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A M-TYPE RISK MODEL WITH MARKOV-MODULATED PREMIUM RATE

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ABSTRACT. In this paper, we consider a m-type risk model with Markov-modulated premium rate. A integral equation for the conditional ruin probability is obtained. A recursive inequality for the ruin probability with the stationary initial distribution and the upper bound for the ruin probability with no initial reserve are given. A system of Laplace transforms of non-ruin probabilities, given the initial environment state, is established from a system of integro-differential equations. In the two-state model, explicit formulas for non-ruin probabilities are obtained when the initial reserve is zero or when both claim size distributions belong to the K_n-family, $n \in N^+$. One example is given with claim sizes that have exponential distributions.

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1. Introduction

The study of the probability of ruin has been the centre of interest of many papers treating actuarial risk theory. The main objective of ruin theory is to obtain exact formulas or approximations of ruin probabilities in various kinds risk models. In this paper we are interested in the ruin probabilities in a Markov-modulated risk model. Models of this type have been investigated, e.g., by Reinhard (1984), Asmussen (1989), Grandell (1991), Jasiulewicz (2001), Lu and Wei (2005).

More recently, Reinhard and Snoussi (2001, 2002) have discussed the severity of ruin and the distribution of the surplus prior to ruin in a discrete semi-Markov risk model, respectively. Wu (1999) develops generalized bounds for the probability of ruin under a Markovian-modulated risk model. Jasiulewicz (2001) considers the probability of ruin under the influence of a premium rate

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which varies with the level of free reserves. Lu and Wei (2005) consider a Markov-modulated risk model in which the claim inter-arrivals, claim sizes and premiums influenced by an external Markovian environment process.

The purpose of this paper is to obtain the explicit formulas of the probability of ruin in a Markov-modulated model where premiums vary according to a Markovian environment. The same problem is studied for some authors, however, there are four main differences than them: first, in this paper, the premium rate is controlled by the Markov process, and the model provides different premium rate against random economic environment so as to get the maximization of economic effects for insurance companies; second, the Laplace transform approach is used to solve the system of integro-differential equations; third, the characteristic equation is fully discussed; fourth, we consider the m-type risk model. By these, explicit formulas for non-ruin probabilities when the initial reserve is zero are derived, in the two-state model, and the explicit results for non-ruin probabilities when both claim amounts distributions belong to K_n-family, $n \in N^+$, are obtained, where K_n-family of distributions includes, as the special cases, Erlang, Coxian, phase-type distributions, as well as the mixture of these distributions.

This paper is organized as follows. In Section 2, definitions and some results for the model are presented. A system of Laplace transforms of non-ruin probabilities is derived in Section 3. This system is fully solved in Section 4 for a two-state model and the two-type risk model when both claim size distributions belong to the K_n-family, $n \in N^+$. One example is also given.

2. The m-type risk reserve process

Let $(\Omega,\ F,\ P)$ be a complete probability space containing the following independent objects:

1. a point process $\{N_l(t) : t \geq 0\}$ with $N_l(0) = 0$, $E[N_l(t)] = \alpha_l t$, $l = 1, 2, \ldots m$.
2. a sequence $\{X_{lk} : k = 1, 2, 3 \ldots\}$ of independent and identically distributed random variables, having a common distribution function $F_l(x)$, with $F_l(0) = 0$ and mean value μ_l, $l = 1, 2, \ldots m$.

The m-type risk reserve process with Markov-modulated premium rate $R = \{R(t) : t \geq 0\}$ is defined by

$$R(t) = R(0) + \int_0^t c_{J_v} dv - \sum_{l=1}^{m} \sum_{k=1}^{N_l(t)} X_{lk}, \tag{1}$$

where $R(0) = u$ is the initial reserve. X_{l1},$X_{l2}, \ldots$represent the values of the successive claims of the lth risk and $N_l(t)$ represents the number of claims of the lth risk on the company during the interval $(0, t]$. Assume $\{J_v, v \geq 0\}$ is a stationary, Ergodic Markov jump process with finite state space $S = \{1, 2, \ldots, n\}$. The premium rate is controlled by the Markov process $\{J_v, v \geq 0\}$, that is , the premium rate at time t is c_{J_t}, and $c_i > 0$ is a constant when $J_v = i$,

$i = 1, 2, \ldots, n$. In addition we assume that $\{X_{lk},\ l = 1, 2, \ldots, m;\ k = 1, 2, \ldots\}$, $\{N_l(t),\ t \geq 0; l = 1, 2, \ldots, m\}$, $\{J_v, v \geq 0\}$ are mutually independent.

We are interested in calculating the probability of ultimate ruin

$$\psi(u) = \Pr(\inf_{0 \leq t < \infty} R(t) < 0 | R(0) = u),\ u \geq 0. \tag{2}$$

Assume that all states of process $\{J_v, v \geq 0\}$ communicate. Let η_i be the rate at which the process $\{J_v, v \geq 0\}$ leaves the state i when $J_v = i$ and p_{ij} be the probability that it then goes to j, i.e., the intensity η_{ij} of transition from i to j is given by

$$\eta_{ij} = \begin{cases} \eta_i p_{ij} \text{ for } i \neq j\ , \\ \\ -\eta_i \text{ for } i = j\ . \end{cases} \tag{3}$$

Let $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be a stationary distribution. Because all states communicate then

$$q_i \eta_i = \sum_{j=1}^{n} q_j \eta_j p_{ji}\ , \tag{4}$$

and the distribution $\mathbf{q}$ is a stationary initial distribution.

Define

$$T_u = \inf\{t > 0 | R(t) < 0\}$$

to be the time of ruin and define the ultimate ruin probabilities, given that the initial environment state is i and the initial reserve is u, by

$$\psi_i(u) = \Pr(T_u < \infty | R(0) = u, J_0 = i), \tag{5}$$

and the ultimate ruin probability in the stationary case by

$$\psi(u) = \sum_{i=1}^{n} q_i \psi_i(u),\ u \geq 0. \tag{6}$$

Their corresponding ultimate survival probabilities, or non-ruin probabilities, are defined, for $u \geq 0$, by $\varphi_i(u) = 1 - \psi_i(u), i \in S$, and $\varphi(u) = 1 - \psi(u)$, respectively.

Let $c_* = \min_{1 \leq i \leq n}\{c_i\}$, then the relative security $\rho = \frac{c_*}{\sum_{l=1}^{m} \alpha_l \mu_l} - 1$ and further assume $\rho > 0$.

Lemma. *If $\rho > 0$, then*

1. $\lim_{u \to \infty} \psi_i(u) = 0$, *for all* $i = 1, 2, \ldots n$,
2. $\lim_{u \to \infty} \psi(u) = 0$.

Proof. Let $Y(t) = R(t) - u$, then $Y(t)$ is risk reserve process.

$$\lim_{t \to \infty} \frac{Y(t)}{t} = \lim_{t \to \infty} \frac{1}{t} \left(\int_0^t c_{J_v} dv - \sum_{l=1}^{m} \sum_{k=1}^{N_l(t)} X_{lk} \right) \geq c_* - \lim_{t \to \infty} \frac{1}{t} \sum_{l=1}^{m} \sum_{k=1}^{N_l(t)} X_{lk}$$

$$= c_* - \sum_{l=1}^{m}\left(\lim_{t\to\infty}\frac{\sum_{k=1}^{N_l(t)}X_{lk}}{N_l(t)}\cdot\lim_{t\to\infty}\frac{N_l(t)}{t}\right) = c_* - \sum_{l=1}^{m}\alpha_l\mu_l > 0,$$

there exists a random variable T, such that $Y(t) > 0$ for all $t > T$. Since only finitely many claims can occur before T it follows that $\inf_{t>0}\{Y(t)\}$ is finite with probability one and thus

$$\lim_{u\to\infty}\psi_i(u) = 0$$

$$\lim_{u\to\infty}\psi(u) = \sum_{i=1}^{n} q_i \lim_{u\to\infty}\psi_i(u) = 0$$

□

Theorem 1. *Under the above assumptions and if $\rho > 0$ then*

$$\begin{aligned}\psi_i(u)c_i &= \psi_i(0)c_i + m\eta_i\int_0^u\psi_i(z)dz - \sum_{l=1}^{m}\alpha_l\int_0^u\overline{F}_l(z)dz \\ &+ \sum_{l=1}^{m}\alpha_l\int_0^u\psi_i(u-z)\overline{F}_l(z)dz - m\eta_i\sum_{j=1}^{n}p_{ij}\int_0^u\psi_j(z)dz,\end{aligned} \tag{7}$$

and

$$\psi_i(0)c_i = \sum_{l=1}^{m}\alpha_l\mu_l - m\eta_i\int_0^\infty\psi_i(u)du + m\eta_i\sum_{j=1}^{n}p_{ij}\int_0^\infty\psi_j(u)du, \tag{8}$$

for $i = 1, 2, \ldots n$

Proof. At first we will find the equations for the probability of non-ruin $\varphi_i(u) = 1 - \psi_i(u)$. It is normal to derive differential equations by a recursive argument. Consider $R(t)$ in a short time interval $(0, h)$ and separate the four possible cases as follows:

1. no claim occurs in $(0, h)$ and no change of state i in $(0, h)$,
2. one claim occurs in $(0, h)$, but the amount to be paid does not cause ruin and no change of state i in $(0, h)$,
3. no claim occurs in $(0, h)$ and change of state i in $(0, h)$,
4. at least one claim occurs in $(0, h)$ and at least one change of state of the process in $(0, h)$.

Using the laws of conditional probability, the non-ruin probability $\varphi_i(u)$ is equal to

$$\begin{aligned}\varphi_i(u) &= \left[\prod_{l=1}^{m}(1-\eta_i h-\alpha_l h+o(h))\right]\varphi_i(u+c_i h) \\ &+\sum_{l=1}^{m}\left[\prod_{k=1,k\neq l}^{m}(1-\eta_i h-\alpha_k h+o(h))\right]\alpha_l h\int_0^{u+c_i h}\varphi_i(u+c_i h-z)dF_l(z) \\ &+\left[\sum_{l=1}^{m}(1-\eta_i h-\alpha_l h+o(h))\eta_i h\right]\sum_{j=1}^{n}p_{ij}\varphi_j(u+c_j h)+o(h).\end{aligned} \tag{9}$$

Since $\varphi_i(u+c_ih)\approx\varphi_i(u)+\varphi_i'(u)c_ih$, after dividing by h, formula (9) reduces to

$$\begin{aligned}\varphi_i'(u)c_i \;=&\left(\sum_{l=1}^m\alpha_l+m\eta_i\right)[\varphi_i(u)+\varphi_i'(u)c_ih]\\ &-\sum_{l=1}^m\left[1-\sum_{k=1,k\neq l,}^m\alpha_kh-(m-1)\eta_ih\right]\alpha_l\int_0^{u+c_ih}\varphi_i(u+c_ih-z)dF_l(z)\\ &-\left[\sum_{l=1}^m(1-\eta_ih-\alpha_lh)\right]\eta_i\sum_{j=1}^n p_{ij}\varphi_j(u+c_jh)+\frac{o(h)}{h}.\end{aligned}$$

Taking $h\to 0$ we have

$$\begin{aligned}\varphi_i'(u)c_i=\left(\sum_{l=1}^m\alpha_l+m\eta_i\right)\varphi_i(u)\; &-\sum_{l=1}^m\alpha_l\int_0^u\varphi_i(u-z)dF_l(z)\\ &-m\eta_i\sum_{j=1}^n p_{ij}\varphi_j(u)\end{aligned}\qquad(10)$$

for $i=1,2,\ldots n$.
Integrating (10) over $(0,t)$ yields

$$\begin{aligned}\varphi_i(t)c_i \;=&\;\varphi_i(0)c_i+\left(\sum_{l=1}^m\alpha_l+m\eta_i\right)\int_0^t\varphi_i(u)du\\ &+\sum_{l=1}^m\alpha_l\int_0^t[\int_0^u\varphi_i(u-z)(1-F_l(z))'dz]du-m\eta_i\sum_{j=1}^n p_{ij}\int_0^t\varphi_j(u)du.\end{aligned}$$

Since

$$\begin{aligned}&\int_0^t[\int_0^u\varphi_i(u-z)(1-F_l(z))'dz]du\\ &=\varphi_i(0)\int_0^t[1-F_l(u)]du-\int_0^t\varphi_i(u)du+\int_0^t[\int_0^u(1-F_l(z))\varphi_i'(u-z)dz]du,\end{aligned}$$

and

$$\begin{aligned}\int_0^t[\int_0^u(1-F_l(z))\varphi_i'(u-z)dz]du \;&=\int_0^t[\int_z^t(1-F_l(z))\varphi_i'(u-z)du]dz\\ &=\int_0^t[1-F_l(z)][\varphi_i(t-z)-\varphi_i(0)]dz.\end{aligned}$$

Thus

$$\varphi_i(t)c_i \;=\varphi_i(0)c_i+m\eta_i\int_0^t\varphi_i(u)du$$

$$+\sum_{l=1}^{m}\alpha_l\int_0^t(1-F_l(z))\varphi_i(t-z)dz-m\eta_i\sum_{j=1}^{n}p_{ij}\int_0^t\varphi_j(u)du. \quad (11)$$

Taking $\varphi_i(t)=1-\psi_i(t)$ and $\overline{F}_l(t)=1-F_l(t)$ in (11) we have

$$\begin{aligned}\psi_i(t)c_i &= \psi_i(0)c_i+m\eta_i\int_0^t\psi_i(u)du-\sum_{l=1}^{m}\alpha_l\int_0^t\overline{F}_l(z)dz\\ &+ \sum_{l=1}^{m}\alpha_l\int_0^t\psi_i(t-z)\overline{F}_l(z)dz-m\eta_i\sum_{j=1}^{n}p_{ij}\int_0^t\psi_j(u)du. \quad (12)\end{aligned}$$

From the Lemma 1 and by the monotone convergence, it follows from (12), that as $t\to\infty$

$$\psi_i(0)c_i=\sum_{l=1}^{m}\alpha_l\mu_l-m\eta_i\int_0^\infty\psi_i(u)du+m\eta_i\sum_{j=1}^{n}p_{ij}\int_0^\infty\psi_j(u)du.$$

□

Theorem 2. *Under the above assumptions*

$$\psi(0)-\psi(u)\le\frac{1}{c_*}\sum_{l=1}^{m}\alpha_l\int_0^u\overline{F}_l(z)[1-\psi(u-z)]dz, \quad (13)$$

and

$$\psi(0)\le\frac{1}{1+\rho}. \quad (14)$$

Proof. Since $c_*=\min_{1\le i\le n}\{c_i\}$, then from (7) we have

$$\begin{aligned}c_*[\psi_i(0)-\psi_i(u)] &\le \sum_{l=1}^{m}\alpha_l\int_0^u\overline{F}_l(z)dz-\sum_{l=1}^{m}\alpha_l\int_0^u\overline{F}_l(z)\psi_i(u-z)dz\\ &- m\eta_i\int_0^u\psi_i(z)dz+m\eta_i\sum_{j=1}^{n}p_{ij}\int_0^u\psi_j(z)dz. \quad (15)\end{aligned}$$

Summing (15) with respect to $\mathbf{q}=(q_1,q_2,\dots,q_n)$,

$$\begin{aligned}c_*[\psi(0)-\psi(u)] &\le\sum_{l=1}^{m}\alpha_l\int_0^u\overline{F}_l(z)dz-\sum_{l=1}^{m}\alpha_l\int_0^u\overline{F}_l(z)\psi(u-z)dz\\ &-m\sum_{i=1}^{n}q_i\eta_i\int_0^u\psi_i(z)dz+m\sum_{i=1}^{n}q_i\eta_i\sum_{j=1}^{n}p_{ij}\int_0^u\psi_j(z)dz. \quad (16)\end{aligned}$$

Since $q_i\eta_i=\sum_{j=1}^{n}q_j\eta_jp_{ji}$, then

$$c_*[\psi(0)-\psi(u)]\le\sum_{l=1}^{m}\alpha_l\int_0^u\overline{F}_l(z)[1-\psi(u-z)]dz,$$

that is,

$$\psi(0)-\psi(u)\leq \frac{1}{c_*}\sum_{l=1}^{m}\alpha_l\int_0^u \overline{F}_l(z)[1-\psi(u-z)]dz. \tag{17}$$

From the Lemma 1 and by the monotone convergence, it follows from (17), that as $u\to\infty$

$$\psi(0)\leq \frac{1}{c_*}\sum_{l=1}^{m}\alpha_l\int_0^\infty \overline{F}_l(z)dz=\frac{\sum_{l=1}^m\alpha_l\mu_l}{c_*}=\frac{1}{1+\rho}.$$

□

3. Laplace transforms

Eq.(8) does not give an explicit value for the probabilities $\varphi_i(0)$ as in the classical case($n=1$). We now apply Laplace transforms to solve it. Let $\widetilde{\varphi}_i$ and $\widetilde{f}_l$ be the Laplace transforms of φ_i and f_l, respectively, i.e.,

$$\widetilde{\varphi}_i(s)=\int_0^\infty e^{-su}\varphi_i(u)du,\ \widetilde{f}_l(s)=\int_0^\infty e^{-su}f_l(u)du,\ i\in S,\ l=1,2,\ldots,m. \tag{18}$$

Taking Laplace transforms on both sides of Eq.(11) yields

$$\varphi_i(0)=[s-\frac{m\eta_i+\sum_{l=1}^m\alpha_l(1-\widetilde{f}_l(s))}{c_i}]\widetilde{\varphi}_i(s)+\frac{m\eta_i}{c_i}\sum_{j=1}^n p_{ij}\widetilde{\varphi}_j(s), \tag{19}$$

or in a matrix form

$$A(s)\overrightarrow{\varphi}(s)=\overrightarrow{\varphi}(0), \tag{20}$$

where

$$A(s)=\begin{pmatrix} s-\beta_1(s) & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & s-\beta_n(s)\end{pmatrix}+\begin{pmatrix} \frac{m\eta_1}{c_1} & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & \frac{m\eta_n}{c_n}\end{pmatrix}\mathrm{P}, \tag{21}$$

$\beta_i(s)=\frac{m\eta_i+\sum_{l=1}^m\alpha_l[1-\widetilde{f}_l(s)]}{c_i}$, $i=1,2,\ldots,n$, $\overrightarrow{\varphi}(s)=[\widetilde{\varphi}_1(s),\ldots,\widetilde{\varphi}_n(s)]^T$, $\overrightarrow{\varphi}(0)=[\varphi_1(0),\ldots,\varphi_n(0)]^T$ and P is given by $\mathrm{P}=(p_{ij})$, with $p_{ii}=0$, for $i\in S$.

Then $\overrightarrow{\varphi}(s)$ can be solved as

$$\overrightarrow{\varphi}(s)=A(s)^{-1}\overrightarrow{\varphi}(0), \tag{22}$$

and

$$\det[A(s)]=0, \tag{23}$$

is the characteristic equation of (20).

4. The two-state model

In this section, we derive explicit expressions for non-ruin probabilities. By discussing analytically the roots of Eq.(23), the Laplace transform of $\overrightarrow{\varphi}(s)$ can be inverted for certain types of claim size distributions.

Now we consider the case when $n = 2$, that is $\{J_v\}$ is a two-state Markov process, which reflects the random environmental effects due to "normal" versus "abnormal", or "high season" versus "low season" conditions. In this case $p_{12} = p_{21} = 1$, $p_{11} = p_{22} = 0$ and $q_1 = \frac{\eta_2}{\eta_1+\eta_2}$, $q_2 = \frac{\eta_1}{\eta_1+\eta_2}$.

4.1 The characteristic equation

In this case matrix (21) has the form

$$A(s) = \begin{pmatrix} s - \frac{m\eta_1+\sum_{l=1}^m \alpha_l}{c_1} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_1} & \frac{m\eta_1}{c_1} \\ \frac{m\eta_2}{c_2} & s - \frac{m\eta_2+\sum_{l=1}^m \alpha_l}{c_2} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_2} \end{pmatrix},$$

and the characteristic Eq.(23) is of form

$$\begin{aligned} Q(s): &= [s - \frac{m\eta_1 + \sum_{l=1}^m \alpha_l}{c_1} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_1}] \\ &\times [s - \frac{m\eta_2 + \sum_{l=1}^m \alpha_l}{c_2} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_2}] \\ &= \frac{m^2\eta_1\eta_2}{c_1c_2}. \end{aligned} \tag{24}$$

Note that $s = 0$ is one root of Eq.(24). The following theorem shows that it also has one and only one positive root, which plays the key role in deriving the non-ruin probabilities $\varphi_i(u)$.

Theorem 3. *Characteristic Eq.(24) has exactly one positive real root, say ξ, on the right half complex plane.*

Proof. For $\delta > 0$, consider the root of equation:

$$\begin{aligned} Q_\delta(s): &= [s - \frac{m\eta_1 + \sum_{l=1}^m \alpha_l + \delta}{c_1} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_1}] \\ &\times [s - \frac{m\eta_2 + \sum_{l=1}^m \alpha_l + \delta}{c_2} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_2}] \\ &= \frac{m^2\eta_1\eta_2}{c_1c_2}. \end{aligned} \tag{25}$$

Noting that $Q_\delta(0) = \frac{[m\eta_1+\delta][m\eta_2+\delta]}{c_1c_2} > \frac{m^2\eta_1\eta_2}{c_1c_2}$, $Q_\delta(+\infty) = +\infty$ and $Q_\delta(s) = 0$ has two positive roots, we conclude that Eq.(25) has at least two positive roots.

Furthermore, if s is on the half circle: $|z| = r(r > 0)$ and $\Re(z) \geq 0$ on the complex plane, then $|Q_\delta(s)| > \frac{m^2\eta_1\eta_2}{c_1c_2}$, for r is sufficiently large, while if s is on the imaginary axis, $\Re(s) = 0$, then $|Q_\delta(s)| \geq \frac{(m\eta_1+\delta)(m\eta_2+\delta)}{c_1c_2} > \frac{m^2\eta_1\eta_2}{c_1c_2}$, which is the right side of Eq.(25). This implies on the boundary of the contour enclosed by the half circle and the imaginary axis, that $|Q_\delta(s)| > \frac{m^2\eta_1\eta_2}{c_1c_2}$. We conclude, by Rouché's Theorem, that on the right half plane, the number of roots to Eq.(25) equals the number of roots of the equation$Q_\delta(s) = 0$. Moreover, by Rouché's Theorem, the latter only has exactly two roots with a positive real part on the right half complex plane.

It follows that Eq.(25) also only has exactly two roots with a positive real part, say, $\xi_1(\delta)$ and $\xi_2(\delta)$, on the right half complex plane.

Finally, as $\delta \to 0^+$, $\xi_j(\delta) \to \xi_j(0), j = 1, 2$, where $\xi_j(0)$ are roots of Eq.(24). Moreover, the fact that $s = 0$ is a root of (24) shows that $\lim_{\delta\to 0^+} \xi_1(\delta) = 0$, $\lim_{\delta\to 0^+} \xi_2(\delta) = \xi > 0$, the unique positive real root of (24). □

4.2 Formulas for $\varphi_1(0)$ and $\varphi_2(0)$

Now Eq.(20) has the form

$$\begin{pmatrix} s - \frac{m\eta_1+\sum_{l=1}^m \alpha_l}{c_1} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_1} & \frac{m\eta_1}{c_1} \\ \frac{m\eta_2}{c_2} & s - \frac{m\eta_2+\sum_{l=1}^m \alpha_l}{c_2} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_2} \end{pmatrix}$$

$$\times \begin{pmatrix} \widetilde{\varphi}_1(s) \\ \widetilde{\varphi}_2(s) \end{pmatrix} = \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix},$$

or

$$\begin{cases} \widetilde{\varphi}_1(s) = \dfrac{\varphi_1(0)[s - \frac{m\eta_2+\sum_{l=1}^m \alpha_l}{c_2} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_2}] - \varphi_2(0)\frac{m\eta_1}{c_1}}{Q(s) - \frac{m^2\eta_1\eta_2}{c_1c_2}}, \\ \widetilde{\varphi}_2(s) = \dfrac{\varphi_2(0)[s - \frac{m\eta_1+\sum_{l=1}^m \alpha_l}{c_1} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(s)}{c_1}] - \varphi_1(0)\frac{m\eta_2}{c_2}}{Q(s) - \frac{m^2\eta_1\eta_2}{c_1c_2}}. \end{cases} \tag{26}$$

Since $\widetilde{\varphi}_1(s)$ and $\widetilde{\varphi}_2(s)$ are finite for all s with $\Re(s) \geq 0$ and $Q(\xi) = \frac{m^2\eta_1\eta_2}{c_1c_2}$, we have that both the numerators in (26) are zero when $s = \xi$, i.e.,

$$\varphi_1(0)[\xi - \frac{m\eta_2 + \sum_{l=1}^m \alpha_l}{c_2} + \frac{\sum_{l=1}^m \alpha_l \widetilde{f_l}(\xi)}{c_2}] = \varphi_2(0)\frac{m\eta_1}{c_1}. \tag{27}$$

Then (26) can be rewritten as

$$\begin{cases} \widetilde{\varphi}_1(s) = \dfrac{\varphi_1(0)[(s-\xi) + \frac{\sum_{l=1}^m \alpha_l [\widetilde{f_l}(s) - \widetilde{f_l}(\xi)]}{c_2}]}{Q(s) - \frac{m^2\eta_1\eta_2}{c_1c_2}}, \\ \widetilde{\varphi}_2(s) = \dfrac{\varphi_2(0)[(s-\xi) + \frac{\sum_{l=1}^m \alpha_l [\widetilde{f_l}(s) - \widetilde{f_l}(\xi)]}{c_1}]}{Q(s) - \frac{m^2\eta_1\eta_2}{c_1c_2}}. \end{cases} \tag{28}$$

On the other hand, from (11), as $t \to \infty$, we have

$$\varphi_i(0) = 1 - \frac{1}{c_i}\sum_{l=1}^{m}\alpha_l\mu_l - \frac{m\eta_i}{c_i}\int_0^{\infty}[\varphi_i(u) - \sum_{j=1}^{2}p_{ij}\varphi_j(u)], \tag{29}$$

then Eq.(29) gives

$$\frac{\eta_2}{c_2}\varphi_1(0) + \frac{\eta_1}{c_1}\varphi_2(0) = \frac{\eta_1}{c_1}(1 - \frac{1}{c_2}\sum_{l=1}^{m}\alpha_l\mu_l) + \frac{\eta_2}{c_2}(1 - \frac{1}{c_1}\sum_{l=1}^{m}\alpha_l\mu_l). \tag{30}$$

Combining (27) with (30), we get the following theorem.

Theorem 4. *For risk model given by (1), with $n = 2$ and $\rho > 0$, the non-ruin probabilities when the initial reserve is zero are given by*

$$\begin{cases} \varphi_1(0) = \dfrac{m[\frac{\eta_1}{c_1}(1-\frac{1}{c_2}\sum_{l=1}^{m}\alpha_l\mu_l)+\frac{\eta_2}{c_2}(1-\frac{1}{c_1}\sum_{l=1}^{m}\alpha_l\mu_l)]}{\xi-\frac{\sum_{l=1}^{m}\alpha_l(1-\widetilde{f}_l(\xi))}{c_2}}, \\ \\ \varphi_2(0) = \dfrac{m[\frac{\eta_1}{c_1}(1-\frac{1}{c_2}\sum_{l=1}^{m}\alpha_l\mu_l)+\frac{\eta_2}{c_2}(1-\frac{1}{c_1}\sum_{l=1}^{m}\alpha_l\mu_l)]}{\xi-\frac{\sum_{l=1}^{m}\alpha_l(1-\widetilde{f}_l(\xi))}{c_1}}. \end{cases} \tag{31}$$

4.3 Explicit results for $\varphi_1(u)$ and $\varphi_2(u)$

In this section, we only consider the case when $m = 2$. The claim size distributions f_1 and f_2 belong to the K_n-family, $n \in N^+$, namely, their Laplace transformations are of the form:

$$\widetilde{f}_1(s) = \frac{p_{k-1}(s)}{p_k(s)},\ \widetilde{f}_2(s) = \frac{q_{r-1}(s)}{q_r(s)},\ k, r \in N^+, \tag{32}$$

where $p_{k-1}(s)$ and $q_{r-1}(s)$ polynomials of degrees $k-1$ and $r-1$ or less, respectively, while $p_k(s)$, $q_r(s)$ are polynomials of degrees k and r, with only negative roots, and satisfying $p_{k-1}(0) = p_k(0)$ and $q_{r-1}(0) = q_r(0)$. This general class of distributions includes, as special cases, the Erlang, Coxian and phase-type distributions, as well as mixtures of these [see (Cohen, 1982; Tijms, 1994)].

It turns out that equations in (28) can be transformed to rational expressions by multiplying both numerators and denominators by $p_k(s)q_r(s)$:

$$\begin{aligned} \widetilde{\varphi}_1(s) &= \frac{\varphi_1(0)(s-\xi)[p_k(s)q_r(s) + \frac{\alpha_1 q_r(s)}{c_2 p_k(\xi)}\cdot\frac{p_{k-1}(s)p_k(\xi)-p_k(s)p_{k-1}(\xi)}{s-\xi}]}{p_k(s)q_r(s)[Q(s) - \frac{4\eta_1\eta_2}{c_1c_2}]} \\ &+ \frac{\varphi_1(0)(s-\xi)\frac{\alpha_2 p_k(s)}{c_2 q_r(\xi)}\cdot\frac{q_{r-1}(s)q_r(\xi)-q_r(s)q_{r-1}(\xi)}{s-\xi}}{p_k(s)q_r(s)[Q(s) - \frac{4\eta_1\eta_2}{c_1c_2}]} \\ &= \frac{\varphi_1(0)(s-\xi)[p_k(s)q_r(s) + \frac{\alpha_1 q_r(s)}{c_2}(p_{k-1}[s,\xi] - \frac{p_{k-1}(\xi)}{p_k(\xi)}p_k[s,\xi])]}{p_k(s)q_r(s)[Q(s) - \frac{4\eta_1\eta_2}{c_1c_2}]} \\ &+ \frac{\varphi_1(0)(s-\xi)\frac{\alpha_2 p_k(s)}{c_2}(q_{r-1}[s,\xi] - \frac{q_{r-1}(\xi)}{q_r(\xi)}q_r[s,\xi])}{p_k(s)q_r(s)[Q(s) - \frac{4\eta_1\eta_2}{c_1c_2}]}, \end{aligned} \tag{33}$$

$$
\begin{aligned}
\widetilde{\varphi}_2(s) &= \frac{\varphi_2(0)(s-\xi)[p_k(s)q_r(s)+\frac{\alpha_1 q_r(s)}{c_1 p_k(\xi)}\cdot\frac{p_{k-1}(s)p_k(\xi)-p_k(s)p_{k-1}(\xi)}{s-\xi}]}{p_k(s)q_r(s)[Q(s)-\frac{4\eta_1\eta_2}{c_1c_2}]} \\
&+ \frac{\varphi_2(0)(s-\xi)\frac{\alpha_2 p_k(s)}{c_1 q_r(\xi)}\cdot\frac{q_{r-1}(s)q_r(\xi)-q_r(s)q_{r-1}(\xi)}{s-\xi}}{p_k(s)q_r(s)[Q(s)-\frac{4\eta_1\eta_2}{c_1c_2}]} \\
&= \frac{\varphi_2(0)(s-\xi)[p_k(s)q_r(s)+\frac{\alpha_1 q_r(s)}{c_1}(p_{k-1}[s,\xi]-\frac{p_{k-1}(\xi)}{p_k(\xi)}p_k[s,\xi])]}{p_k(s)q_r(s)[Q(s)-\frac{4\eta_1\eta_2}{c_1c_2}]} \\
&+ \frac{\varphi_2(0)(s-\xi)\frac{\alpha_2 p_k(s)}{c_1}(q_{r-1}[s,\xi]-\frac{q_{r-1}(\xi)}{q_r(\xi)}q_r[s,\xi])}{p_k(s)q_r(s)[Q(s)-\frac{4\eta_1\eta_2}{c_1c_2}]}, \qquad (34)
\end{aligned}
$$

where $p_{k-1}[s,\xi] := \frac{p_{k-1}(s)-p_{k-1}(\xi)}{s-\xi}$, a polynomial of degree $k-2$, is the first order divided difference of $p_{k-1}(s)$ with respect to ξ and $p_k[s,\xi]$, $q_{r-1}[s,\xi]$, and $q_r[s,\xi]$ are defined similarly. It is clear that both numerators of (33) and (34) are now polynomials of degree $k+r+1$.

For simplicity, let $D_{k+r+2}(s)$ be the common denominator of (33) and (34), which is clearly a polynomial of degree $k+r+2$ with the leading coefficient 1. Then equation $D_{k+r+2}(s)=0$ has $k+r+2$ roots on the complex plane and all of them are in pairs of conjugate forms. Noting that $s=0$ and $s=\xi$ are of two roots, then

$$
D_{k+r+2}(s) = s(s-\xi)\prod_{j=1}^{k+r}(s+\gamma_j). \qquad (35)
$$

We remark that all γ_j's have a positive real parts, since, otherwise, they would also be roots to the characteristic Eq.(25), which is a contradiction to the conclusion in Theorem 3 that there is only one root on the right half complex plane.

Then (33) and (34) can be simplified to

$$
\begin{aligned}
\widetilde{\varphi}_1(s) &= \frac{\varphi_1(0)[p_k(s)q_r(s)+\frac{\alpha_1 q_r(s)}{c_2}(p_{k-1}[s,\xi]-\frac{p_{k-1}(\xi)}{p_k(\xi)}p_k[s,\xi])]}{s\prod_{j=1}^{k+r}(s+\gamma_j)} \\
&+ \frac{\varphi_1(0)\frac{\alpha_2 p_k(s)}{c_2}(q_{r-1}[s,\xi]-\frac{q_{r-1}(\xi)}{q_r(\xi)}q_r[s,\xi])}{s\prod_{j=1}^{k+r}(s+\gamma_j)} \\
&= \frac{\varphi_1(0)g_{k+r}(s)}{s\prod_{j=1}^{k+r}(s+\gamma_j)}, \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\varphi}_2(s) &= \frac{\varphi_2(0)[p_k(s)q_r(s)+\frac{\alpha_1 q_r(s)}{c_1}(p_{k-1}[s,\xi]-\frac{p_{k-1}(\xi)}{p_k(\xi)}p_k[s,\xi])]}{s\prod_{j=1}^{k+r}(s+\gamma_j)} \\
&+ \frac{\varphi_2(0)\frac{\alpha_2 p_k(s)}{c_1}(q_{r-1}[s,\xi]-\frac{q_{r-1}(\xi)}{q_r(\xi)}q_r[s,\xi])}{s\prod_{j=1}^{k+r}(s+\gamma_j)}
\end{aligned}
$$

$$= \frac{\varphi_2(0)h_{k+r}(s)}{s\prod_{j=1}^{k+r}(s+\gamma_j)}, \tag{37}$$

where

$$\begin{aligned} g_{k+r}(s) &= p_k(s)q_r(s) + \frac{\alpha_1 q_r(s)}{c_2}(p_{k-1}[s,\xi] - \frac{p_{k-1}(\xi)}{p_k(\xi)}p_k[s,\xi]) \\ &+ \frac{\alpha_2 p_k(s)}{c_2}(q_{r-1}[s,\xi] - \frac{q_{r-1}(\xi)}{q_r(\xi)}q_r[s,\xi]), \end{aligned} \tag{38}$$

$$\begin{aligned} h_{k+r}(s) &= p_k(s)q_r(s) + \frac{\alpha_1 q_r(s)}{c_1}(p_{k-1}[s,\xi] - \frac{p_{k-1}(\xi)}{p_k(\xi)}p_k[s,\xi]) \\ &+ \frac{\alpha_2 p_k(s)}{c_1}(q_{r-1}[s,\xi] - \frac{q_{r-1}(\xi)}{q_r(\xi)}q_r[s,\xi]). \end{aligned} \tag{39}$$

Then if $\gamma_j, j = 1, 2, \ldots, k+r$, are distinct numbers, we obtain, by partial fractions, that

$$\widetilde{\varphi}_1(s) = \varphi_1(0)[\frac{g_0}{s} + \sum_{j=1}^{k+r}\frac{g_j}{s+\gamma_j}],\ \widetilde{\varphi}_2(s) = \varphi_2(0)[\frac{h_0}{s} + \sum_{j=1}^{k+r}\frac{h_j}{s+\gamma_j}], \tag{40}$$

and accordingly,

$$\varphi_1(u) = \varphi_1(0)[g_0 + \sum_{j=1}^{k+r} g_j e^{-\gamma_j u}],\ \varphi_2(u) = \varphi_2(0)[h_0 + \sum_{j=1}^{k+r} h_j e^{-\gamma_j u}], \tag{41}$$

where $g_0 = \frac{g_{k+r}(0)}{\prod_{j=1}^{k+r}\gamma_j}$, $h_0 = \frac{h_{k+r}(0)}{\prod_{j=1}^{k+r}\gamma_j}$ and

$$g_j = \frac{-g_{k+r}(-\gamma_j)}{\gamma_j\prod_{l=1,l\neq j}^{k+r}(\gamma_l - \gamma_j)},\ h_j = \frac{-h_{k+r}(-\gamma_j)}{\gamma_j\prod_{l=1,l\neq j}^{k+r}(\gamma_l - \gamma_j)},\ j = 1, 2, \ldots, k+r. \tag{42}$$

From (41), we immediately have that $g_0 = \frac{1}{\varphi_1(0)}$, $h_0 = \frac{1}{\varphi_2(0)}$, since $\lim_{u\to\infty}\varphi_i(u) = 1$, $i = 1, 2$, and $\sum_{j=0}^{k+r} g_j = 1$, $\sum_{j=0}^{k+r} h_j = 1$.

We summarize the above results in the following theorem.

Theorem 5. *For risk models given by (1), with $n = 2$ and $\rho > 0$, if the claim size distributions belong to the k_n-family (32) and if $D_{k+r+2}(s) = 0$ has $k+r$ distinct roots $-\gamma_1, -\gamma_2, \ldots, -\gamma_{k+r}$, with negative real parts, then the non-ruin probabilities are given by*

$$\varphi_1(u) = 1 + \varphi_1(0)\sum_{j=1}^{k+r} g_j e^{-\gamma_j u},\ \varphi_2(u) = 1 + \varphi_2(0)\sum_{j=1}^{k+r} h_j e^{-\gamma_j u}, \tag{43}$$

where $\varphi_1(0)$ and $\varphi_2(0)$ are given by (31), while g_j, h_j are given by (42).

We remark that if some of γ_j's come in pairs of complex forms, the non-ruin probabilities may contain damped trigonometric functions, which can be seen in the next section.

4.4 Example

In this section, we illustrate the non-ruin probabilities $\varphi_1(u)$ and $\varphi_2(u)$ given by (42), with one example. In the example, we assume that both claim sizes are exponentially distributed with mean μ_1 and μ_2, respectively.

Example. Let $f_l(x) = \frac{1}{\mu_l}e^{-\frac{x}{\mu_l}}$, for $x \geq 0$ and $l = 1, 2$. Then their Laplace transforms are of forms $\widetilde{f_l}(s) = \frac{\frac{1}{\mu_l}}{s+\frac{1}{\mu_l}}$ for $l = 1, 2$, i.e., $p_0(s) = \frac{1}{\mu_1}$, $p_1(s) = s+\frac{1}{\mu_1}$, $q_0(s) = \frac{1}{\mu_2}$, $q_1(s) = s + \frac{1}{\mu_2}$.

The explicit expressions of $\varphi_1(0)$ and $\varphi_2(0)$, given by (31), are now obtained as

$$\varphi_1(0) = \frac{2[\frac{\eta_1}{c_1}(1 - \frac{1}{c_2}(\alpha_1\mu_1 + \alpha_2\mu_2)) + \frac{\eta_2}{c_2}(1 - \frac{1}{c_1}(\alpha_1\mu_1 + \alpha_2\mu_2))]}{\xi[1 - \frac{\alpha_1\mu_1(\mu_2\xi+1)+\alpha_2\mu_2(\mu_1\xi+1)}{c_2(\mu_1\xi+1)(\mu_2\xi+1)}]}, \tag{44}$$

$$\varphi_2(0) = \frac{2[\frac{\eta_1}{c_1}(1 - \frac{1}{c_2}(\alpha_1\mu_1 + \alpha_2\mu_2)) + \frac{\eta_2}{c_2}(1 - \frac{1}{c_1}(\alpha_1\mu_1 + \alpha_2\mu_2))]}{\xi[1 - \frac{\alpha_1\mu_1(\mu_2\xi+1)+\alpha_2\mu_2(\mu_1\xi+1)}{c_1(\mu_1\xi+1)(\mu_2\xi+1)}]}, \tag{45}$$

where ξ is the positive root of equation:

$$\begin{aligned} Q(s) &= [s - \frac{2\eta_1 + \alpha_1 + \alpha_2}{c_1} + \frac{\alpha_1\frac{\frac{1}{\mu_1}}{s+\frac{1}{\mu_1}} + \alpha_2\frac{\frac{1}{\mu_2}}{s+\frac{1}{\mu_2}}}{c_1}] \\ &\times [s - \frac{2\eta_1 + \alpha_1 + \alpha_2}{c_2} + \frac{\alpha_1\frac{\frac{1}{\mu_1}}{s+\frac{1}{\mu_1}} + \alpha_2\frac{\frac{1}{\mu_2}}{s+\frac{1}{\mu_2}}}{c_2}] \\ &= \frac{4\eta_1\eta_2}{c_1c_2}, \end{aligned} \tag{46}$$

which is equivalent to

$$\begin{aligned} D_4(s) &= [(s - \frac{2\eta_1 + \alpha_1 + \alpha_2}{c_1})(s + \frac{1}{\mu_1})(s + \frac{1}{\mu_2}) + \frac{\frac{\alpha_1}{\mu_1}(s + \frac{1}{\mu_1}) + \frac{\alpha_2}{\mu_2}(s + \frac{1}{\mu_1})}{c_1}] \\ &\times [s - \frac{2\eta_1 + \alpha_1 + \alpha_2}{c_2} + \frac{\alpha_1\frac{\frac{1}{\mu_1}}{s+\frac{1}{\mu_1}} + \alpha_2\frac{\frac{1}{\mu_2}}{s+\frac{1}{\mu_2}}}{c_2}] \\ &- \frac{4\eta_1\eta_2}{c_1c_2}(s + \frac{1}{\mu_1})(s + \frac{1}{\mu_2}) = 0, \end{aligned} \tag{47}$$

and it has exactly 4 roots, $s = 0$, $s = \xi$, $s = -\gamma_1$ and $s = -\gamma_2$. Now

$$g_2(s) = (s + \frac{1}{\mu_1})(s + \frac{1}{\mu_2}) - \frac{\alpha_1}{c_2}(s + \frac{1}{\mu_2})\frac{1}{\mu_1\xi + 1} - \frac{\alpha_2}{c_2}(s + \frac{1}{\mu_1})\frac{1}{\mu_2\xi + 1}, \tag{48}$$

$$h_2(s) = (s + \frac{1}{\mu_1})(s + \frac{1}{\mu_2}) - \frac{\alpha_1}{c_1}(s + \frac{1}{\mu_2})\frac{1}{\mu_1\xi + 1} - \frac{\alpha_2}{c_1}(s + \frac{1}{\mu_1})\frac{1}{\mu_2\xi + 1}, \tag{49}$$

and (42) yield,

$$g_1 = \frac{(\gamma_1 - \frac{1}{\mu_1})(\frac{1}{\mu_2} - \gamma_1) - \frac{\alpha_1}{c_2}(\frac{1}{\mu_2} - \gamma_1)\frac{1}{\mu_1\xi+1} - \frac{\alpha_2}{c_2}(\frac{1}{\mu_1} - \gamma_1)\frac{1}{\mu_2\xi+1}}{\gamma_1(\gamma_2 - \gamma_1)}, \tag{50}$$

$$g_2 = \frac{(\gamma_2 - \frac{1}{\mu_1})(\frac{1}{\mu_2} - \gamma_2) - \frac{\alpha_1}{c_2}(\frac{1}{\mu_2} - \gamma_2)\frac{1}{\mu_1\xi+1} - \frac{\alpha_2}{c_2}(\frac{1}{\mu_1} - \gamma_2)\frac{1}{\mu_2\xi+1}}{\gamma_2(\gamma_1 - \gamma_2)}, \tag{51}$$

$$h_1 = \frac{(\gamma_1 - \frac{1}{\mu_1})(\frac{1}{\mu_2} - \gamma_1) - \frac{\alpha_1}{c_1}(\frac{1}{\mu_2} - \gamma_1)\frac{1}{\mu_1\xi+1} - \frac{\alpha_2}{c_1}(\frac{1}{\mu_1} - \gamma_1)\frac{1}{\mu_2\xi+1}}{\gamma_1(\gamma_2 - \gamma_1)}, \tag{52}$$

$$h_2 = \frac{(\gamma_2 - \frac{1}{\mu_1})(\frac{1}{\mu_2} - \gamma_2) - \frac{\alpha_1}{c_1}(\frac{1}{\mu_2} - \gamma_2)\frac{1}{\mu_1\xi+1} - \frac{\alpha_2}{c_1}(\frac{1}{\mu_1} - \gamma_2)\frac{1}{\mu_2\xi+1}}{\gamma_2(\gamma_1 - \gamma_2)}. \tag{53}$$

Therefore, the non-ruin probabilities $\varphi_1(u)$ and $\varphi_2(u)$ are given by

$$\begin{cases} \varphi_1(u) = 1 + \varphi_1(0)[g_1 e^{-\gamma_1 u} + g_2 e^{-\gamma_2 u}], \\ \\ \varphi_2(u) = 1 + \varphi_2(0)[h_1 e^{-\gamma_1 u} + h_2 e^{-\gamma_2 u}], \end{cases} \quad u \geq 0 \tag{54}$$

where $\varphi_1(0)$ and $\varphi_2(0)$ are given by (44) and (45).

5. Conclusions

In this paper, we have generalized results in Jasiulewicz(2001). Compared to Jasiulewicz(2001), our results are more general and allow for a wider range of models for the aggregate claims process, in particular those for which premium rate varies with random economic environment. In this model, the premium rate is controlled by the Markov process, and the model provides different premium rate against random economic environment so as to get the maximization of economic effects for insurance companies.

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