J. Appl. Math. & Informatics Vol. 27(2009), No. 5 - 6, pp. 1001 - 1015
Website: http://www.kcam.biz

HYBRID DIFFERENCE SCHEMES FOR A SYSTEM OF SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATIONS.

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ABSTRACT. In this paper, two hybrid difference schemes on the Shishkin mesh are constructed for solving a weakly coupled system of two singularly perturbed convection-diffusion second order ordinary differential equations with a small parameter multiplying the highest derivative. We prove that the schemes are almost second order convergence in the supremum norm independent of the diffusion parameter. Error bounds for the numerical solution and its derivative are established. Numerical results are provided to illustrate the theoretical results.

AMS Mathematics Subject Classification : 65L10, CR G1.7
Key words and phrases : Singular perturbation problems, Piecewise uniform mesh, Hybrid difference scheme, Mid-point scheme, Cubic spline method, Scaled derivative.

1. Introduction

Robust parameter-uniform numerical methods for system of singularly perturbed ordinary differential equations have been considered widely in the literature (see [4] - [6], [8] - [10], [12] - [14] and references therein). While many finite difference methods have been proposed to approximate such solutions, there has been much less research into the finite difference approximation of their derivatives, even though such approximations are desirable in certain applications (flux or drag). As far as authors' knowledge goes only very few works have been reported in the literature (see [2], [3], [7], [14], [15] and references therein) for finding approximation to scaled derivatives of the solution for problems involving singularly perturbed second order ordinary differential equation with smooth/non-smooth data. For a singularly perturbed convection-diffusion

Received March 27, 2008. Revised June 6, 2008. Accepted June 27, 2008. *Corresponding author. The first author wishes to acknowledge the Council of Scientific and Industrial Research (CSIR), New Delhi, India for its financial support.

problem, M. Stynes and H. -G. Roos [1] presented a numerical method that is composed of the central difference scheme in the layer region $(1-\tau, 1)$ and mid-point scheme on outer region $(0, 1-\tau]$ defined on the Shishkin mesh with $\tau = \min\{0.5, \frac{2\varepsilon}{\alpha}\ln N\}$. This is a monotone method and when $\tau < 0.5$, it satisfies a parameter-uniform error bound of the form

$$\| U - u \| \leq \begin{cases} CN^{-1}(\varepsilon + N^{-1}), & \text{if} \quad x_i \in [0, 1-\tau], \\ C(N^{-1}\ln N)^2, & \text{if} \quad x_i \in (1-\tau, 1]. \end{cases}$$

In [13], the authors have analyzed a robust computational method that uses cubic spline scheme in the fine mesh region and classical central difference scheme in the coarse mesh region for singularly perturbed coupled system of reaction-diffusion boundary value problems. Motivated by these works, in this paper, two hybrid difference schemes are proposed to approximate the solution and its scaled first derivative of weakly coupled system of two singularly perturbed convection-diffusion equations. Here, bounds on the errors in approximating the first derivative of the solution with a weight in the fine mesh and without a weight in the coarse mesh are also obtained.

Note: Through out this paper, C denotes a generic constant is independent of the singular perturbation parameter ε and the dimension of the discrete problem N. Note that C can take different values at different place, even in the same argument. Let $y : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the supremum norm $\| y \|_D = \sup_{x\in D} |y(x)|$. In case of vectors $\bar{y} = (y_1, y_2)^T$, we define $|\bar{y}(x)| = (|y_1(x)|, |y_2(x)|)^T$ and $\| \bar{y} \|_D = \max\{\| y_1 \|_D, \| y_2 \|_D\}$.

Assumption : We shall assume that $\varepsilon \leq CN^{-1}$ throughout the paper as is generally the case for discretization of convection-dominated problem [1].

2. Continuous problem

Find $u_1, u_2 \in Y \equiv C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{aligned} &L_1\bar{u} \equiv -\varepsilon u_1'' + a_1(x)u_1' + b_{11}(x)u_1 + b_{12}(x)u_2 = f_1(x),\ x \in \Omega = (0,1), \\ &L_2\bar{u} \equiv -\varepsilon u_2'' + a_2(x)u_2' + b_{21}(x)u_1 + b_{22}(x)u_2 = f_2(x),\ x \in \Omega, \\ &u_1(0) = A_1, \quad u_2(0) = A_2, \quad u_1(1) = B_1, \quad u_2(1) = B_2, \end{aligned} \tag{1}$$

$$\begin{aligned} a_1(x) \geq \alpha_1 > 0, &\quad a_2(x) \geq \alpha_2 > 0, \\ b_{12}(x) \leq 0, &\quad b_{21}(x) \leq 0, \\ b_{11}(x) + b_{12}(x) \geq 0, &\quad b_{22}(x) + b_{21}(x) \geq 0, \end{aligned}$$

where the functions $a_i(x)$, $b_{ij}(x)$ and $f_i(x)$, $i, j = 1, 2$ are sufficiently smooth on $\overline{\Omega} = [0,1], 0 < \varepsilon \ll 1$. Let $\alpha = \min\{\alpha_1, \alpha_2\}$. The above system can be written in the vector form as

$$\mathbf{L}\bar{u} \equiv \begin{pmatrix} L_1\bar{u} \\ L_2\bar{u} \end{pmatrix} \equiv \begin{pmatrix} -\varepsilon\frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon\frac{d^2}{dx^2} \end{pmatrix}\bar{u} + \begin{pmatrix} a_1(x)\frac{d}{dx} & 0 \\ 0 & a_2(x)\frac{d}{dx} \end{pmatrix}\bar{u}$$
$$+ \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}\bar{u} = \bar{f}(x), \quad x \in \Omega,$$

$\bar{u}(0) = (A_1, A_2)^T$, $\bar{u}(1) = (B_1, B_2)^T$, where $\bar{u}(x) = (u_1(x), u_2(x))^T$ and $\bar{f}(x) = (f_1(x), f_2(x))^T$.

2.1. Analytical results.

Lemma 1. [14]. *Suppose that a function $\bar{u}(x) = (u_1(x), u_2(x))^T$, u_1, $u_2 \in Y$ satisfies $\bar{u}(0) \geq \bar{0}$, $\bar{u}(1) \geq \bar{0}$ and $\mathbf{L}\bar{u}(x) \geq \bar{0}$, for all $x \in \Omega$. Then $\bar{u}(x) \geq \bar{0}$, for all $x \in \overline{\Omega}$.*

In the rest of the problem for continuous case the norm $\| \,.\, \|$ means $\| \,.\, \|_{\overline{\Omega}}$.

Lemma 2. [14]. *If $u_1, u_2 \in Y$ then for $j = 1, 2$,*

$$|u_j(x)| \leq C\max\{|u_1(0)|, |u_2(0)|, |u_1(1)|, |u_2(1)|, \| L_1\bar{u} \|_\Omega, \| L_2\bar{u} \|_\Omega\}, x \in \bar{\Omega}.$$

The sharper bounds on the derivatives of the solution are obtained by decomposing the solution $\bar{u}$ into regular and singular components as, $\bar{u} = \bar{v} + \bar{w}$, where $\bar{v} = (v_1, v_2)^T$ and $\bar{w} = (w_1, w_2)^T$. The regular component $\bar{v}$ can be written in the form $\bar{v} = \bar{v}_0 + \varepsilon\bar{v}_1 + \varepsilon^2\bar{v}_2$, where $\bar{v}_0 = (v_{01}, v_{02})^T$, $\bar{v}_1 = (v_{11}, v_{12})^T$, $\bar{v}_2 = (v_{21}, v_{22})^T$ are defined respectively to be the solutions of the problems

$$\begin{pmatrix} a_1(x)\frac{d}{dx} & 0 \\ 0 & a_2(x)\frac{d}{dx} \end{pmatrix}\bar{v}_0 + \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}\bar{v}_0 = \bar{f}(x),\ \bar{v}_0(0) = \bar{u}(0),$$

$$\begin{pmatrix} a_1(x)\frac{d}{dx} & 0 \\ 0 & a_2(x)\frac{d}{dx} \end{pmatrix}\bar{v}_1 + \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}\bar{v}_1 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}\bar{v}_0,\ \bar{v}_1(0) = \bar{0}$$

and

$$\mathbf{L}\bar{v}_2 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}\bar{v}_1,\ \bar{v}_2(0) = \bar{0},\ \bar{v}_2(1) = \bar{0}.$$

Thus the regular component $\bar{v}$ is the solution of

$$\mathbf{L}\bar{v} = \bar{f}, \quad \bar{v}(0) = \bar{u}(0), \quad \bar{v}(1) = \bar{v}_0(1) + \varepsilon\bar{v}_1(1) + \varepsilon^2\bar{v}_2(1).$$

Then the singular component $\bar{w}$ is the solution of

$$\mathbf{L}\bar{w} = \bar{0}, \quad \bar{w}(0) = \bar{0}, \quad \bar{w}(1) = \bar{u}(1) - \bar{v}(1).$$

The following lemma provides the bound on the derivatives of the regular and singular components of the solution $\bar{u}$.

Lemma 3. *The regular and singular components $\bar{v}$ and $\bar{w}$ and their derivatives satisfy the bounds for $0 \leq k \leq 4$, $j = 1, 2$,*

$$\| v_j^{(k)} \| \leq C(1 + \varepsilon^{3-k}) \quad \text{and} \quad |w_j^{(k)}(x)| \leq C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon}, \quad \forall \quad x \in \overline{\Omega}.$$

Proof. In [14], a proof is given for $k = 3$. A similar argument holds good for $k = 4$. □

3. Discrete problem

In this section, first we derive a cubic spline scheme on a variable mesh. Let $x_0 = 0$, $x_N = 1$, $x_i = \sum_{k=0}^{i-1} h_k$, $h_k = x_{k+1} - x_k$, $i = 1, ..., N-1$ be the mesh. For given values $U_j(x_0), U_j(x_1), ..., U_j(x_N)$ of a function $u_j(x)$, $j = 1, 2$ at the nodal points $x_0, x_1, ..., x_N$ there exist interpolating cubic spline functions $S_1(x)$ and $S_2(x)$ with the following properties: For $j = 1, 2$,

(i) $S_j(x)$ coincides with a polynomial of degree three on each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, ..., N-1$; (ii) $S_j(x) \in C^2(\bar{\Omega})$; (iii) $S_j(x_i) = U_j(x_i)$, $i = 0, 1, ..., N$.

Then the cubic spline functions are given by

$$\begin{aligned} S_j(x) = {} & \frac{(x_{i+1}-x)^3}{6h_i} M_{j,i} + \frac{(x-x_i)^3}{6h_i} M_{j,i+1} + (U_j(x_i) - \frac{h_i^2}{6} M_{j,i})(\frac{x_{i+1}-x}{h_i}) \\ & + (U_j(x_{i+1}) - \frac{h_i^2}{6} M_{j,i+1})(\frac{x-x_i}{h_i}),\ x_i \le x \le x_{i+1},\ i = 0, 1, ..., N-1 \end{aligned}$$

where $M_{j,i} = S_j''(x_i)$, $i = 0, ..., N$, $j = 1, 2$. From the basic properties of spline, it should satisfy the following condition of continuity for $j = 1, 2$, $i = 1, ..., N-1$

$$\frac{h_{i-1}}{6} M_{j,i-1} + \frac{h_i + h_{i-1}}{3} M_{j,i} + \frac{h_i}{6} M_{j,i+1} = \frac{U_j(x_{i+1}) - U_j(x_i)}{h_i} - \frac{U_j(x_i) - U_j(x_{i-1})}{h_{i-1}}. \tag{2}$$

In order to obtain a second order approximation for the first order derivative of $u_j(x)$, $j = 1, 2$, we use the Taylor series expansion for U_j around x_i. That is,

$$U_j(x_{i+1}) \simeq U_j(x_i) + h_i U_j'(x_i) + \frac{h_i^2}{2} U_j''(x_i) \tag{3}$$

$$U_j(x_{i-1}) \simeq U_j(x_i) - h_{i-1} U_j'(x_i) + \frac{h_{i-1}^2}{2} U_j''(x_i). \tag{4}$$

Multiplying (4) by h_i^2/h_{i-1}^2, then subtracting from (3) and multiplying (4) by h_i/h_{i-1}, then adding it to (3), we get the following approximation for $U_j'(x_i)$ and $U_j''(x_i)$, $j = 1, 2$ respectively

$$U_j'(x_i) \simeq \frac{1}{h_i h_{i-1}(h_i + h_{i-1})} (h_{i-1}^2 U_j(x_{i+1}) + (h_i^2 - h_{i-1}^2) U_j(x_i) - h_i^2 U_j(x_{i-1})),$$

$$U_j''(x_i) \simeq \frac{2}{h_i h_{i-1}(h_i + h_{i-1})} (h_{i-1} U_j(x_{i+1}) - (h_i + h_{i-1}) U_j(x_i) + h_i U_j(x_{i-1})).$$

Using these approximations in $U_j'(x_{i+1}) \simeq U_j'(x_i) + h_i U_j''(x_i)$ and $U_j'(x_{i-1}) \simeq U_j'(x_i) - h_{i-1} U_j''(x_i), j = 1, 2$, we get the following approximation

$$\begin{aligned} U_j'(x_{i+1}) \simeq {} & \frac{1}{h_i h_{i-1}(h_i + h_{i-1})} [(h_{i-1}^2 + 2h_i h_{i-1}) U_j(x_{i+1}) - (h_i + h_{i-1})^2 U_j(x_i) \\ & + h_i^2 U_j(x_{i-1})], \end{aligned}$$

$$U_j'(x_{i-1}) \simeq \frac{1}{h_i h_{i-1}(h_i + h_{i-1})}[-h_{i-1}^2 U_j(x_{i+1}) + (h_i + h_{i-1})^2 U_j(x_i) - (h_i^2 + 2h_i h_{i-1})U_j(x_{i-1})].$$

Substituting

$$-\varepsilon M_{1,i} + a_1(x_i)U_1'(x_i) + b_{11}(x_i)U_1(x_i) + b_{12}(x_i)U_2(x_i) = f_1(x_i),\ i = 1, ..., N-1$$

and

$$-\varepsilon M_{2,i} + a_2(x_i)U_2'(x_i) + b_{21}(x_i)U_1(x_i) + b_{22}(x_i)U_2(x_i) = f_2(x_i),\ i = 1, ..., N-1$$

in (2), we get the following two linear system of equations, for $i = 1, ..., N-1$

$$r_{1,i}^- U_1(x_{i-1}) + r_{1,i}^c U_1(x_i) + r_{1,i}^+ U_1(x_{i+1}) + q_{1,i}^- U_2(x_{i-1}) + q_{1,i}^c U_2(x_i) + q_{1,i}^+ U_2(x_{i+1}) = F_1(x_i),$$
$$r_{2,i}^- U_2(x_{i-1}) + r_{2,i}^c U_2(x_i) + r_{2,i}^+ U_2(x_{i+1}) + q_{2,i}^- U_1(x_{i-1}) + q_{2,i}^c U_1(x_i) + q_{2,i}^+ U_1(x_{i+1}) = F_2(x_i), \quad (5)$$

where

$$r_{1,i}^- = \frac{h_i^2}{2h_{i-1}(h_i + h_{i-1})} a_1(x_{i+1}) - \frac{h_i}{h_{i-1}} a_1(x_i) - \frac{(h_i + 2h_{i-1})}{2(h_i + h_{i-1})} a_1(x_{i-1}) + \frac{h_{i-1}}{2} b_{11}(x_{i-1}) - \frac{3\varepsilon}{h_{i-1}},$$

$$r_{1,i}^c = \frac{-(h_i + h_{i-1})}{2h_{i-1}} a_1(x_{i+1}) + \frac{(h_i^2 - h_{i-1}^2)}{h_i h_{i-1}} a_1(x_i) + \frac{(h_i + h_{i-1})}{2h_i} a_1(x_{i-1}) + (h_i + h_{i-1}) b_{11}(x_i) + \frac{3\varepsilon(h_i + h_{i-1})}{h_i h_{i-1}},$$

$$r_{1,i}^+ = \frac{(2h_i + h_{i-1})}{2(h_i + h_{i-1})} a_1(x_{i+1}) + \frac{h_{i-1}}{h_i} a_1(x_i) - \frac{h_{i-1}^2}{2h_i(h_i + h_{i-1})} a_1(x_{i-1}) + \frac{h_i}{2} b_{11}(x_{i+1}) - \frac{3\varepsilon}{h_i},$$

$$q_{1,i}^- = \frac{h_{i-1} b_{12}(x_{i-1})}{2}, \quad q_{1,i}^c = (h_i + h_{i-1}) b_{12}(x_i), \quad q_{1,i}^+ = \frac{h_i b_{12}(x_{i+1})}{2},$$

$$F_1(x_i) = F_{1,i}^- f_1(x_{i-1}) + F_{1,i}^c f_1(x_i) + F_{1,i}^+ f_1(x_{i+1}),$$

$$F_{1,i}^- = \frac{h_{i-1}}{2}, \quad F_{1,i}^c = (h_i + h_{i-1}), \quad F_{1,i}^+ = \frac{h_i}{2},$$

$$r_{2,i}^- = \frac{h_i^2 a_2(x_{i+1})}{2h_{i-1}(h_i + h_{i-1})} - \frac{h_i}{h_{i-1}} a_2(x_i) - \frac{(h_i + 2h_{i-1})}{2(h_i + h_{i-1})} a_2(x_{i-1}) + \frac{h_{i-1}}{2} b_{22}(x_{i-1}) - \frac{3\varepsilon}{h_{i-1}},$$

$$r_{2,i}^c = \frac{-(h_i + h_{i-1})}{2h_{i-1}} a_2(x_{i+1}) + \frac{(h_i^2 - h_{i-1}^2)}{h_i h_{i-1}} a_2(x_i) + \frac{(h_i + h_{i-1})}{2h_i} a_2(x_{i-1}) + (h_i + h_{i-1}) b_{22}(x_i) + \frac{3\varepsilon(h_i + h_{i-1})}{h_i h_{i-1}},$$

$$r_{2,i}^+ = \frac{(2h_i + h_{i-1})}{2(h_i + h_{i-1})} a_2(x_{i+1}) + \frac{h_{i-1}}{h_i} a_2(x_i) - \frac{h_{i-1}^2 a_2(x_{i-1})}{2h_i(h_i + h_{i-1})} + \frac{h_i}{2} b_{22}(x_{i+1}) - \frac{3\varepsilon}{h_i},$$

$$q_{2,i}^- = \frac{h_{i-1}}{2} b_{21}(x_{i-1}), \quad q_{2,i}^c = (h_i + h_{i-1}) b_{21}(x_i), \quad q_{2,i}^+ = \frac{h_i}{2} b_{21}(x_{i+1}),$$

$$F_2(x_i) = F_{2,i}^- f_2(x_{i-1}) + F_{2,i}^c f_2(x_i) + F_{2,i}^+ f_2(x_{i+1}),$$
$$F_{2,i}^- = \frac{h_{i-1}}{2}, \quad F_{2,i}^c = (h_i + h_{i-1}), \quad F_{2,i}^+ = \frac{h_i}{2}.$$

3.1. Hybrid difference schemes: On Ω a piecewise-uniform mesh of N mesh interval is constructed as follows. The domain $\overline{\Omega}$ is subdivided into the two subintervals $[0, 1-\sigma] \cup [1-\sigma, 1]$ for some σ that satisfy $0 < \sigma \leq \frac{1}{2}$. On each subinterval a uniform mesh with $N/2$ mesh-intervals is placed. The interior points of the mesh are denote by $\overline{\Omega}^N = \{x_i \mid \ x_i = 2i(1-\sigma)/N,\ 0 \leq i \leq N/2; \quad x_i = x_{i-1} + 2\sigma/N,\ N/2 < i \leq N\}$ condensing at the boundary point $x_N = 1$. In the literature usually one takes the transition parameter σ as $\min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\}$. But for our analysis we assume that $\sigma = \frac{2\varepsilon}{\alpha}\ln N$, since otherwise N^{-1} is exponentially small compared with ε. Then the mesh widths are

$$h_i = \begin{cases} H_1 = 2(1-\sigma)/N, & i = 1, ..., N/2, \\ H_2 = 2\sigma/N, & i = N/2+1, ..., N. \end{cases}$$

On the above Shishkin mesh, we propose two hybrid difference schemes that use the central difference, cubic spline and mid-point scheme. The discrete problem is:

$$\begin{aligned} &\mathbf{L}^N \bar{U}(x_i) \equiv \begin{pmatrix} L_1^N \bar{U}(x_i) \\ L_2^N \bar{U}(x_i) \end{pmatrix} = \begin{pmatrix} F_1(x_i) \\ F_2(x_i) \end{pmatrix} \\ &U_1(0) = u_1(0), \quad U_2(0) = u_2(0), \quad U_1(1) = u_1(1), \quad U_2(1) = u_2(1). \end{aligned} \tag{6}$$

Hybrid Difference Scheme - I: In this scheme, we use the central finite difference scheme in the fine mesh region and mid-point difference scheme in the coarse region, that is,

$$L_1^N \bar{U}(x_i) = \begin{cases} -\varepsilon\delta^2 U_1(x_i) + a_{1,i-1/2} D^- U_1(x_i) + b_{11,i-1/2}\hat{U}_1(x_i) + \\ b_{12,i-1/2}\hat{U}_2(x_i) = F_1(x_i), \quad 0 < i \leq N/2, \\ -\varepsilon\delta^2 U_1(x_i) + a_{1,i}\, D^0 U_1(x_i) + b_{11,i}\, U_1(x_i) + \\ b_{12,i}\, U_2(x_i) = F_1(x_i), \quad N/2 < i < N, \end{cases}$$

$$L_2^N \bar{U}(x_i) = \begin{cases} -\varepsilon\delta^2 U_2(x_i) + a_{2,i-1/2} D^- U_2(x_i) + b_{21,i-1/2}\hat{U}_1(x_i) + \\ b_{22,i-1/2}\hat{U}_2(x_i) = F_2(x_i), \quad 0 < i \leq N/2, \\ -\varepsilon\delta^2 U_2(x_i) + a_{2,i} D^0 U_2(x_i) + b_{21,i}\, U_1(x_i) + \\ b_{22,i} U_2(x_i) = F_2(x_i), \quad N/2 < i < N, \end{cases}$$

$$F_1(x_i) = \begin{cases} f_{1,\,i-1/2}, & 1 \leq i \leq N/2 \\ f_{1,\,i}, & N/2+1 \leq i \leq N-1 \end{cases}$$

$$F_2(x_i) = \begin{cases} f_{2,\,i-1/2}, & 1 \leq i \leq N/2, \\ f_{2,\,i}, & N/2+1 \leq i \leq N-1, \end{cases}$$

where $\delta^2 U_j(x_i) = \frac{2}{h_i+h_{i-1}}(\frac{U_j(x_{i+1})-U_j(x_i)}{h_i} - \frac{U_j(x_i)-U_j(x_{i-1})}{h_{i-1}})$,
$D^0 U_j(x_i) = \frac{U_j(x_{i+1})-U_j(x_{i-1})}{h_i+h_{i-1}}$, $D^- U_j(x_i) = \frac{U_j(x_i)-U_j(x_{i-1})}{h_{i-1}}$ and

$$\hat{U}_j(x_i) \equiv (U_j(x_i) + U_j(x_{i-1}))/2, a_{j,i-1/2} \equiv a_j((x_{i-1}+x_i)/2), a_{j,i} \equiv a_j(x_i);$$

similarly for $b_{1j,i-1/2}$, $b_{2j,i-1/2}$, $f_{j,i-1/2}, b_{1j,i}$, $b_{2j,i}$ and $f_{j,i}, j = 1, 2$.

Hybrid Difference Scheme - II: In this scheme, we use the cubic spline difference scheme defined by (5) in the fine mesh region and the mid-point difference scheme in the coarse mesh region.

$$L_1^N \bar{U}(x_i) = \begin{cases} -\varepsilon\delta^2 U_1(x_i) + a_{1,i-1/2} D^- U_1(x_i) + b_{11,i-1/2}\hat{U}_1(x_i) + \\ b_{12,i-1/2}\hat{U}_2(x_i) = F_1(x_i), \quad 0 < i \le N/2, \\ r^-_{1,i}U_1(x_{i-1}) + r^c_{1,i}U_1(x_i) + r^+_{1,i}U_1(x_{i+1}) + q^-_{1,i}U_2(x_{i-1}) + \\ q^c_{1,i}U_2(x_i) + q^+_{1,i}U_2(x_{i+1}) = F_1(x_i), \ N/2 < i \le N-1. \end{cases}$$

$$L_2^N \bar{U}(x_i) = \begin{cases} -\varepsilon\delta^2 U_2(x_i) + a_{2,i-1/2} D^- U_2(x_i) + b_{21,i-1/2}\hat{U}_1(x_i) + \\ b_{22,i-1/2}\hat{U}_2(x_i) = F_2(x_i), \quad 0 < i \le N/2, \\ r^-_{2,i}U_2(x_{i-1}) + r^c_{2,i}U_2(x_i) + r^+_{2,i}U_2(x_{i+1}) + q^-_{2,i}U_1(x_{i-1}) + \\ q^c_{2,i}U_1(x_i) + q^+_{2,i}U_1(x_{i+1}) = F_2(x_i), \ N/2 < i \le N-1. \end{cases}$$

3.2. Numerical solution estimates. To guarantee the monotonicity property of the difference operator $\mathbf{L}^N$, we impose the following mild assumption on the minimum number of mesh points [1],

$$\frac{N}{\ln N} \ge 2\max\{\frac{\| a_1 \|}{\alpha}, \frac{\| a_2 \|}{\alpha}\}. \tag{7}$$

Lemma 4. [1, 14]. *Assume that the inequality (7) holds true. The finite difference scheme (5) satisfies a discrete maximum principle of the form: For any mesh function $\bar{Z}(x_i) = (Z_1(x_i), Z_2(x_i))^T$, if $\bar{Z}(x_0) \ge \bar{0}$, $\bar{Z}(x_N) \ge \bar{0}$ and $L_j^N \bar{Z}(x_i) \ge 0$, $j = 1, 2$, for all $i = 1, ..., N-1$, then $\bar{Z}(x_i) \ge \bar{0}$, for all $i = 0, ..., N$.*

In the rest of the problem for discrete case the norm $\| . \|$ means $\| . \|_{\overline{\Omega}^N}$.

Lemma 5. *For any mesh function $\bar{Z}(x_i)$, for $j = 1, 2$*

$$\| Z_j \| \le C\max\{|Z_1(x_0)|, |Z_1(x_N)|, |Z_2(x_0)|, |Z_2(x_N)|, \| L_1^N \bar{Z} \|_{\Omega^N}, \| L_2^N \bar{Z} \|_{\Omega^N}\}.$$

Using the procedure adopted in [1], [11, §4], we can deduce the following error estimate for the Hybrid Difference Scheme - I. For $j = 1, 2$, we have

$$|L_j^N(\bar{U} - \bar{u})(x_i)| \le \begin{cases} C\varepsilon H_1 \| u_j^{(3)} \| + CH_1^2(\| u_j^{(3)} \| + \| u_j^{(2)} \|), \ i = 1, ..., N/2, \\ C\varepsilon H_2^2 \| u_j^{(4)} \| + \| a_1 \| H_2^2 \| u_j^{(3)} \|, \ i = N/2+1, ..., N-1. \end{cases} \tag{8}$$

For the Hybrid Difference Scheme - II, the truncation error for $i = N/2 + 1, ..., N-1$, is given by

$$L_1^N(\bar{U}-\bar{u})(x_i) = r_{1,i}^- U_1(x_{i-1}) + r_{1,i}^c U_1(x_i) + r_{1,i}^+ U_1(x_{i+1}) + q_{1,i}^- U_2(x_{i-1})$$
$$+ q_{1,i}^c U_2(x_i) + q_{1,i}^+ U_2(x_{i+1}) - [F_{1,i}^- f_1(x_{i-1}) + F_{1,i}^c f_1(x_i) + F_{1,i}^+ f_1(x_{i+1})],$$
$$L_2^N(\bar{U}-\bar{u})(x_i) = r_{2,i}^- U_2(x_{i-1}) + r_{2,i}^c U_2(x_i) + r_{2,i}^+ U_2(x_{i+1}) + q_{2,i}^- U_1(x_{i-1})$$
$$+ q_{2,i}^c U_1(x_i) + q_{2,i}^+ U_1(x_{i+1}) - [F_{2,i}^- f_2(x_{i-1}) + F_{2,i}^c f_2(x_i) + F_{2,i}^+ f_2(x_{i+1})].$$

Using (1) for f_j, $j = 1,2$ in $L_j^N(\bar{U}-\bar{u})(x_i)$, $j = 1,2$ and using the Taylor's series expansion, we have

$$L_1^N(\bar{U}-\bar{u})(x_i) = T_{0,i}U_1(x_i) + T_{1,i}U_1'(x_i) + T_{2,i}U_1''(x_i) + T_{3,i}U_1^{(3)}(x_i) + T_{4,i}U_1^{(4)}(x_i)$$
$$+ ... + Q_{0,i}U_2(x_i) + Q_{1,i}U_2'(x_i) + Q_{2,i}U_2''(x_i) + Q_{3,i}U_2^{(3)}(x_i) + T_{4,i}U_2^{(4)}(x_i) + ...,$$

where

$$\begin{aligned}
T_{0,i} &= r_{1,i}^- + r_{1,i}^c + r_{1,i}^+ - [b_{11}(x_{i-1})F_{1,i}^- + b_{11}(x_i)F_{1,i}^c + b_{11}(x_{i+1})F_{1,i}^+],\\
T_{1,i} &= -h_{i-1}r_{1,i}^- + h_i r_{1,i}^+ - [F_{1,i}^- a_1(x_{i-1}) + F_{1,i}^c a_1(x_i) + F_{1,i}^+ a_1(x_{i+1})]\\
&+ h_{i-1}F_{1,i}^- b_{11}(x_{i-1}) - h_i F_{1,i}^+ b_{11}(x_{i+1}),\\
T_{2,i} &= \frac{h_{i-1}^2}{2!}r_{1,i}^- + \frac{h_i^2}{2!}r_{1,i}^+ + \varepsilon[F_{1,i}^- + F_{1,i}^c + F_{1,i}^+] + h_{i-1}a_1(x_{i-1})F_{1,i}^-\\
&- h_i a_1(x_i)F_{1,i}^+ - \frac{h_{i-1}^2}{2!}b_{11}(x_{i-1})F_{1,i}^- - \frac{h_i^2}{2!}b_{11}(x_{i+1})F_{1,i}^+,\\
T_{3,i} &= -\frac{h_{i-1}^3}{3!}r_{1,i}^- + \frac{h_i^3}{3!}r_{1,i}^+ + \varepsilon[h_{i-1}F_{1,i}^- + h_i F_{1,i}^+] - [\frac{h_{i-1}^2}{2!}a_1(x_{i-1})F_{1,i}^-\\
&+ \frac{h_i^2}{2!}a_1(x_{i+1})F_{1,i}^+] + \frac{h_{i-1}^3}{3!}F_{1,i}^- b_{11}(x_{i-1}) - \frac{h_i^3}{3!}F_{1,i}^+ b_{11}(x_{i+1}),\\
T_{4,i} &= \frac{h_{i-1}^4}{4!}r_{1,i}^- + \frac{h_i^4}{4!}r_{1,i}^+ + \varepsilon[\frac{h_{i-1}^2}{2!}F_{1,i}^- + \frac{h_i^2}{2!}F_{1,i}^+] + [\frac{h_{i-1}^3}{3!}F_{1,i}^- a_1(x_{i-1})\\
&- \frac{h_i^3}{3!}F_{1,i}^+ a_1(x_{i+1})] - \frac{h_{i-1}^4}{4!}F_{1,i}^- b_{11}(x_{i-1}) - \frac{h_i^4}{4!}F_{1,i}^+ b_{11}(x_{i+1}),\\
Q_{0,i} &= q_{1,i}^- + q_{1,i}^c + q_{1,i}^+ - [b_{12}(x_{i-1})F_{1,i}^- + b_{12}(x_i)F_{1,i}^c + b_{12}(x_{i+1})F_{1,i}^+],\\
Q_{1,i} &= -h_{i-1}q_{1,i}^- + h_i q_{1,i}^+ + h_i b_{12}(x_{i-1})F_{1,i}^- - h_i b_{12}(x_{i+1})F_{1,i}^+,\\
Q_{2,i} &= \frac{h_{i-1}^2}{2!}q_{1,i}^- + \frac{h_i^2}{2!}q_{1,i}^+ - \frac{h_{i-1}^2}{2!}b_{12}(x_{i-1})F_{1,i}^- + \frac{h_i^2}{2!}b_{12}(x_{i+1})F_{1,i}^+,\\
Q_{3,i} &= -\frac{h_{i-1}^3}{3!}q_{1,i}^- + \frac{h_i^3}{3!}q_{1,i}^+ + \frac{h_{i-1}^3}{3!}b_{12}(x_{i-1})F_{1,i}^- - \frac{h_i^3}{3!}b_{12}(x_{i+1})F_{1,i}^+,\\
Q_{4,i} &= \frac{h_{i-1}^4}{4!}q_{1,i}^- + \frac{h_i^4}{4!}q_{1,i}^+ - \frac{h_{i-1}^4}{4!}b_{12}(x_{i-1})F_{1,i}^- - \frac{h_i^4}{4!}b_{12}(x_{i+1})F_{1,i}^+.
\end{aligned}$$

It can be easily seen that $T_{0,i} = T_{1,i} = T_{2,i} = T_{3,i} = Q_{0,i} = Q_{1,i} = Q_{2,i} = Q_{3,i} =$

$Q_{4,i} = 0$ and $T_{4,i} = -3\varepsilon(\frac{h_i^3 + h_{i-1}^3}{h_i + h_{i-1}})(\frac{1}{4!} - \frac{1}{2!\,6})u_1^{(4)} + O(N^{-3})$. Similarly one can obtain the truncation error for $L_2^N(\bar{U} - \bar{u})(x_i)$.

We can deduce the following estimate for the Hybrid Difference Scheme - II. For $j = 1, 2$, we have

$$|L_j^N(\bar{U} - \bar{u})(x_i)| \leq C\varepsilon H_2^2 \parallel u_j^{(4)} \parallel, \quad i = N/2 + 1, ..., N - 1. \tag{9}$$

3.3. Error analysis. The discrete solution $\bar{U}(x_i)$ can be decomposed into the sum $\bar{U}(x_i) = \bar{V}(x_i) + \bar{W}(x_i)$ where $\bar{V}(x_i)$ and $\bar{W}(x_i)$ are regular and singular components respectively defined as

$$\mathbf{L}^N \bar{V}(x_i) = \bar{f}(x_i), \quad i = 1, ..., N-1, \qquad \bar{V}(0) = \bar{v}(0), \quad \bar{V}(1) = \bar{v}(1)$$
$$\text{and} \quad \mathbf{L}^N \bar{W}(x_i) = \bar{0}, \quad i = 1, ..., N-1, \qquad \bar{W}(0) = \bar{0}, \quad \bar{W}(1) = \bar{w}(1).$$

The error in the numerical solution can be written in the form $(\bar{U} - \bar{u})(x_i) = (\bar{V} - \bar{v})(x_i) + (\bar{W} - \bar{w})(x_i)$.

Lemma 6. *At each mesh point* $x_i \in \overline{\Omega}^N$, *the error of the regular component satisfies the estimate*

$$|(\bar{V} - \bar{v})(x_i)| \leq \begin{pmatrix} CN^{-2}(1 + x_i) \\ CN^{-2}(1 + x_i) \end{pmatrix}.$$

Proof. Using $\varepsilon \leq CN^{-1}$, (8), (9) and the bounds on the derivatives of $\bar{v}$, we have for $j = 1, 2$

$$\begin{aligned} |L_j^N(\bar{V} - \bar{v})(x_i)| &\leq \begin{cases} CN^{-1}(\varepsilon + N^{-1}), & i = 1, ..., N/2 \\ CN^{-2}, & i = N/2 + 1, ..., N - 1 \end{cases} \\ &\leq CN^{-2}, \quad i = 1, ..., N - 1. \end{aligned}$$

Consider the barrier functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} CN^{-2}(1 + x_i) \\ CN^{-2}(1 + x_i) \end{pmatrix} \pm (\bar{V} - \bar{v})(x_i).$$

Then, we have $\bar{\Psi}^{\pm}(x_0) \geq \bar{0}$ and $\bar{\Psi}^{\pm}(x_N) > 0$. For $i = 1, ..., N/2$, we have for $j = 1, 2$

$$L_j^N \bar{\Psi}^{\pm}(x_i) = CN^{-2} a_{j,i-1/2} + CN^{-2}(b_{j1,i-1/2} + b_{j2,i-1/2})(1 + \frac{x_i + x_{i-1}}{2}) \pm CN^{-2} \geq 0.$$

For $i = N/2 + 1, ..., N - 1$; for the Hybrid Difference Scheme - I, we have for $j = 1, 2$

$$L_j^N \bar{\Psi}^{\pm}(x_i) = CN^{-2} a_{j,i} + CN^{-2}(b_{j1,i} + b_{j2,i})(1 + x_i) \pm CN^{-2} > 0$$

and for the Hybrid Difference Scheme - II, we have for $j = 1, 2$

$$\begin{aligned} L_j^N \bar{\Psi}^{\pm}(x_i) &= CN^{-2}(r_{j,i}^- + r_{j,i}^c + r_{j,i}^+ + q_{j,i}^- + q_{j,i}^c + q_{j,i}^+) + CN^{-2}([r_{j,i}^- \\ &+ q_{j,i}^-](x_{i-1}) + [r_{j,i}^c + q_{j,i}^c](x_i) + [r_{j,i}^+ + q_{j,i}^+](x_{i+1})) \pm CN^{-2} > 0. \end{aligned}$$

Applying Theorem 4 to $\bar{\Psi}^{\pm}(x_i)$, $x_i \in \overline{\Omega}^N$, we get the required result. □

Lemma 7. *At each mesh point $x_i \in \overline{\Omega}^N$, the error of the singular component satisfies the estimate*

$$|(\bar{W} - \bar{w})(x_i)| \leq \begin{pmatrix} CN^{-2}(\ln N)^3 \\ CN^{-2}(\ln N)^3 \end{pmatrix}.$$

Proof. Suppose $\sigma = \dfrac{2\varepsilon}{\alpha}\ln N$, the mesh is non-uniform. We split the argument into two cases depending on the localization of the mesh point. In the first case $x_i \in \overline{\Omega}^N \cap (0, 1-\sigma]$, using the arguments in [14, Lemma 6], for $0 < i \leq N/2$ we have

$$|(\bar{W} - \bar{w})(x_i)| \leq \begin{pmatrix} CN^{-2} \\ CN^{-2} \end{pmatrix}.$$

Now for $x_i \in \overline{\Omega}^N \cap (1-\sigma, 1)$, using (8), (9) and the bounds on the derivatives of $\bar{w}$, we have for $j = 1, 2$

$$|L_j^N(\bar{W} - \bar{w})(x_i)| \quad \leq \quad C(H_2^2 + \frac{H_2^2}{\varepsilon^3}\exp(-(1-x_i)\alpha/\varepsilon)).$$

For all i, $N/2 \leq i \leq N$, we introduce the mesh functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} CN^{-2} + C\dfrac{\sigma^2}{\varepsilon^3 N^2}(x_i - (1-\sigma)) \\ CN^{-2} + C\dfrac{\sigma^2}{\varepsilon^3 N^2}(x_i - (1-\sigma)) \end{pmatrix} \pm (\bar{W} - \bar{w})(x_i).$$

It is easy to show that $\bar{\Psi}^{\pm}(x_{N/2}) \geq \bar{0}$ and $\bar{\Psi}^{\pm}(x_N) > \bar{0}$.

For the Hybrid Difference Scheme - I, we have for $j = 1, 2$

$$L_j^N \bar{\Psi}^{\pm}(x_i) = C\frac{\sigma^2}{\varepsilon^3 N^2}a_{j,i} + (CN^{-2} + C\frac{\sigma^2}{\varepsilon^3 N^2})(b_{j1,i} + b_{j2,i})(x_i - (1-\sigma)) \pm C\frac{\sigma^2}{\varepsilon^3 N^2} > 0$$

and for the Hybrid Difference Scheme - II, we have for $j = 1, 2$

$$\begin{aligned} L_j^N \bar{\Psi}^{\pm}(x_i) &= CN^{-2}(r_{j,i}^- + r_{j,i}^c + r_{j,i}^+ + q_{j,i}^- + q_{j,i}^c + q_{j,i}^+) + C\frac{\sigma^2}{\varepsilon^3 N^2}([r_{j,i}^- + q_{j,i}^-](x_{i-1} \\ &-(1-\sigma)) + [r_{j,i}^c + q_{j,i}^c](x_i - (1-\sigma)) + [r_{j,i}^+ + q_{j,i}^+](x_{i+1} - (1-\sigma)) \pm C\frac{\sigma^2}{\varepsilon^3 N^2} > 0. \end{aligned}$$

Then by Theorem 4 we get $\bar{\Psi}^{\pm}(x_i) \geq \bar{0}$. Thus we get the required result. □

Theorem 1. *Let $\bar{u}(x) = (u_1(x), u_2(x))^T$, $x \in \bar{\Omega}$ be the solution of (1) and let $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$, $x_i \in \bar{\Omega}^N$ be the numerical solution of problem (6). Then we have*

$$\sup_{0<\varepsilon\leq 1} \| U_1 - u_1 \|_{\Omega^N} \leq CN^{-2}(\ln N)^3 \textit{ and } \sup_{0<\varepsilon\leq 1} \| U_2 - u_2 \|_{\Omega^N} \leq CN^{-2}(\ln N)^3.$$

Proof. The proof of the theorem follows immediately, if one applies the above Lemmas 6 and 7 to $\bar{U} - \bar{u} = \bar{V} - \bar{v} + \bar{W} - \bar{w}$. □

3.4. Numerical derivative estimates. Here, we approximate the first derivative $\bar{u}'$ of the solution of the problem (1) by the discrete derivative $D^-\bar{U}(x_i)$ in the coarse mesh region and by the scaled central discrete derivative $\varepsilon D^0\bar{U}(x_i)$ in the fine mesh region, where $\bar{U}$ is the numerical solution given by the Hybrid Difference Scheme - I. We note that the errors $e_j(x_i) \equiv U_j(x_i) - u_j(x_i)$ satisfy the equations for $j = 1, 2$

$$[-\varepsilon\delta^2 + a_{j,i}D^-]e_j(x_i) = -[b_{j1,i}e_1(x_i) + b_{j2,i}e_2(x_i)] + \text{truncation error}, 0 < i \leq \frac{N}{2}, \quad (10)$$

$$[-\varepsilon\delta^2 + a_{j,i}D^0]e_j(x_i) = -[b_{j1,i}e_1(x_i) + b_{j2,i}e_2(x_i)] + \text{truncation error}, \frac{N}{2} < i < N, \quad (11)$$

where, by Theorem 1, $[b_{j1}(x_i)e_1(x_i) + b_{j2}(x_i)e_2(x_i)] = O(N^{-2}(\ln N)^3)$. Hence the analysis carried out in [14, §4] can be applied immediately to get the following theorem.

Theorem 2. *Let $\bar{u}(x)$ be the solution of (1) and $\bar{U}(x_i)$ be the numerical solution of (6). Then, for $j = 1, 2$, and $x \in \bar{\Omega}_i = [x_{i-1}, x_i]$, we have*

$$\sup_{0<\varepsilon\leq 1} \| D^-U_j(x_i) - u_j' \|_{\overline{\Omega}_i} \leq CN^{-1}\ln N, \quad 1 \leq i \leq N/2.$$

Theorem 3. *Let $\bar{u}$ be the solution of (1) and $\bar{U}$ be the numerical solution of (6). Then, for $j = 1, 2$, and $x \in \bar{\Omega}_i = [x_{i-1}, x_i]$, we have*

$$\sup_{0<\varepsilon\leq 1} \| \varepsilon(D^0U_j(x_i) - u_j') \|_{\overline{\Omega}_i} \leq CN^{-2}(\ln N)^2, \quad N/2 + 1 \leq i \leq N - 1.$$

Proof. The proof of the theorem follows immediately, if one applies the arguments and method of proof adopted in [15, §2] for (11). □

Remark Let $\widetilde{U}_j$, $j = 1, 2$, denote the piecewise linear interpolant of the finite difference solution $\{U_j(x_i)\}_{i=0}^N$. As done in [2, p.66], we get for $j = 1, 2$ and $x \in \bar{\Omega}_i = [x_{i-1}, x_i]$,

$$\sup_{0<\varepsilon\leq 1} \| \widetilde{D}^-U_j - u_j' \|_{\overline{\Omega}_i} \leq CN^{-1}\ln N, \quad i = 1, ..., N/2,$$

and

$$\sup_{0<\varepsilon\leq 1} \| \varepsilon(\widetilde{D}^0U_j - u_j') \|_{\overline{\Omega}_i} \leq CN^{-2}(\ln N)^2, \quad i = N/2 + 1, ..., N$$

where, $\widetilde{D}^-U_j(x) = D^-U_j(x_i)$, for all $x \in (x_{i-1}, x_i]$, $i = 1, ..., N/2$ and $\widetilde{D}^0U_j(x) = D^0U_j(x_i)$, for all $x \in (x_{i-1}, x_i]$, $i = N/2 + 1, ..., N$.

4. Numerical results

In this section, we consider the following example to illustrate the results obtained in the paper.

$$\begin{aligned} &-\varepsilon u_1''(x) + 7u_1'(x) + (9 + x)u_1(x) - 8u_2(x) = 2 + e^{-x} \\ &-\varepsilon u_2''(x) + 7u_2'(x) - 4u_1(x) + (5 + x)u_2(x) = 1 + e^{-x} \\ &u_1(0) = 1, \quad u_1(1) = 0, \quad u_2(0) = 1, \quad u_2(1) = 0. \end{aligned}$$

Let U_j^N be a numerical approximation for the exact solution u_j on the mesh Ω^N and N is the number of mesh points. The exact solution to the test problem is not available, so for all integers $N, 2N \in R_N = [64, 128, 256, 512]$ and for a finite set of values $\varepsilon \in R_\varepsilon = [2^{-21}, 2^{-11}]$, we compute the maximum pointwise two-mesh difference for $j = 1, 2$,

$$S_{\varepsilon,j}^N = \| U^N - \widetilde{U}^{2N} \|_{\Omega^N}$$

$$D_{\varepsilon,j}^N = \begin{cases} \max |(D^- U_j^N - \widetilde{D}^- U_j^{2N})(x_i)|, & \text{for} \quad 1 \le i \le N/2 \\ \max |\varepsilon(D^0 U_j^N - \widetilde{D}^0 U_j^{2N})(x_i)|, & \text{for} \quad N/2 + 1 \le i \le N - 1, \end{cases}$$

where U^N and $\widetilde{U}^{2N}$ denote respectively, the numerical solutions obtained using N and $2N$ mesh intervals. From these values the $\varepsilon-$uniform maximum pointwise two-mesh difference $S_j^N = \max_{\varepsilon \in R_\varepsilon} D_{\varepsilon,j}^N$, $D_j^N = \max_{\varepsilon \in R_\varepsilon} D_{\varepsilon,j}^N$, $j = 1, 2$ are formed for each available value of N satisfying N, $2N \in R_N$. Approximations to the $\varepsilon-$uniform order of local convergence are defined, for all $N, 4N \in R_N$, by

$$r_j^N = \log_2(\frac{S_j^N}{S_j^{2N}}), \quad p_j^N = \log_2(\frac{D_j^N}{D_j^{2N}}), \quad j = 1, 2.$$

Surface plots of the maximum error for the solution as well as scaled first derivative of the above test problem are presented. In Figures 1 and 2, respectively we observe that as $\varepsilon \to 0$, the maximum error for the numerical approximation U_1, U_2 and $\varepsilon D^0 U_1$, $\varepsilon D^0 U_2$ to the exact solution u_1, u_2 and $\varepsilon u_1'$, $\varepsilon u_2'$ respectively decreases and gets stabilized at a constant value. Tables 2 and 3 present $\varepsilon-$uniform maximum pointwise two-mesh difference and $\varepsilon-$uniform order of local convergence to the scaled derivatives in the fine mesh region and the non scaled derivative in the coarse mesh region.

TABLE 1. *Values of S_1^N, r_1^N and S_2^N, r_2^N for the solution u_1 and u_2 respectively*

ε	Number of mesh points N				
	32	64	128	256	512
Hybrid Difference Scheme - I					
S_1^N	8.3495e-3	3.0032e-3	1.0647e-3	3.3753e-4	1.0428e-4
r_1^N	**1.4752**	**1.4961**	**1.6574**	**1.6946**	-
S_2^N	7.8640e-3	2.8296e-3	1.0036e-3	3.1821e-4	9.8308e-5
r_2^N	**1.4747**	**1.4954**	**1.6571**	**1.6946**	-
Hybrid Difference Scheme - II					
S_1^N	8.3496e-3	3.0033e-3	1.0647e-3	3.3753e-4	1.0428e-4
r_1^N	**1.4752**	**1.4961**	**1.6574**	**1.6946**	-
S_2^N	7.8641e-3	2.8296e-3	1.0036e-3	3.1822e-4	9.8309e-5
r_2^N	**1.4747**	**1.4954**	**1.6571**	**1.6946**	-

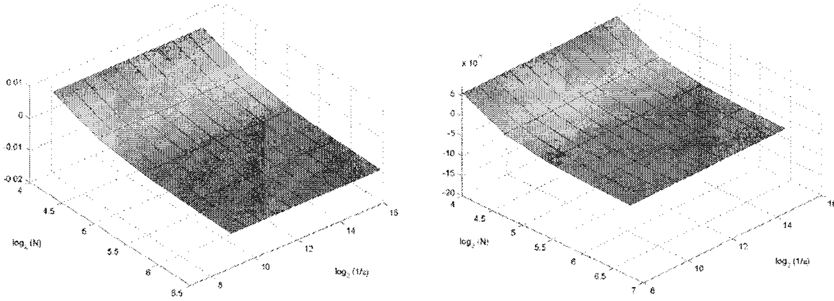

FIGURE 1. Surface plot of the maximum pointwise errors as a function of N and ε for the solution U_1 and U_2.

TABLE 2. *Values of* D_1^N, p_1^N *and* D_2^N, p_2^N *for the scaled derivative components* $\varepsilon u_1'$ *and* $\varepsilon u_2'$ *respectively in the fine mesh region*

ε	Number of mesh points N				
	32	64	128	256	512
Hybrid Difference Scheme - I					
D_1^N	9.0917e-2	4.5900e-2	1.9367e-2	7.2509e-3	2.5043e-3
p_1^N	**9.8606e-1**	**1.2449**	**1.4174**	**1.5338**	-
D_2^N	8.5611e-2	4.3251e-2	1.8254e-2	6.8351e-3	2.3608e-3
p_2^N	**9.8506e-1**	**1.2445**	**1.4172**	**1.5337**	-
Hybrid Difference Scheme - II					
D_1^N	9.0919e-2	4.5901e-2	1.9367e-2	7.2510e-3	2.5043e-3
p_1^N	**9.8606e-1**	**1.2449**	**1.4173**	**1.5338**	-
D_2^N	8.5611e-2	4.3251e-2	1.8254e-2	6.8351e-3	2.3608e-3
p_2^N	**9.8506e-1**	**1.2445**	**1.4172**	**1.5337**	-

5. Conclusion

A weakly coupled system of two singularly perturbed convection-diffusion second order ordinary differential equations subject to Dirichlet boundary conditions was examined. Two hybrid difference schemes on the Shishkin mesh were constructed for solving this problem which generates $\varepsilon-$uniform convergent numerical approximation to the solution as well as to the scaled first derivative of the solution. Numerical results were presented, which are in agreement with the theoretical results.

TABLE 3. *Values of* D_1^N, p_1^N *and* D_2^N, p_2^N *for the derivative components* u_1' *and* u_2' *respectively in the coarse mesh region*

ε	Number of mesh points N				
	32	64	128	256	512
Hybrid Difference Scheme - I					
D_1^N	7.5223e-3	3.7941e-3	1.9052e-3	9.5462e-4	4.7781e-4
p_1^N	**9.8742e-1**	**9.9382e-1**	**9.9694e-1**	**9.9849e-1**	-
D_2^N	3.4903e-3	1.7525e-3	8.7801e-4	4.3945e-4	2.1984e-4
p_2^N	**9.9394e-1**	**9.9711e-1**	**9.9854e-1**	**9.9925e-1**	-
Hybrid Difference Scheme - II					
D_1^N	7.5223e-3	3.7941e-3	1.9052e-3	9.5462e-4	4.7781e-4
p_1^N	**9.8742e-1**	**9.9382e-1**	**9.9694e-1**	**9.9849e-1**	-
D_2^N	3.4903e-3	1.7525e-3	8.7801e-4	4.3945e-4	2.1984e-4
p_2^N	**9.9394e-1**	**9.9711e-1**	**9.9854e-1**	**9.9925e-1**	-

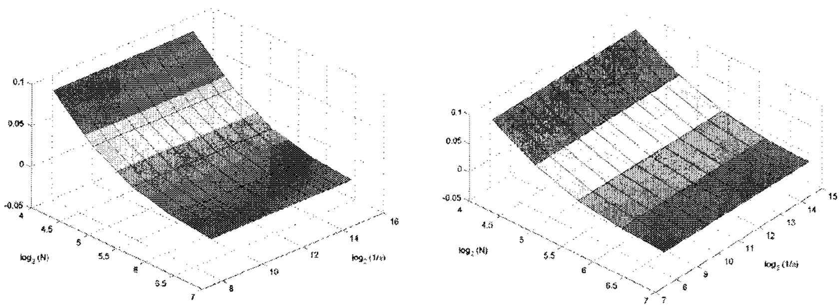

FIGURE 2. Surface plot of the maximum pointwise errors as a function of N and ε for the solution $\varepsilon D^0 U_1$ and $\varepsilon D^0 U_2$.

REFERENCES

1. M. STYNES AND H.G. ROOS, *The mid-point upwind scheme,* Applied Numerical Mathametics, **23**, (1997), 361-374.
2. P. A. FARRELL, A. F. HEGARTY, J. J. H.MILLER, E. O'RIORDAN, G. I. SHISHKIN , *Robust computational techniques for boundary layers* , Chapman and Hall/CRC Press, Boca Raton, U.S.A., (2000).
3. N. KOPTEVA, M. STYNES, *Approximation of derivatives in a convection-diffusion two-point bondary value problem,* Applied Numerical Mathematics, **39**(1), (2001), 47-60.
4. S.MATHEWS, E. O'RIORDAN, G. I. SHISHKIN, *A numerical method for a system of singularly perturbed reaction-diffusion equations,* Journal of Computational and Applied Mathematics, **145**(1), (2002), 151-166.
5. N. MADDEN, M. STYNES, *A uniform convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problem,* IMA, Journal of Numerical Analysis, **23**(4), (2003), 627-644.

6. S. BELLEW, E. O'RIORDAN , *A parameter robust numerical method for a system of two singularly perturbed convection-diffusion equations*, Applied Numerical Mathematics, **51**(2-3), (2004), 171-186.
7. P. A. FARRELL, A. F. HEGARTY, J. J. H.MILLER, E. O'RIORDAN, G. I. SHISHKIN , *Singularly perturbed convection- diffusion problem with boundary and weak interior layers.* Journal of Computational Applied Mathematics, **166**(1), (2004), 133-151.
8. T. VALANARASU, N. RAMANUJAM, *Asymptotic initial value method for a system of singularly perturbed second order ordinary differential equations of convection diffusion type,* International International Journal of Computer Mathematics, **81**(11), (2004), 1381-1393.
9. Z. CEN ,*Parameter-uniform finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations*, International International Journal of Computer Mathematics, **82**(2), (2005), 177-192.
10. TORSTEN LIN ß, *Analysis of an upwind finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations,* Preprint MATH- NM-08-2006, TU Dresden, July (2006).
11. J. L. GRACIA, E. O'RIORDAN, M. L. PICKETT, *A parameter robust highedr order numerical method for a singularly perturbed two-parameter problem*, Applied Numerical Mathematics, **56**, (2006), 962-980.
12. A. TAMILSELVAN, N. RAMANUJAM AND V. SHANTHI, *A numerical methods for singularly perturbed weakly coupled system of two second order ordinary differential equations with discontinuous source term,* Journal of Computational and Applied Mathematics, **202**, (2007), 203-216.
13. S. NATESAN AND B. SUNDAR DEB, *A robust computational method for singularly perturbed coupled system of reaction-diffusion boundary-value problems,* Journal of Applied Mathematics and Computation, **188**, (2007), 353-364.
14. R. MYTHILI PRIYADHARSHINI, N. RAMANUJAM AND V. SHANTHI, *Approximation of derivative in a system of singularly perturbed convection-diffusion equations,* To appear in Journal of Applied Mathematics & Computing, (2007).
15. R. MYTHILI PRIYADHARSHINI AND N. RAMANUJAM, *Approximation of Derivative to a singularly perturbed second -order ordinary differential equation with discontinuous convection coefficient using hybrid difference scheme,* To appear in International Journal of Computer Mathematics, (2007).

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