \in R \tag{11}$$

Then $h(t_0) = R_n(a)$ and $h(a) = 0$. Our assumptions imply that h is relatively continuous and finite on T, and differentiable on $T - Q$. Differentiating (15), we see that all cancels out except for one term

$$h'(t) = -\frac{u^{(n+1)}(t)}{n!}(a-t)^n, \quad t \in T - Q. \tag{12}$$

then we get

$$-h(t) = \int_t^a \frac{u^{(n+1)}(t)}{n!}(a-t)^n dt, \quad t \in T.$$

and

$$\int_{t_0}^a \frac{u^{(n+1)}(t)}{n!}(a-t)^n dt = -h(a) + h(t_0) = R_n(a), \quad t \in T.$$

As $x = a$, (14) is proved. Next, let $M = \|u^{(n+1)}(t)\|_\infty$, If $M = +\infty$, the (14) is valid. If $M < +\infty$, define $g(t) = M\frac{(t-a)^{n+1}}{(n+1)!}$ for $t \ge a$, and $g(t) = -M\frac{(a-t)^{n+1}}{(n+1)!}$ for $t \le a$. In both cases,

$$g'(t) = M\frac{|t - a|^n}{n!} \ge \|h'(t)\|_\infty, \quad t \in T - Q.$$

then we get, $\|h(t_0) - h(a)\|_\infty \le \|g(t_0) - g(a)\|_\infty$, or $\|R_n(a)\|_\infty \le M\frac{|a-t_0|^{n+1}}{(n+1)!}$, Thus (14) follows, because a is arbitrary value. $\square$

4. Applications and numerical results

Example 1. Consider the 2×2 matrix differential Riccati equations (1):

$$X'(t) = A_{21}(t) + A_{22}(t)X(t) - X(t)A_{11}(t) - X(t)A_{12}(t)X(t),$$
$$X(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \tag{13}$$

where $A_{21}(t) = \begin{bmatrix} e^t & e^t(t+2) \\ 0 & -t(t-1)e^{-t} \end{bmatrix}, A_{12}(t) = \begin{bmatrix} t & 0 \\ 0 & e^t \end{bmatrix}, A_{11}(t) = \begin{bmatrix} e^t & 0 \\ 0 & t \end{bmatrix},$

$$A_{22}(t) = \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix},$$

By applying matrix differential transform method on Eq.(13), for $k = 0, 1, 2,$ ..., N, we get

$$(k+1)\mathbf{X}(k+1) = \mathbf{A}_{21}(k) + \mathbf{A}_{22}(k) \otimes \mathbf{X}(k) - \mathbf{X}(k) \otimes \mathbf{A}_{11}(k)$$
$$-\mathbf{X}(k) \otimes \mathbf{A}_{12}(k) \otimes \mathbf{X}(k),$$
$$\mathbf{X}(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{14}$$

where $\mathbf{X}(k), \mathbf{A}_{21}(k), \mathbf{A}_{22}(k), \mathbf{A}_{11}(k)$ and $\mathbf{A}_{12}(k)$ are the matrix differential transformation of $X(t), A_{21}(t), A_{22}(t), A_{11}(t)$ and $A_{12}(t)$, respectively.

By using the differential transform operator listed on Theorems. 1-5, we can rewrite Eq. (14) in to

$$\mathbf{X}(k+1) = \frac{1}{(k+1)} \Big\{ \mathbf{A}_{21}(k) + \sum_{l=0}^{k} \mathbf{A}_{22}(l)\mathbf{X}(k-l) - \sum_{l=0}^{k} \mathbf{X}(l)\mathbf{A}_{11}(k-l)$$
$$- \sum_{l=0}^{k} \Big(\sum_{r=0}^{l} \mathbf{X}(r)\mathbf{A}_{12}(l-r) \Big) \mathbf{X}(k-l) \Big\}, \tag{15}$$

From matrix functions $A(t), B(t)$ and $C(t)$, and using the concept of differential transformation operator, we get

$$A_{21}(t) = \sum_{k=0}^{\infty} \mathbf{A}_{21}(k)t^k = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}t + \begin{bmatrix} \frac{1}{2} & 2 \\ 0 & -2 \end{bmatrix}t^2 + \begin{bmatrix} \frac{1}{6} & \frac{5}{6} \\ 0 & \frac{3}{2} \end{bmatrix}t^3 + ...,$$

$$A_{12}(t) = \sum_{k=0}^{\infty} \mathbf{A}_{12}(k)t^k = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}t + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}t^2 + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}t^3 + ...,$$

$$A_{11}(t) = \sum_{k=0}^{\infty} \mathbf{A}_{11}(k)t^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}t + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}t^2 + \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{bmatrix}t^3 + ...,$$

and

$$A_{22}(t) = \sum_{k=0}^{\infty} \mathbf{A}_{22}(k)t^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}t + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}t^2 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}t^3 + ...,$$

From Eqs. (15), and by taking $N = 3$, the following system for $k = 0, 1, 2, 3$, is obtained:

$$\mathbf{X}(1) = \mathbf{A}_{21}(0) + \mathbf{A}_{22}(0)\mathbf{X}(0) - \mathbf{X}(0)\mathbf{A}_{11}(0) - \mathbf{X}(0)\mathbf{A}_{12}(0)\mathbf{X}(0) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$

$$\mathbf{X}(2) = \tfrac{1}{2}\mathbf{A}_{21}(1) + \tfrac{1}{2}\mathbf{A}_{22}(1)\mathbf{X}(0) + \tfrac{1}{2}\mathbf{A}_{22}(0)\mathbf{X}(1) - \tfrac{1}{2}\mathbf{X}(1)\mathbf{A}_{11}(0) - \tfrac{1}{2}\mathbf{X}(0)\mathbf{A}_{11}(1)$$
$$- \tfrac{1}{2}\mathbf{X}(0)\mathbf{A}_{12}(1)\mathbf{X}(0) - \mathbf{X}(0)\mathbf{A}_{12}(0)\mathbf{X}(1) = \begin{bmatrix} 0 & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} \end{bmatrix},$$

$$\mathbf{X}(3) = \tfrac{1}{3}\mathbf{A}_{21}(2) + \tfrac{1}{3}\mathbf{A}_{22}(2)\mathbf{X}(0) + \tfrac{1}{3}\mathbf{A}_{22}(1)\mathbf{X}(1) + \tfrac{1}{3}\mathbf{A}_{22}(0)\mathbf{X}(2) - \tfrac{1}{3}\mathbf{X}(2)\mathbf{A}_{11}(0)$$
$$- \tfrac{1}{3}\mathbf{X}(1)\mathbf{A}_{11}(1) - \tfrac{1}{3}\mathbf{X}(0)\mathbf{A}_{11}(2) - \tfrac{2}{3}\mathbf{X}(0)\mathbf{A}_{12}(0)\mathbf{X}(2) - \tfrac{2}{3}\mathbf{X}(0)\mathbf{A}_{12}(1)\mathbf{X}(1)$$
$$- \tfrac{1}{3}\mathbf{X}(1)\mathbf{A}_{12}(0)\mathbf{X}(1) - \tfrac{1}{3}\mathbf{X}(0)\mathbf{A}_{12}(2)\mathbf{X}(0) = \begin{bmatrix} 0 & \tfrac{1}{6} \\ 0 & -\tfrac{1}{6} \end{bmatrix}, \tag{16}$$

$$\mathbf{X}(4) = \tfrac{1}{4}\mathbf{A}_{21}(3) + \tfrac{1}{4}\mathbf{A}_{22}(3)\mathbf{X}(0) + \tfrac{1}{4}\mathbf{A}_{22}(2)\mathbf{X}(1) + \tfrac{1}{4}\mathbf{A}_{22}(1)\mathbf{X}(2) + \tfrac{1}{4}\mathbf{A}_{22}(0)\mathbf{X}(3)$$
$$- \tfrac{1}{4}\mathbf{X}(3)\mathbf{A}_{11}(0) - \tfrac{1}{4}\mathbf{X}(2)\mathbf{A}_{11}(1) - \tfrac{1}{4}\mathbf{X}(1)\mathbf{A}_{11}(2) - \tfrac{1}{4}\mathbf{X}(0)\mathbf{A}_{11}(3) - \tfrac{1}{2}\mathbf{X}(0)\mathbf{A}_{12}(0)\mathbf{X}(3)$$
$$- \tfrac{1}{2}\mathbf{X}(0)\mathbf{A}_{12}(1)\mathbf{X}(2) - \tfrac{1}{2}\mathbf{X}(1)\mathbf{A}_{12}(0)\mathbf{X}(2) - \tfrac{1}{2}\mathbf{X}(0)\mathbf{A}_{12}(2)\mathbf{X}(1) - \tfrac{1}{4}\mathbf{X}(1)\mathbf{A}_{12}(1)\mathbf{X}(1)$$
$$- \tfrac{1}{4}\mathbf{X}(0)\mathbf{A}_{12}(3)\mathbf{X}(0) = \begin{bmatrix} 0 & \tfrac{1}{24} \\ 0 & \tfrac{1}{24} \end{bmatrix},$$

$$\vdots$$

By substituted of $\mathbf{X}(:)$ obtained from list (16) in (5), we get

$$X(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} t + \begin{bmatrix} 0 & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} \end{bmatrix} t^2 + \begin{bmatrix} 0 & \tfrac{1}{6} \\ 0 & -\tfrac{1}{6} \end{bmatrix} t^3 + \begin{bmatrix} 0 & \tfrac{1}{24} \\ 0 & \tfrac{1}{24} \end{bmatrix} t^4,$$

Similarly, utilizing the recurrence relations in Eq. (15), $\mathbf{X}(k)$ are obtained for $k = 0, 1, 2, ..., N$ and then, by using the inverse transformation rule in Eq.(5), the closed form of solution can be obtained

$$X(t) = \begin{bmatrix} 1 & e^t \\ 0 & e^{-t} \end{bmatrix}.$$

which is exactly the same as the exact solution.

Example 2. Consider the 3×2 matrix differential Riccati equations (1):

$$X'(t) = A_{21}(t) + A_{22}(t)X(t) - X(t)A_{12}(t)X(t),$$
$$X(0) = \begin{bmatrix} 0 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}, \tag{17}$$

where

$$A_{21}(t) = \begin{bmatrix} 0 & -t^7 + t^4 + 3t^2 \\ -t^2 + 2t & -t^4 e^t + 2e^t + 1 \\ t - 1 & -t^4 + t + e^t \end{bmatrix}, A_{12}(t) = \begin{bmatrix} 0 & -1 & 0 \\ -t & 0 & 0 \end{bmatrix}, A_{22}(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

Similar on previous Example, by applying matrix differential transform method on Eq.(17), for $k = 0, 1, 2, ..., N$, we get

$$(k + 1)\mathbf{X}(k + 1) = \mathbf{A}_{21}(k) + \mathbf{A}_{22}(k) \otimes \mathbf{X}(k) - \mathbf{X}(k) \otimes \mathbf{A}_{12}(k) \otimes \mathbf{X}(k),$$
$$\mathbf{X}(0) = \begin{bmatrix} 0 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}, \tag{18}$$

where $\mathbf{X}(k), \mathbf{A}_{21}(k), \mathbf{A}_{22}(k)$ and $\mathbf{A}_{12}(k)$ are the differential transform version of $X(t), A_{21}(t), A_{22}(t),$ and $A_{12}(t),$ respectively.

From Eqs.(18), and by taking $N = 3$, we get

$$\mathbf{X}(1) = \mathbf{A}_{21}(0) + \mathbf{A}_{22}(0)\mathbf{X}(0) - \mathbf{X}(0)\mathbf{A}_{12}(0)\mathbf{X}(0) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{X}(2) = \tfrac{1}{2}\mathbf{A}_{21}(1) + \tfrac{1}{2}\mathbf{A}_{22}(0)\mathbf{X}(1) + \tfrac{1}{2}\mathbf{A}_{22}(1)\mathbf{X}(0) - X(0)\mathbf{A}_{12}(0)\mathbf{X}(1) - \tfrac{1}{2}\mathbf{X}(0)\mathbf{A}_{12}(1)\mathbf{X}(0)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \tfrac{1}{2} \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{X}(3) = \tfrac{1}{3}\mathbf{A}_{21}(2) + \tfrac{1}{3}\mathbf{A}_{22}(0)\mathbf{X}(2) + \tfrac{1}{3}\mathbf{A}_{22}(1)\mathbf{X}(1) + \tfrac{1}{3}\mathbf{A}_{22}(2)\mathbf{X}(0) - \tfrac{2}{3}\mathbf{X}(0)\mathbf{A}_{12}(0)\mathbf{X}(2)$$

$$- \tfrac{2}{3}\mathbf{X}(0)\mathbf{A}_{12}(1)\mathbf{X}(1) - \tfrac{1}{3}\mathbf{X}(1)\mathbf{A}_{12}(0)\mathbf{X}(1) - \tfrac{1}{3}\mathbf{X}(0)\mathbf{A}_{12}(2)\mathbf{X}(0) = \begin{bmatrix} 0 & 0 \\ 0 & \tfrac{1}{6} \\ 0 & 0 \end{bmatrix}, \qquad (19)$$

$$\mathbf{X}(4) = \tfrac{1}{4}\mathbf{A}_{21}(3) + \tfrac{1}{4}\mathbf{A}_{22}(0)\mathbf{X}(3) + \tfrac{1}{4}\mathbf{A}_{22}(1)\mathbf{X}(2) + \tfrac{1}{4}\mathbf{A}_{22}(2)\mathbf{X}(1) + \tfrac{1}{4}\mathbf{A}_{22}(3)\mathbf{X}(0)$$

$$- \tfrac{1}{2}\mathbf{X}(0)\mathbf{A}_{12}(0)\mathbf{X}(3) - \tfrac{1}{2}\mathbf{X}(2)\mathbf{X}(0)\mathbf{A}_{12}(1) - \tfrac{1}{2}\mathbf{X}(2)X(1)\mathbf{A}_{12}(0) - \tfrac{1}{2}\mathbf{X}(1)\mathbf{X}(0)\mathbf{A}_{12}(2)$$

$$- \tfrac{1}{4}\mathbf{X}(1)\mathbf{A}_{12}(1)\mathbf{X}(1) - \tfrac{1}{4}\mathbf{X}(0)\mathbf{A}_{12}(3)\mathbf{X}(0) = \begin{bmatrix} 0 & 0 \\ 0 & \tfrac{1}{24} \\ 0 & 0 \end{bmatrix},$$

$$\vdots$$

By substituted of $\mathbf{X}(:)$ obtained from list (19) in (5), we get

$$X(t) = \begin{bmatrix} 0 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & 0 \\ 0 & \tfrac{1}{2} \\ 0 & 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 & 0 \\ 0 & \tfrac{1}{6} \\ 0 & 0 \end{bmatrix} t^3 + \begin{bmatrix} 0 & 0 \\ 0 & \tfrac{1}{24} \\ 0 & 0 \end{bmatrix} t^4,$$

Similarly, utilizing the recurrence relations in Eq. (18), $\mathbf{X}(k)$ are obtained for $k = 0, 1, 2, ..., N$ and then, by using the inverse transformation rule in Eq.(5), the closed form of solution can be obtained

$$X(t) = \begin{bmatrix} 0 & t^3 - 1 \\ t - 1 & e^t \\ 0 & 1 \end{bmatrix}.$$

which is exactly the same as the exact solution.

Example 3. Consider the following time-varying second kind matrix differential Riccati equations:

$$X(t)X'(t) = C(t) + A(t)X(t) + X(t)B(t)X(t),$$
$$X(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad (20)$$

where

$$A(t) = \begin{bmatrix} -te^{2t} & e^{-t} \\ 0 & -te^{4t} \end{bmatrix}, B(t) = \begin{bmatrix} 1 & 2te^{-t} - te^{-2t} \\ 0 & t \end{bmatrix}, C(t) = \begin{bmatrix} te^{3t} & 0 \\ 0 & -te^{4t} \end{bmatrix},$$

Similar on previous Examples, the matrix differential transform version of Eq. (20), for $k = 0, 1, 2, ..., N$, is

$$\mathbf{X}(k) \otimes \mathbf{X}'(k) = \mathbf{C}(k) + \mathbf{A}(k) \otimes \mathbf{X}(k) + \mathbf{X}(k) \otimes \mathbf{B}(k) \otimes \mathbf{X}(k),$$
$$\mathbf{X}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad (21)$$

where $\mathbf{X}(k), \mathbf{A}(k), \mathbf{B}(k)$ and $\mathbf{C}(k)$ are the differential transform of $X(t)$, $A(t)$, $B(t)$ and $C(t)$, respectively, we can rewrite Eq.(21) in to

$$
\begin{aligned}
&\sum_{l=0}^{k}(k-l+1)\mathbf{X}(l)\mathbf{X}(k-l+1) - \mathbf{C}(k) - \sum_{l=0}^{k}\mathbf{A}(l)\mathbf{X}(k-l) \\
&\quad - \sum_{l=0}^{k}\left[\sum_{r=0}^{l}\mathbf{X}(r)\mathbf{A}(l-r)\right]\mathbf{X}(k-l) = 0,
\end{aligned} \tag{22}
$$

by taking $N = 3$, the following system for $k = 0, 1, 2, 3$, is obtained:

$$\mathbf{X}(0)\mathbf{X}(1) - \mathbf{A}(0)\mathbf{X}(0) - \mathbf{X}(0)\mathbf{B}(0)\mathbf{X}(0) - \mathbf{C}(0) = \mathbf{0}_{2\times 2},$$

$$
\begin{aligned}
&\mathbf{X}(1)^2 + 2\mathbf{X}(0)\mathbf{X}(2) - \mathbf{A}(1)\mathbf{X}(0) - \mathbf{A}(0)\mathbf{X}(1) - \left[\mathbf{X}(1)\mathbf{B}(0) + \mathbf{X}(0)\mathbf{B}(1)\right]\mathbf{X}(0) \\
&-\mathbf{X}(0)\mathbf{B}(0)\mathbf{X}(1) - \mathbf{C}(1) = \mathbf{0}_{2\times 2},
\end{aligned}
$$

$$
\begin{aligned}
&3\mathbf{X}(2)\mathbf{X}(1) + 3\mathbf{X}(0)\mathbf{X}(3) - \mathbf{A}(2)\mathbf{X}(0) - \mathbf{A}(1)\mathbf{X}(1) - \mathbf{A}(0)\mathbf{X}(2) - \left[\mathbf{X}(2)\mathbf{B}(0)\right. \\
&\left.+\mathbf{X}(1)\mathbf{B}(1) + \mathbf{X}(0)\mathbf{B}(2)\right]\mathbf{X}(0) - \left[\mathbf{X}(1)\mathbf{B}(0) + \mathbf{X}(0)\mathbf{B}(1)\right]\mathbf{X}(1) - \mathbf{X}(0)\mathbf{B}(0)\mathbf{X}(2) \\
&-\mathbf{C}(2) = \mathbf{0}_{2\times 2},
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
&4\ \mathbf{X}(3)\ \mathbf{X}(1) + 2\ \mathbf{X}(2)^2 + 4\ \mathbf{X}(0)\ \mathbf{X}(4) - \mathbf{A}(3)\ \mathbf{X}(0) - \mathbf{A}(2)\mathbf{X}(1) - \mathbf{A}(1)\mathbf{X}(2) \\
&-\mathbf{A}(0)\mathbf{X}(3) - \left[\mathbf{X}(3)\mathbf{B}(0) + \mathbf{X}(2)\mathbf{B}(1) + \mathbf{X}(1)\mathbf{B}(2) + \mathbf{X}(0)\mathbf{B}(3)\right]\mathbf{X}(0) \\
&-\left[\mathbf{X}(2)\mathbf{B}(0) + \mathbf{X}(1)\mathbf{B}(1) + \mathbf{X}(0)\mathbf{B}(2)\right]\mathbf{X}(1) - \left[\mathbf{X}(1)\mathbf{B}(0) + \mathbf{X}(0)\mathbf{B}(1)\right]\mathbf{X}(2) \\
&-\mathbf{X}(0)\mathbf{B}(0)\mathbf{X}(3) - \mathbf{C}(3) = \mathbf{0}_{2\times 2},
\end{aligned}
$$

$$\vdots$$

Solving the above system and using the inverse transformation rule (5), we get the following series solution

$$
X(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} t + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} t^2 + \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{4}{3} \end{bmatrix} t^3 + \begin{bmatrix} \frac{1}{24} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} t^4,
$$

Similarly, utilizing the recurrence relations in Eq. (22), $\mathbf{X}(k)$ are obtained for $k = 0, 1, 2, ..., N$ and then, by using the inverse transformation rule in Eq.(5), the closed form of solution can be obtained

$$
X(t) = \begin{bmatrix} e^t & t \\ 0 & e^{2t} \end{bmatrix}.
$$

which is exactly the same as the exact solution.

Example 4. In the end example, consider the following time-varying second kind matrix differential Riccati equations:

$$
\begin{aligned}
&\left[S(t) + X(t)\right]X'(t) = D(t) + A(t)X(t) - X(t)B(t) + X(t)C(t)X(t), \\
&X(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\end{aligned} \tag{24}
$$

where

$$
A(t) = \begin{bmatrix} e^{-t} & 1 \\ -1 & e^{-t} \end{bmatrix}, B(t) = \begin{bmatrix} e^{-t} & -e^t \\ e^{-t} & e^{-t} \end{bmatrix}, C(t) = \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix},
$$

and

$$
D(t) = \begin{bmatrix} 3 + 2e^t - 2te^t & t - e^{2t} \\ -1 + t^2 & 2t + 2e^t \end{bmatrix}, S(t) = \begin{bmatrix} 1 & t^3 \\ te^{-t} & t^3 - 2t^2 \end{bmatrix}.
$$

Similar on previous Examples, by applying matrix differential transform method on Eq.(24), for $k = 0, 1, 2, ..., N$, we get

$$\big[\mathbf{S}(k)+\mathbf{X}(k)\big]\otimes\mathbf{X}'(k)=\mathbf{D}(k)+\mathbf{A}(k)\otimes\mathbf{X}(k)-\mathbf{X}(k)\otimes\mathbf{B}(k)+\mathbf{X}(k)\otimes\mathbf{C}(k)\otimes\mathbf{X}(k),$$
$$\mathbf{X}(0) = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right], \tag{25}$$

where $\mathbf{X}(k), \mathbf{A}(k), \mathbf{B}(k), \mathbf{C}(k), \mathbf{D}(k)$ and $\mathbf{S}(k)$ are the differential transform version of $X(t), A(t), B(t), C(t), D(t)$ and $S(t)$, respectively.

By using the differential transform operator listed on Theorems. (1)-(5), we can rewrite Eq. (25) in to

$$\sum_{l=0}^{k}(k-l+1)\big[\mathbf{S}(l)+\mathbf{X}(l)\big]\mathbf{X}(k-l+1)-\mathbf{D}(k)-\sum_{l=0}^{k}\mathbf{A}(l)\mathbf{X}(k-l)$$
$$+\sum_{l=0}^{k}\mathbf{X}(l)\mathbf{B}(k-l)-\sum_{l=0}^{k}\big[\sum_{r=0}^{l}\mathbf{X}(r)\mathbf{C}(l-r)\big]\mathbf{X}(k-l)=\mathbf{0}_{2\times2}, \tag{26}$$

From Eq.(26), and by taking $N = 3$, the following system for $k = 0, 1, 2, 3$, is obtained:

$$\big[\mathbf{S}(0) + \mathbf{X}(0)\big]\mathbf{X}(1) - \mathbf{A}(0)\mathbf{X}(0) + \mathbf{X}(0)\mathbf{B}(0) - \mathbf{X}(0)\mathbf{C}(0)\mathbf{X}(0) = \mathbf{0}_{2\times2},$$

$$\big[\mathbf{S}(1) + \mathbf{X}(1)\big]\mathbf{X}(1) + 2\big[\mathbf{S}(0) + \mathbf{X}(0)\big]\mathbf{X}(2) - \mathbf{A}(1)\mathbf{X}(0) - \mathbf{A}(0)\mathbf{X}(1) + \mathbf{X}(1)\mathbf{B}(0)$$
$$+\mathbf{X}(0)\mathbf{B}(1) - \big[\mathbf{X}(1)\mathbf{C}(0) + \mathbf{X}(0)\mathbf{C}(1)\big]\mathbf{X}(0) - \mathbf{X}(0)\mathbf{C}(0)\mathbf{X}(1) = \mathbf{0}_{2\times2},$$

$$\big[\mathbf{S}(2) + \mathbf{X}(2)\big]\mathbf{X}(1) + 2\big[\mathbf{S}(1) + \mathbf{X}(1)\big]\mathbf{X}(2) + 3\big[\mathbf{S}(0) + \mathbf{X}(0)\big]\mathbf{X}(3) - \mathbf{A}(2)\mathbf{X}(0)$$
$$-\mathbf{A}(1)\mathbf{X}(1) - \mathbf{A}(0)\mathbf{X}(2) + \mathbf{X}(2)\mathbf{B}(0) + \mathbf{X}(1)\mathbf{B}(1) + \mathbf{X}(0)\mathbf{B}(2) - \big[\mathbf{X}(2)\mathbf{C}(0)$$
$$+\mathbf{X}(1)\mathbf{C}(1) + \mathbf{X}(0)\mathbf{C}(2)\big]\mathbf{X}(0) - \big[\mathbf{X}(1)\mathbf{C}(0) + \mathbf{X}(0)\mathbf{C}(1)\big]\mathbf{X}(1) - \mathbf{X}(0)\mathbf{C}(0)\mathbf{X}(2)$$
$$= \mathbf{0}_{2\times2}, \tag{27}$$

$$\big[\mathbf{S}(3)+\mathbf{X}(3)\big]\mathbf{X}(1)+2\big[\mathbf{S}(2)+\mathbf{X}(2)\big]\mathbf{X}(2)+3\big[\mathbf{S}(1)+\mathbf{X}(1)\big]\mathbf{X}(3)+4\big[\mathbf{S}(0)+\mathbf{X}(0)\big]\mathbf{X}(4)$$
$$-\mathbf{A}(3)\mathbf{X}(0) - \mathbf{A}(2)\mathbf{X}(1) - \mathbf{A}(1)\mathbf{X}(2) - \mathbf{A}(0)\mathbf{X}(3) + \mathbf{X}(3)\mathbf{B}(0) + \mathbf{X}(2)\mathbf{B}(1)$$
$$+\mathbf{X}(1)\mathbf{B}(2) + \mathbf{X}(0)\mathbf{B}(3) - [\mathbf{X}(3)\mathbf{C}(0) + \mathbf{X}(2)\mathbf{C}(1) + \mathbf{X}(1)\mathbf{C}(2) + \mathbf{X}(0)\mathbf{C}(3)]\mathbf{X}(0)$$
$$-\big[\mathbf{X}(2)\mathbf{C}(0) + \mathbf{X}(1)\mathbf{C}(1) + \mathbf{X}(0)\mathbf{C}(2)\big]\mathbf{X}(1) - \big[\mathbf{X}(1)\mathbf{C}(0) + \mathbf{X}(0)\mathbf{C}(1)\big]\mathbf{X}(2)$$
$$-\mathbf{X}(0)\mathbf{C}(0)\mathbf{X}(3) = \mathbf{0}_{2\times2},$$
$$\vdots$$

Solving the above system and using the inverse transformation rule (5), we get the following series solution

$$X(t)=\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right]+\left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right]t+\left[\begin{array}{cc} 0 & \frac{1}{2} \\ 0 & 0 \end{array}\right]t^2+\left[\begin{array}{cc} 0 & \frac{1}{6} \\ 0 & 0 \end{array}\right]t^3+\left[\begin{array}{cc} 0 & \frac{1}{24} \\ 0 & 0 \end{array}\right]t^4,$$

Similarly, utilizing the recurrence relations in Eq. (26), $\mathbf{X}(k)$ are obtained for $k = 0, 1, 2, ..., N$ and then, by using the inverse transformation rule in Eq.(5), the closed form of solution can be obtained

$$X(t) = \left[\begin{array}{cc} t & e^t \\ t - 1 & -t \end{array}\right].$$

which is exactly the same as the exact solution.

REFERENCES

1. H. Kwakernaak and R. Sivan,*Linear Optimal Control Systems*, Wiley Interscience, NewYork, 1972.

2. L. Dieci,*Numerical integration of the differential Riccati equation and some related issues*, SIAM J. Numer. Anal.**29**(3)(1992), 781-815.

3. L. Dieci, M. R. Osborne, and R. D. Russell,*A Riccati transformation method for solving linear BVPs. I: Theoretical aspects*, SIAM Journal on Numerical Analysis, **25** (1988), 1055-1073.

4. C.S. Kenney, R.B. Leipnik,*Numerical integration of the differential matrix Riccati equation*, IEEE Trans. Automat. Contr. **AC-30** (1985), 962-970.

5. Chiu H. Choi,*A survey of numerical methods for solving matrix Riccati differential equations*, Southeastcon90 Proc. (1990), 696-700.

6. C. Choi Chiu, J. Laub Alan,*Efficient matrix-valued algorithms for solving stiff Riccati differential equations*, IEEE Trans. Automat. Contr. **35** (7) (1990), 770-776.

7. A. Borhanifar, Reza Abazari,*Numerical Solution of Second-Order Matrix Differential Models Using Cubic Matrix Splines*, Applied Mathematical Sciences **59** (2007), 2927-2937.

8. M.M. Tunga, E. Defez, J. Sastre,*Numerical solutions of second-order matrix models using cubic-matrix splines*, Comput. Math. Appl. http://doi:10.1016/j.camwa.2008.05.022.

9. J.K. Zhou,*Differential Transformation and its Application for Electrical Circuits*, Huazhong University Press, Wuhan, China, 1986.

10. Fatma Ayaz,*Solutions of the system of differential equations by differential transform method*, Appl. Math. Comput. **147** (2004), 547-567.

11. F. Ayaz,*Application of differential transform method to differential-algebraic equations*, Appl. Math. Comput. **152** (2004), 649-657.

12. Figen Kangalgil, Fatma Ayaz,*Solitary wave solutions for the KdV and mKdV equations by differential transform method*, Chaos, Solitons & Fractals, **41** (2009), 464-472.

13. Momani S, Odibat Z, Hashim I,*Algorithms for nonlinear fractional partial differential equations: A selection of numerical methods*, Topological method in Nonlinear Analysis, **31** (2008), 211-226.

14. Y. Keskin, A. Kurnaz, M.E. Kiris, G. Oturanc,*Approximate solutions of generalized pantograph equations by the differential transform method*, Int. J. Nonlinear Sci. **8** (2007), 159-164.

15. A. Arikoglu, I. Ozkol,*Solution of boundary value problems for integro-differential equations by using differential transform method*, Appl. Math. Comput. **168** (2005), 1145-1158.

Reza Abazari received her B.Sc from the University of Pnu(Ardabil, Iran, June 2005) in pure mathematics under the direction of Dr J. Saffar Ardabili and M. Sc. at Mohaghegh university of Ardabil under the direction of Dr A. Borhanifar in applied mathematics(Ardabil, Iran, June 2007).

Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran.
Tel:+98 451 5514703; Fax:+98 451 5514701.
e-mail: abazari-r@uma.ac.ir