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MULTIPLE SYMMETRIC POSITIVE SOLUTIONS OF A NEW KIND STURM-LIOUVILLE-LIKE BOUNDARY VALUE PROBLEM WITH ONE DIMENSIONAL p-LAPLACIAN

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ABSTRACT. In this paper, we are concerned with the following four point boundary value problem with one-dimensional p-Laplacian,

$$\begin{cases} (\phi_p(x'(t)))' + h(t)f(t, x(t), |x'(t)|) = 0, & 0 < t < 1, \\ x'(0) - \delta x(\xi) = 0, \quad x'(1) + \delta x(\eta) = 0, \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ \delta > 0,\ 1 > \eta > \xi > 0,\ \xi + \eta = 1$. By using a fixed point theorem in a cone, we obtain the existence of at least three symmetric positive solutions. The interesting point is that the boundary condition is a new Sturm-Liouville-like boundary condition, which has rarely been treated up to now.

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1. Introduction

In this paper, we study the existence of multiple symmetric positive solutions for the following boundary value problem(BVP) with one-dimensional p-Laplacian

$$\begin{cases} (\phi_p(x'(t)))' + h(t)f(t, x(t), |x'(t)|) = 0, & 0 < t < 1, \\ x'(0) - \delta x(\xi) = 0, \quad x'(1) + \delta x(\eta) = 0, \end{cases} \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ \delta > 0,\ 1 > \eta > \xi > 0,\ \xi + \eta = 1$. By $\phi_q(s)$ we denote the inverse to $\phi_p(s)$, $\frac{1}{p} + \frac{1}{q} = 1$. Throughout, we assume $h,\ f$ satisfy:

(C_1) $h \in L^1[0, 1]$ is nonnegative on $(0, 1)$, and $h(t) \not\equiv 0$ on any subinterval of $(0, 1)$, $h(t) = h(1 - t)$ for $t \in [0, 1]$.

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(C_2) $f \in C([0,1]\times[0,+\infty)\times R, [0,+\infty))$, and $f(t,u,v) = f(1-t,u,v)$, $(t,u,v) \in [0,1] \times [0,+\infty) \times R$.

There is much current attention focused on questions of positive solutions of boundary value problems for ordinary differential equations, see[1,5,12,13] and the references therein. At the same time, efforts to obtain necessary and sufficient conditions for the existence of symmetric solutions of BVPs can also be found in the literature. For a small sample of such work, we refer the readers to the papers [2,8,9,10]. And in the vast field of the research of differential boundary value problems, particular attention has been focused on the Sturm-Liouville BVP

$$\begin{cases} x''(t) + \lambda f(t,x,x') = 0, & 0 < t < 1, \\ \alpha x(0) - \beta x'(0) = 0, & \gamma x(1) + \delta x'(1) = 0, \end{cases} \tag{1.2}$$

where $\alpha, \beta, \gamma, \delta > 0$, $\alpha\gamma + \alpha\delta + \beta\gamma > 0$, many results for the existence of positive solutions of (1.2) have been obtained, see [4,6,7] and the references therein.

Recently, in [11], the authors studied the following Sturm-Liouville-like four-point boundary value problem

$$\begin{cases} (\varphi_p(u'(t)))' + f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) - \alpha u'(\xi) = 0, & u(1) + \beta u'(\eta) = 0, \end{cases} \tag{1.3}$$

where $\alpha, \beta > 0, 1 > \eta > \xi > 0$. By giving sufficient conditions, the authors get the existence of positive solutions.

It is obvious that the boundary condition $x'(0) - \alpha x(\xi) = 0$, $x'(1) + \beta x(\eta) = 0$ can also be called Sturm-Liouville-like boundary condition. However, up to now, there are few works devoted to such kind of four-point BVP. In addition, the treatments of symmetric cases of such BVPs are not known to the authors as well, so we intend to fill in such gaps in the literature.

2. Background material and the fixed point theorem

In this section, for convenience, we present the main definitions and theorem that will be used in this paper.

Definition 2.1. Let E be a real Banach Space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(1) $au \in P$ for all $a \geq 0$,
(2) $u, -u \in P$ implies u=0.

Every cone $P \subset E$ induces an ordering in E given by $x \geq y$, if and only if $y - x \in P$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Definition 2.3. The map α is said to be a nonnegative concave(convex) continuous function provided that α: $P \to [0, \infty)$ is continuous and

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y)$$
$$(\alpha(\lambda x + (1-\lambda)y) \leq \lambda\alpha(x) + (1-\lambda)\alpha(y))$$

for all $x, y \in P$ and $0 \leq \lambda \leq 1$.

Let α, γ, θ, ψ be nonnegative continuous maps on P with α concave, and θ, γ convex. Then for positive numbers a, b, c, d we define the following subsets of P

$$P(\gamma, d) = \{x \in P \mid \gamma(x) < d\},$$
$$P(\alpha, \gamma, b, d) = \{x \in \overline{P(\gamma, d)} \mid \alpha(x) \geq b\},$$
$$P(\alpha, \theta, \gamma, b, c, d) = \{x \in \overline{P(\gamma, d)} \mid \alpha(x) \geq b,\ \theta(x) \leq c\},$$
$$R(\psi, \gamma, a, d) = \{x \in \overline{P(\gamma, d)} \mid \psi(x) \geq a\},$$

then it is obvious that $P(\gamma, d)$, $P(\alpha, \gamma, b, d)$ and $P(\alpha, \theta, \gamma, b, c, d)$ are convex and $R(\psi, \gamma, a, d)$ are closed.

Next we state Avery-Peterson fixed point theorem.

Theorem 2.1. ([3]) *Let P be a cone in Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying*

$$\psi(\lambda x) \leq \lambda\psi(x) \text{ for all } 0 \leq \lambda \leq 1.$$

and

$$\alpha(x) \leq \psi(x), \quad \|x\| \leq M\gamma(x) \text{ for all } x \in \overline{P(\gamma, d)}.$$

with M, d be positive numbers. Suppose that $T : P \to P$ is completely continuous and there exist positive numbers a, b, c, with $a < b$ such that

- (A_1) *$\{x \in P(\alpha, \theta, \gamma, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\alpha, \theta, \gamma, b, c, d)$;*
- (A_2) *$\alpha(Tx) > b$ for $x \in P(\alpha, \gamma, b, d)$ with $\theta(Tx) > c$;*
- (A_3) *$0 \notin R(\psi, \gamma, a, d)$ and $\psi(Tx) < a$ for $x \in R(\psi, \gamma, a, d)$ with $\psi(x) = a$.*

Then T has at least three fixed points x_1, x_2, $x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \leq d, \quad i = 1, 2, 3; \quad \psi(x_1) < a; \quad \psi(x_2) > a \quad \text{with } \alpha(x_2) < b; \quad \alpha(x_3) > b.$$

3. Existence of triple symmetric positive solutions

let $X = C^1[0, 1]$ be endowed with the maximum norm,

$$\|x\| = \max\left\{\max_{0 \leq t \leq 1} |x(t)|,\ \max_{0 \leq t \leq 1} |x'(t)|\right\}.$$

Cone $P \subset X$ is defined as

$$P = \{x \in X \mid x(t) \geq 0 \text{ is concave and symmetric on } [0, 1],\ x'(0) - \delta x(\xi) = 0\}.$$

We define an operator $T:\ P\to X$,
$(Tx)(t) =$

$$\begin{cases} \frac{1}{\delta}\phi_q\int_0^{\frac{1}{2}}h(s)f(s,x(s),|x'(s)|)\mathrm{d}s-\int_0^{\xi}\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s \\ +\int_0^{t}\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s, \qquad t\in[0,\frac{1}{2}], \\ \frac{1}{\delta}\phi_q\int_{\frac{1}{2}}^{1}h(s)f(s,x(s),|x'(s)|)\mathrm{d}s-\int_{\eta}^{1}\phi_q\int_{\frac{1}{2}}^{s}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s \\ +\int_t^{1}\phi_q\int_{\frac{1}{2}}^{s}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s, \qquad t\in[\frac{1}{2},1]. \end{cases}$$

Let

$$\begin{aligned}&(T^1x)(t)\\ &=\frac{1}{\delta}\phi_q\int_0^{\frac{1}{2}}h(s)f(s,x(s),|x'(s)|)\mathrm{d}s-\int_0^{\xi}\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\\ &\quad+\int_0^{t}\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s,\ t\in[0,\frac{1}{2}],\end{aligned}$$

$$\begin{aligned}(T^2x)(t)=&\frac{1}{\delta}\phi_q\int_{\frac{1}{2}}^{1}h(s)f(s,x(s),|x'(s)|)\mathrm{d}s-\int_{\eta}^{1}\phi_q\int_{\frac{1}{2}}^{s}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\\ &+\int_t^{1}\phi_q\int_{\frac{1}{2}}^{s}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s,\ t\in[\frac{1}{2},1].\end{aligned}$$

Obviously, $(T^1x)(t)$ is continuous on $t\in[0,\frac{1}{2}]$, and $(T^2x)(t)$ is continuous on $t\in[\frac{1}{2},1]$, $(T^1x)(\frac{1}{2})=(T^2x)(\frac{1}{2})$, so $(Tx)(t)$ is continuous on $t\in[0,1]$.

Let $k>2$, the nonnegative continuous concave functional α, the nonnegative continuous convex functional θ,γ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\begin{aligned}&\gamma(x)=\max_{0\le t\le 1}|x'(t)|, \qquad \psi(x)=\theta(x)=\max_{0\le t\le 1}|x(t)|,\\ &\alpha(x)=\min_{\frac{1}{k}\le t\le(1-\frac{1}{k})}|x(t)|, \quad \text{for} \quad x\in P.\end{aligned}$$

Lemma 3.2. $T:\ P\to P$ *is completely continuous.*

Proof. Step 1: We firstly prove that $T:P\to P$ is well defined. $(Tx)(t)\in C^1[0,1]$, $(Tx)''(t)=-h(t)f(t,x(t),x'(t))\le 0, (Tx)'(0)-\delta(Tx)(\xi)=0$. And it can be easily seen that $(Tx)(t)=(Tx)(1-t)$. In fact, the assumption (C_1) (C_2) yield that for $0\le t\le\frac{1}{2}$,

$$
\begin{aligned}
&(Tx)(t)\\
&=\frac{1}{\delta}\phi_q\int_0^{\frac{1}{2}}h(s)f(s,x(s),|x'(s)|)\mathrm{d}s-\int_0^{\xi}\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\\
&\quad+\int_0^t\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\\
&=\frac{1}{\delta}\phi_q\int_{\frac{1}{2}}^1 h(1-s)f(1-s,x(1-s),|x'(1-s)|)\mathrm{d}s-\int_\eta^1\phi_q\int_{1-s}^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),\\
&\qquad\qquad|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s+\int_{1-t}^1\phi_q\int_{1-s}^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\\
&=\frac{1}{\delta}\phi_q\int_{\frac{1}{2}}^1 h(s)f(s,x(s),|x'(s)|)\mathrm{d}s-\int_\eta^1\phi_q\int_{\frac{1}{2}}^{s}h(1-\tau)f(1-\tau,x(1-\tau),\\
&\qquad\quad|x'(1-\tau)|)\mathrm{d}\tau\mathrm{d}s+\int_{1-t}^1\phi_q\int_{\frac{1}{2}}^{s}h(1-\tau)f(1-\tau,x(1-\tau),|x'(1-\tau)|)\mathrm{d}\tau\mathrm{d}s\\
&=\frac{1}{\delta}\phi_q\int_{\frac{1}{2}}^1 h(s)f(s,x(s),|x'(s)|)\mathrm{d}s-\int_\eta^1\phi_q\int_{\frac{1}{2}}^{s}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\\
&\quad+\int_{1-t}^1\phi_q\int_{\frac{1}{2}}^{s}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s=(Tx)(1-t).
\end{aligned}
$$

With the same argument, we can get that $(Tx)(t)=(Tx)(1-t)$ also holds for $t\in[1/2,1]$. Noticing $0<\delta<\frac{1}{\xi}$, $(Tx)(t)\geq 0$ on $t\in[0,1]$ is obvious. Thus $T:\ P\to P$ is well defined.

Step 2: $(Tx)(t)$ is continuous on $x\in P$. Let $x_n\to x$, as $n\to\infty$ in P. Then there exists a real number r such that $\sup_{n\in N\setminus\{0\}}\|x\|\leq r$, and according to (C_2), we have

$$S_r:=\sup\{|f(t,x(t),|x'(t)|)|\ \ 0\leq t\leq 1,\ 0\leq x(t)\leq r,|x'(t)|\leq r\}<+\infty.$$

Considering (C_1), as $t\in[0,\frac{1}{2}]$, we see that

$$
\begin{aligned}
&\left|\frac{1}{\delta}\phi_q\int_0^{\frac{1}{2}}h(s)f(s,x(s),|x'(s)|)\mathrm{d}s-\int_0^{\xi}\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\right.\\
&\quad\left.+\int_0^t\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\right|<+\infty,\\
&\left|\phi_q\int_t^{\frac{1}{2}}h(s)f(s,x(s),|x'(s)|)\mathrm{d}s\right|<+\infty.
\end{aligned}
$$

By Lebesgue dominated convergence theorem and the continuity of ϕ_q, we have

$$\begin{aligned}
&\lim_{n\to+\infty}(Tx_n)(t)\\
&=\lim_{n\to+\infty}\Bigg[\frac{1}{\delta}\phi_q\int_0^{\frac{1}{2}}h(s)f(s,x_n(s),|x_n'(s)|)\mathrm{d}s-\int_0^{\xi}\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x_n(\tau),\\
&\qquad |x_n'(\tau)|)\mathrm{d}\tau\mathrm{d}s+\int_0^{t}\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x_n(\tau),|x_n'(\tau)|)\mathrm{d}\tau\mathrm{d}s\Bigg]\\
&=\frac{1}{\delta}\phi_q\int_0^{\frac{1}{2}}\lim_{n\to+\infty}h(s)f(s,x_n(s),|x_n'(s)|)\mathrm{d}s-\int_0^{\xi}\phi_q\int_s^{\frac{1}{2}}\lim_{n\to+\infty}h(\tau)f(\tau,x_n(\tau),\\
&\qquad |x_n'(\tau)|)\mathrm{d}\tau\mathrm{d}s+\int_0^{t}\phi_q\int_s^{\frac{1}{2}}\lim_{n\to+\infty}h(\tau)f(\tau,x_n(\tau),|x_n'(\tau)|)\mathrm{d}\tau\mathrm{d}s\\
&=(Tx)(t).
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to+\infty}(Tx_n)'(t)=\lim_{n\to+\infty}\int_t^{\frac{1}{2}}h(s)f(s,x_n(s),|x_n'(s)|)\mathrm{d}s\\
&=\int_t^{\frac{1}{2}}\lim_{n\to+\infty}h(s)f(s,x_n(s),|x_n'(s)|)\mathrm{d}s=(T(x))'(t).
\end{aligned}$$

Similarly, when $\frac{1}{2}\le t\le 1$, $\lim_{n\to+\infty}(Tx_n)(t)=(Tx)(t)$, $\lim_{n\to+\infty}(Tx_n)'(t)=(Tx)'(t)$. So we have T is continuous on $x\in P$.

Step 3: In this step, we aim to prove that T maps bounded sets into pre-compact sets on $t\in[0,1]$. Let Ω be a bounded set on X, in what following, we only need to prove that $T\Omega$ is a pre-compact set. From step 2 we have known that for any $x\in\Omega$, Tx is bounded and the boundary has nothing to do with x. So it remains to prove $(Tx)(t)$ is equi-continuous on Ω.

For $\forall x\in\Omega$, $t_1,t_2\in[0,\frac{1}{2}]$, we have

$$\begin{aligned}
|(Tx)(t_1)-(Tx)(t_2)|&=\left|\int_{t_1}^{t_2}\phi_q\int_s^{\frac{1}{2}}h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\right|\\
&\to 0\quad\text{as } t_1\to t_2.
\end{aligned}$$

$$\begin{aligned}
|(Tx)'(t_1)-(Tx)'(t_2)|&=\left|\phi_q\int_{t_1}^{t_2}h(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\right|\\
&\to 0\quad\text{as } t_1\to t_2.
\end{aligned}$$

By the same reasoning, we see that, when $t_1,t_2\in[\frac{1}{2},1]$, $|(Tx)(t_1)-(Tx)(t_2)|\to 0$ as $t_1\to t_2$. When $t_1\in[0,\frac{1}{2}],t_2\in[\frac{1}{2},1],(Tx)(t_2)=(Tx)(1-t_2)$, so when $t_1\to t_2$, we have $1-t_2\to t_1$, thus in this case we also have $|(Tx)(t_1)-(Tx)(t_2)|\to 0$ as $t_1\to t_2$.

Combining the above three steps, we have $T:P\to P$ is completely continuous. This completes the proof. □

Lemma 3.3. *For any $x \in P$, there exists a real number $\overline{M} > 0$ such that*

$$\max_{0\le t\le 1} |x(t)| \le \overline{M} \max_{0\le t\le 1} |x'(t)|,$$

where $\overline{M} = \max\{1, 1/2 + 1/\delta\}$.

Proof. For any $x \in P$, we have

$$\max_{0\le t\le 1} |x(t)| = \left|x(\frac{1}{2})\right| = \left|x(\xi) + \int_{\xi}^{\frac{1}{2}} x'(t)\mathrm{dt}\right| \le \left|\frac{1}{\delta}x'(0) + \int_0^{\frac{1}{2}} |x'(t)|\mathrm{dt}\right|$$
$$\le (\frac{1}{2} + \frac{1}{\delta}) \max_{0\le t\le 1} |x'(t)| \le \overline{M} \max_{0\le t\le 1} |x'(t)|.$$

which gives us the desired result. □

Lemma 3.4. *If $x \in P$, then $x(t) \ge 2\min\{t, (1-t)\} \max_{0\le t\le 1} |x(t)|$.*

Proof. Firstly, when $t \in [0, \frac{1}{2}]$, we have

$$x(t) \ge x(\frac{1}{2}) + \frac{x(1/2) - x(0)}{1/2}(t - 1/2) = 2tx(\frac{1}{2}) + (1 - 2t)x(0) \ge 2tx(\frac{1}{2}). \tag{3.1}$$

Secondly, when $t \in [\frac{1}{2}, 1]$, we have

$$x(t) \ge x(\frac{1}{2}) + \frac{x(1/2) - x(1)}{1/2 - 1}(t - 1/2)$$
$$= 2(1 - t)x(\frac{1}{2}) + (2t - 1)x(1) \ge 2(1 - t)x(\frac{1}{2}). \tag{3.2}$$

Combining (3.1) and (3.2), we have $x(t) \ge 2\min\{t, (1-t)\} \max_{0\le t\le 1} |x(t)|$, which completes our proof. □

Now, we are ready to apply Avery-Peterson fixed point theorem to the operator T, and will give sufficient conditions for the existence of at least three symmetric positive solutions to problem (1.1). Let

$$L = \phi_q \int_0^{\frac{1}{2}} h(s)\mathrm{ds}, \quad M = \int_{\frac{1}{k}}^{\frac{1}{2}} \phi_q \int_s^{\frac{1}{2}} h(\tau)\mathrm{d}\tau\ \mathrm{ds},$$
$$N = \frac{1}{\delta}\phi_q \int_0^{\frac{1}{2}} h(s)\mathrm{ds} - \int_0^{\xi} \phi_q \int_s^{\frac{1}{2}} h(\tau)\mathrm{d}\tau\mathrm{ds} + \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} h(\tau)\mathrm{d}\tau\mathrm{ds},$$
$$C = (\xi - \frac{1}{2})^2 + \frac{1}{\delta}, \quad D = \frac{kC}{2\xi^2}.$$

Remark 3.1. It is obvious that $D > 1$, we will not mention it in what follows.

Theorem 3.5. *Assume $(C_1) - (C_2)$ hold, let $0 < a < b \le md$, suppose further that f satisfies the following conditions:*

(H_1) $f(t,u,|v|) \le \phi_p(d/L)$ *for* $(t,u,|v|) \in [0,1] \times [0, \overline{M}d] \times [0,d]$;
(H_2) $f(t,u,|v|) > \phi_p(kb/(2M))$, *for* $(t,u,|v|) \in [\frac{1}{k}, 1-\frac{1}{k}] \times [b, Db] \times [0,d]$;
(H_3) $f(t,u,|v|) < \phi_p(a/N)$, *for* $(t,u,|v|) \in [0,1] \times [0,a] \times [0,d]$.

Where $m = \min\{\overline{M}, \frac{2\xi^2}{k}\}$, *then boundary value problem (1.1) has at least three symmetric positive solutions* x_1, x_2, x_3 *such that*

$$\begin{aligned} &\max_{0\le t\le 1} |x_i'(t)| \le d, \quad \text{for } i=1,2,3;\\ &\max_{0\le t\le 1} |x_1(t)| < a; \quad \max_{0\le t\le 1} |x_2(t)| > a;\\ &\min_{\frac{1}{k}\le t\le (1-\frac{1}{k})} |x_2(t)| < b; \quad \min_{\frac{1}{k}\le t\le (1-\frac{1}{k})} |x_3(t)| > b. \end{aligned} \tag{3.3}$$

Proof. Problem (1.1) has a solution $x = x(t)$ if and only if it solves the operator equation $x = Tx$. Thus we set out to verify the operator T satisfy Avery-Peterson fixed point theorem which will prove the existence of at least three fixed points of T.

Now, we will prove the main theorem step by step.

Step 1: Obviously, $\psi(\lambda x) \le \lambda\psi(x)$, $\alpha(x) \le \psi(x)$. Lemma 3.2 yields that $\|x\| \le \overline{M}\gamma(x)$ for all $x \in \overline{P(\gamma,d)}$. Then assumption ($H_1$) implies that $f(t,u,|v|) \le \phi_p(d/L)$. On the other hand, for any $x \in P$, there is $Tx \in P$, thus Tx is concave and symmetric on $[0,1]$. And $\max_{0\le t\le 1} |(Tx)'(t)| = |(Tx)'(0)|$, then we have

$$\begin{aligned} \gamma(Tx) &= \max_{0\le t\le 1}((Tx)'(t)) = (Tx)'(0) = \phi_q \int_0^{\frac{1}{2}} h(s) f(s, x(s), |x'(s)|)\mathrm{d}s\\ &\le \frac{d}{L}\cdot L = d. \end{aligned}$$

Hence, $T: \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

Step 2: In order to check that (A_1) of theorem 2.1 is satisfied, we choose $x(t) = [-(t-\frac{1}{2})^2 + C]\cdot \frac{kb}{2\xi^2}$. Obviously,

$$\begin{aligned} \alpha(x) &= \min_{\frac{1}{k}\le t\le (1-\frac{1}{k})} |x(t)| > b,\\ \theta(x) &= \max_{0\le t\le 1} |x(t)| = x(\frac{1}{2}) = \frac{kC}{2\xi^2} b = Db,\\ \gamma(x) &= \max_{0\le t\le 1} |x'(t)| = x'(0) = \frac{kb}{2\xi^2} \le d. \end{aligned}$$

Thus, $x \in P(\alpha,\theta,\gamma,b,Db,d)$ and $x \in \{P(\alpha,\theta,\gamma,b,Db,d) | \alpha(x) > b\} \ne \emptyset$. If $x \in P(\alpha,\theta,\gamma,b,Db,\ d)$, we have $b \le x(t) \le Db$, $|x'(t)| \le d$. Hence, from assumption (H_2) and lemma 3.3, it follows that

$$\alpha(Tx) = \min_{\frac{1}{k}\le t\le (1-\frac{1}{k})} |(Tx)(t)| \ge 2\min\{\frac{1}{k}, (1-\frac{1}{k})\} \max_{0\le t\le 1} |(Tx)(t)| = \frac{2}{k}\left|(Tx)(\frac{1}{2})\right|$$

$$\geq \frac{2}{k}\left[\int_{\frac{1}{k}}^{\frac{1}{2}} \phi_q \int_s^{\frac{1}{2}} h(\tau)f(\tau,x(\tau),|x'(\tau)|)\mathrm{d}\tau\mathrm{d}s\right]$$
$$\geq \frac{2}{k}\cdot\frac{kb}{2M}\cdot M = b.$$

This shows that condition (A_1) of theorem 2.1 is satisfied.

Step 3: In view of lemma 3.3, we see that

$$\alpha(Tx) \geq \frac{2}{k}\theta(Tx) = \frac{2}{k}Db = \frac{2}{k}\cdot\frac{kC}{2\xi^2}b > b.$$

for all $x \in P(\alpha,\gamma,b,d)$ with $\theta(Tx) > Db$.

Step 4: We finally show that (A_3) of theorem 2.1 holds. Clearly, $\psi(0) = 0 < a$, there holds $0 \notin R(\psi,\gamma,a,d)$. Suppose that $x \in R(\psi,\gamma,a,d)$ with $\psi(x) = a$. Then, in view of assumption (H_3), we have

$$\psi(Tx) = \max_{0\leq t\leq 1}|(Tx)(t)| = |(Tx)(\frac{1}{2})|$$
$$= \frac{1}{\delta}\phi_q\int_0^{\frac{1}{2}} h(s)f(s,x(s),x'(s))\mathrm{d}s - \int_0^{\xi}\phi_q\int_s^{\frac{1}{2}} h(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s$$
$$+\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} h(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s$$
$$< \frac{a}{N}\cdot N = a.$$

Hence, the condition (A_3) of theorem 2.1 is also satisfied. Therefore, an application of theorem 2.1 implies the boundary value problem (1.1) has at least three symmetric positive solutions x_1, x_2, x_3 satisfying (3.3). □

4. Example

In this section, we give an example to illustrate our main result.

Consider the following four point boundary value problem:

Example 4.6.

$$\begin{cases} (\phi_3(x'))' + f(t,x(t),|x'(t)|) = 0, \\ x'(0) - 2x(1/4) = 0, x'(1) + 2x(3/4) = 0, \end{cases} \tag{4.1}$$

where $f(t,u,v) = \begin{cases} t(1-t) + u^5 + \left|\frac{v}{400}\right|, & 0 \leq u \leq 12, \\ t(1-t) + 12^5 + \left|\frac{v}{400}\right|, & u \geq 12. \end{cases}$

We can see that $p = 3, h(t) = 1, \delta = 2, \xi = \frac{1}{4}, \eta = \frac{3}{4}$. let $k = 4, a = 1, b = 10, d = 400$. Then $q = \frac{3}{2}, \overline{M} = 1, C = \frac{9}{16}, D = 18, m = \frac{1}{32}, L < 0.8, M = \frac{1}{12}, N < 0.65$.

$$f(t,u,v) \leq 2.5\times 10^5 < \phi_3(d/L),\ \text{ for } (t,u,v) \in [0,1]\times[0,400]\times[-400,400];$$

$$f(t,u,v) > 10^5 > 5.76 \times 10^4 = \phi_3(4b/(2M)),$$
$$\text{for } (t,u,v) \in [\frac{1}{4},\frac{3}{4}] \times [10,180] \times [-400,400];$$
$$f(t,u,v) \le 0.25 + 1 + 1 = 2.25 \le \phi_3(a/N),$$
$$\text{for } (t,u,v) \in [0,1] \times [0,1] \times [-400,400].$$

Then the conditions in theorem 2.1 are all satisfied. So BVP (4.1) has at least three positive solutions x_1, x_2, x_3 such that

$$\max_{0\le t\le 1} |x_i'(t)| \le 400, \quad \text{for } i = 1,2,3;$$
$$\max_{0\le t\le 1} |x_1(t)| < 1; \quad 1 < \max_{0\le t\le 1} |x_2(t)| < 180;$$
$$\min_{\frac{1}{k}\le t\le(1-\frac{1}{k})} |x_2(t)| < 10; \quad \min_{\frac{1}{k}\le t\le(1-\frac{1}{k})} |x_3(t)| > 10.$$

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