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An Operational Analysis for Solving Linear Equation Problems

Shin, Jae Hong* · Lee, Joong Kweon**

In this study, an operational analysis in the context of linear equations is presented. For the analysis, several second-order models concerning students' whole number knowledge and fraction knowledge based on teaching experiment methodology were employed, in addition to our first-order analysis. This ontogenetic analysis begins with students' Explicitly Nested number Sequence (ENS) and proceeds on through various forms of linear equations. This study shows that even in the same representational forms of linear equations, the mathematical knowledge necessary for solving those equations might be different based on the type of coefficients and constants the equation consists of. Therefore, the pedagogical implications are that teachers should be able to differentiate between different types of linear equation problems and propose them appropriately to students by matching the required mathematical knowledge to the students' potential constructs.

Ⅰ. Introduction

There have been tremendous efforts in mathematics education research for investigating the process of solving linear equations and students' corresponding mathematical knowledge (cf. Linchevski & Herscovics, 1996; Usiskin, 1988; Van Amerom, 2003; Wagner & Parker, 1993). Most research has been focused on students' manipulating literal symbols and understanding the concept of a variable along with their operations on those symbols. However, although understanding literal symbols as a variable plays a critical role in students' solving linear equation problems, Wagner and Parker (1993) reported that rash introduction of literal symbols in the context of linear equations forced students to suffer unexpected difficulties (e.g. assuming letters as abbreviations or as representing a specific value). Recently, several studies have attempted to explore how students learn to operate explicitly on unknown quantities based in scheme theoretic approaches. Hackenberg (2005) investigated how students reverse their quantitative reasoning with fractions, aim of which was to understand how students construct schemes and operations that underlie the construction and solution of basic linear equations of the form $ax=b$. The key finding of her study

* University of Georgia, jhshin@uga.edu
** Dongguk University, joonglee@dongguk.edu

was that the interiorization of three levels of units was critical for the construction of schemes to solve a linear equation problem like $3/5x=7$. Tunç-Pekkan (2008) conducted a teaching experiment specifically to investigate students' construction of reciprocal reasoning in stating and solving equations of the form $ax=b$ where a and b are both fractional numbers. The findings of her study indicated that the construction of *measurement units* were critical for producing and operating with equivalency between both sides of a linear equation problem and the construction of measurement units involves the coordination of sequences of different composite units. In addition, Tunç-Pekkan found that the students' construction of a symbolic fraction multiplication scheme was critical for students' construction of reciprocal reasoning. However, the research based in scheme theoretic approaches is still in its early stage and has yet to provide more comprehensive explanations on how students solve various kinds of linear equation problems.

The present study was initiated by the necessity of suggesting an encompassing, theoretical basis fundamental to the scheme theoretic approaches in solving linear equations. We claim that operational analysis of linear equations comes to be meaningful in instruction when distinct types of linear equations are identified in terms of the required mathematical knowledge for each of the distinguished types of problems. In other words, even in the same representational forms of linear equations, say, $ax=b$, the mathematical knowledge necessary for solving those equations might be different based on the type of coefficients and constants the equation consists of. Thus, although all linear equations can be constructed through modifications of the two forms as $x+a=b$ and $ax=b$, our analysis of linear equations would be elaborated under seven categories of linear equations, each of which seems to require students to perform a distinct level of conceptual operations. Also note that Hackenberg (2005) did not identify solving a word problem involving a linear equation structure like "Twenty inches string is five times as long as yours. How long is your string?" with solving it through explicit manipulation of the symbolic expression, $5x=20$ because the latter requires students to compress their reasoning with quantities into a symbolic way of operating where the goal is to determine an unknown in a quantitative situation involving a reversible relationship. That is to say, such transition from the former to the latter likely requires the interiorization of notation so that algebraic symbols can be used in ways that are independent of the sequence of operations used to generate the symbols (Hackenberg, 2005). Thus, as in Hackenberg's study, the schemes and operations that sustain writing and solving a linear equation problem are the center of attention in this study, rather than the students' skill in manipulating numeral and literal symbols.

II. Theoretical Background for Analysis

To fully appreciate the present study, there are needs to understand some scholarly issues around this operational analysis. Thus, this section consists of three parts for the purpose. The first part addresses the notion of first-order knowledge/models and second-order knowledge/ models which identify the nature of this study. The second part explicates students' whole number knowledge, which would be the foundation of students' fraction knowledge. Lastly, the third part provides the research results concerning how students, especially with an explicitly nested number sequence (ENS) or a generalized number sequence (GNS), constructed their fraction knowledge through modification of their abstract whole number sequences.

First-order Models and Second-order Models

First-order mathematical knowledge is an individual's own mathematical knowledge. That is, an individual constructs his or her first-order models to organize, comprehend, and control his or her experience. On the other hand, second-order mathematical knowledge are the models which observers may construct of the observed person's knowledge in order to explain their observation (Steffe, 2009e). Distinguishing between first-order and second-order models is crucial because we attribute mathematical realities to students that are independent of our own mathematical realities (Steffe & Thompson, 2000). We, as researchers and teachers, have no way to directly access students' mathematical knowledge and construct first-order models of students. What we can best do is to build our scheme of the students' mathematical knowledge, that is, second-order models of students (Steffe, 2009e).

Our hypothetical analysis can be regarded as a combination of the first-order analysis based on the researchers' individual mathematical knowledge and the second-order analysis based on the models constructed for students' mathematical developments through observation of their mathematical behavior. Through one year of teaching experiment with a pair of seventh grade students, it became clear to us that it was inappropriate to attribute our mathematical concepts to students because it often conflated the mathematical knowledge of researchers' and students' (Olive & Steffe, 2002). Olive and Steffe also argue that mathematics for students cannot be specified a priori and must be experientially abstracted from the observed modifications students make in their schemes. Thus, it should be acknowledged that our ontogenetic analysis from particular researchers' point of view without considering social interactions with students would be much exposed to the risk of such conflation. However, we claim that such kind of theoretical analysis will help, to a certain degree, establishing a possible orienting point that we want students to reach. Olive and Steffe (2002) also assert that our first-order models of mathematics do play fundamental roles in formulating the second-order models, called mathematics of students as well as in orienting us as we formulate mathematics for students and how to interact with them.

Students' Whole Number Knowledge

Steffe, Cobb, and von Glasersfeld (1988) conducted a teaching experiment with young students for their development of whole number knowledge. They identified three successive number sequences from students' construction of mathematical schemes and operations: the initial number sequence (INS), the tacitly nested number sequence (TNS), and the explicitly nested number sequence. Each new number sequence is the result of a reinteriorization of the previous number sequence and generates more abstract units with which the student can operate (Olive, 1999). That is to say, a gradual decrease in students' dependence on their immediate experimental world can characterize the learning stages of number sequences and it is the operations that students can perform using their number sequences that distinguish among distinct stages of the number sequences (Steffe, 2009c). Later, the notion of the generalized number sequence ensued while seeing how students who had constructed the ENS might use that number sequence to construct schemes to solve multiplying and dividing situations (Steffe, 2009b). For this paper, the explication of the ENS and the GNS is the main focus because students are expected to have constructed the ENS or the GNS for solving the seven prototypes of linear equations presented in this study.

A crucial step for the construction of an ENS is the establishment of an abstract unit item "one" as an iterable unit (Olive, 1999). The iterable one can be produced through repeatedly applying the "one more item" operation when double counting. After the construction of an iterable unit item, a student can engage in part-whole reasoning. When the unit of one is iterable, a number word refers to a composite unit containing a unit which can be iterated the number of times indicated by the number word. This iterability of one "opens possibility for a child to collapse a composite unit into a unit structure containing a singleton unit which can be iterated so many times" (Steffe, 2009c, p. 24). Such characteristic of the ENS enables students to establish multiplicative schemes that involve two levels of units. Further, they can generate a numerical composite of composite unit items as a result of those operations, but they have yet to interiorize or symbolize them so that the numerical composite of composite unit items can be used as given input for further operations (Olive, 1999).

The reinteriorization of the ENS results in iterable composite units. When students have constructed composite units as iterable, they can be regarded as at least in the process of reorganizing their ENS into the GNS (Steffe, 1992). In other words, the GNS is a generalization of the operations on units of the ENS to composite units. "Speaking metaphorically, children are in a 'composite units' world rather than a 'units of one' world" (Steffe, 2009c, p.24). In a GNS, a composite unit is iterable, that is, any composite unit can be taken as the basic unit of the sequence. For a composite unit to be judged as iterable, a student should be able to represent and combine iterations of the composite unit prior to activity. Students' interiorization process from their ENS to the GNS arises when they operate with composite units to solve complex multiplicative problems that require recursive applications of their units-coordinating operations to the results of those operations (Olive, 1999). Therefore, students who have constructed a GNS can take units of units of units rather than simply units of units as given.

Construction of Diverse Fraction Schemes

Although a partitioning operation, mentally projecting a concept of a whole number into an unmarked line segment, is fundamental to students' development of fraction knowledge, the *Fractions Project* (Steffe & Olive, 1990) showed there needs to be several distinctions among students' partitioning operations for detailed descriptions of their constructive itinerary of fraction schemes. First of all, a student's fragmenting of a continuous unit can not be judged as an *equi-partitioning* until "the operating child intends to fragment the continuous unit into equal sized parts and can use any one of these equal sized parts in iteration to produce a connected but segmented unit of the same size as the original unit" (Steffe, 2009a, p. 30). The modification of the equi-partitioning scheme entails a *partitive fractional scheme*, which is regarded as the first genuine fractional scheme. With the partitive fractional scheme, a student can disembed any subcollection of elements from the original partitioned whole without destroying it and constitute a composite unit in its own right by uniting them together. This establishes the classical numerical part-to-whole operation that serves as a fundamental operation in the construction of fractional schemes (Steffe, 2009a). However, the limited understanding of fractions as parts of a specific partitioned whole constrains students' construction of invariant, multiplicative relation between the sizes of the unit fraction and the referent whole. In other words, for the construction of "thirteen-twelfths" as a fractional quantity students should transcend the part-whole meaning of fractions, which requires the construction of *splitting operation*. The splitting operation is qualitatively different than the operations carried on in the equi-partitioning scheme. The splitting operation is a simultaneous composition of partitioning and iterating whereas in the equi-partitioning scheme the two operations are performed sequentially. With the splitting operation, a unit fraction, say, one-twelfth becomes a fractional number freed from its containing whole and available for use in the construction of thirteen-twelfths. The partitive fraction scheme, upon the emergence of the splitting operation, is considered as an *iterative fraction scheme* that can be used to produce improper fractions (Steffe, 2009d).

A *unit fraction composition scheme* emerges when recursive partitioning is embedded in a reversible part-whole fraction scheme. The emergence of the scheme is on the basis of a student's construction of a unit of unit of unit as an assimilating structure whose units can be used as material in further operation. That is to say, attributing to a student a unit fraction composition scheme is to see whether partitioning, say, one-fourth of a stick into three equal parts symbolizes partitioning each one of the four-fourths into three parts (Steffe, 2009f). The construction of a *unit commensurate fraction scheme* can be confirmed if a student could make unit fractional parts of a composite unit in the form of a connected number and transform these unit fractional parts into commensurate fractions (Olive & Steffe, 2009). Olive and Steffe argue that taking three levels of units as given is necessary for the construction of commensurate fractions. For instance, establishing four as an iterable unit in the context of the connected number, say, twenty-four makes possible for a student to be aware that a 4/12-stick can be iterated three times to make a 12/12-stick before repeating it. Then the student can transform the fraction, four-twelfths into one-third and set the commensurate relation between the two fractions, which undergirds the students' construction of *fraction adding scheme.*

Ⅲ. Teaching Experiment Methodology

All excerpted data in this paper, including our data, were grounded in a teaching experiment methodology (Steffe & Thompson, 2000), most of which were collected through interactions with one or two students ranging from third to seventh grade in a computer microworld, JavaBars (Olive, 2007). Therefore, brief descriptions of a teaching experiment methodology and of the program, JavaBars clarify our analysis. The teaching experiment is a methodology for conducting scientific research on mathematics learning whose primary purpose is for researchers to experience students' mathematical learning and reasoning. In other words, we assume that there would be no basis for understanding the mathematical concepts and operations students construct without the experiences afforded by teaching (Steffe & Thompson, 2000). Researchers who do not engage in the teaching of students run the risk that their models will be distorted to reflect their own mathematical knowledge (Cobb & Steffe, 1983). The teaching experiment methodology is deeply rooted in radical constructivism in the sense that researchers conducting teaching experiments attribute mathematical realities to students that are independent of their own mathematical realities and, therefore, a primary goal of the teacher in a teaching experiment is to establish living models of students'mathematics. Steffe and Thompson (2000) argue that the goal of establishing living models is sensible only when the idea of teaching is predicated on an understanding of human beings as self-organizing and self-regulating. That is, mathematics should be regarded as a product of the function of human intelligence (Piaget, 1980, as cited in Steffe and Thompson, 2000) rather than as a product of impersonal, universal, and ahistorical reason.

The JavaBars was specially designed for teaching experiments. Most of whole number and fraction problems in the cited teaching experiments were posed on JavaBars screen and thus most of research findings were the results of retrospective analysis in students' mathematical activities conducted on JavaBars. JavaBars provides students with possibilities for enacting their mathematical operations with whole numbers and fractions. It also provides the teacher/researcher with opportunities to provoke perturbations in students' mathematical schemes and observe students' mathematical thinking in action (Olive & Lobato, 2008). The software consists of on-screen manipulatives like rectangular regions on which students can perform engaging in the fundamental operations in the development of the previously outlined fraction schemes. For example, using JavaBars, a student can make a bar and partition the bar into 4 equal parts, disembed one of the parts by pulling it out of the bar, and then use the REPEAT action to iterate this one part to make a bar that is 5/4 of the original whole bar (cf. Figure 1).

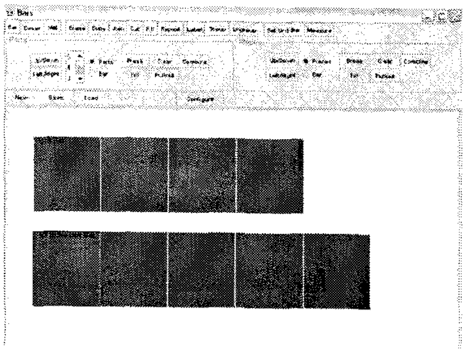

Figure 1. A computer microworld, JavaBars

Ⅳ. Operational Analysis for Linear Equations

The Form of x+a=b

In the case that a and b are whole numbers, an ENS student is expected to be able to solve this sort of linear equation problem. Numerical part-to-whole reasoning is the identifying characteristic of the ENS (Steffe, 1992, p. 290). Steffe (1992) showed, in his teaching experiment with an ENS student, Johanna, that she was able to solve this type of problems. When Steffe asked her to take twelve blocks, told her that together they had nineteen, and asked her how many he had, Johanna said "seven" after sitting silently for about 20 seconds. Her explanation indicated that she first decomposed nineteen into two parts, ten and nine. She then transformed the parts into twelve and seven by adding two to ten and compensated by subtracting two from nine. Steffe argued that this example indicated that Johanna could disembed numerical parts from a numerical whole, use these parts as material for further operating, and reconstitute that result as the original numerical whole because of her ENS. Also, in order to construct a reversible adding scheme, that is, to realize the subtraction b-a as a necessary operations for specifying the unknown, the ENS should be assumed. That is, such reversibility in solving linear equations of the form of x+a=b requires more than the TNS because Steffe (1992) asserts that "constructing the tacitly nested number sequence is not sufficient for establishing addition as the inversion of subtraction. A third level of interiorization of the number sequence, which yields the ENS, is necessary to establish this inversion" (p. 305). However, if a or b is a fractional quantity or number, that is, in the case of the solution being fractional, an ENS is not sufficient. Students must construct schemes that enable them to add and subtract fractional quantities. Throughout the following analysis for linear equation prototypes 1 and 2, which both involve at least one fractional quantity, mathematical constructs necessary for dealing with fractional quantities beyond the ENS are indicated.

Prototype 1: *There is a cup containing 7/5 ounces of juice. If I poured my juice into the cup, the sum of juice in the cup and my juice would equal to 3 ounces. How much juice did I pour into the cup?*

Once the student constructs a goal to find the unknown quantity in the problem situation, she is expected to transform the situation into the subtraction problem of 7/5 ounces from 3 ounces for getting the unknown amount of juice. Then, for the subtraction, the student must have constructed a *commensurate fractional scheme* in order to transform 3 into 15/5. The commensurate fractional scheme seems to call for students' construction of a three-levels-of-units structure, which is the characteristic of a GNS. Steffe (2003) claimed that, in his teaching experiments with a fifth-grade student, Jason, *recursive partitioning* was the basic operation underpinning the act of creating a new partition that could be used to make a fraction commensurate with one third and recursive partitioning is the inverse operation of first producing a composite unit, making multiple copies of this composite unit, and then uniting the copies into a unit of units of units. Thus, producing a recursive partitioning implies that a student can engage in the operations that produce a unit of units of units, but in the reverse direction (Steffe, 2003). Nevertheless, the fact that a student constructed commensurate fractional scheme might not be enough to explain her transforming of 3 into 15/5. In addition to the scheme, the student should have a sense of logical necessity to trigger her constructed commensurate fractional scheme when assimilating this problem situation. That is, to choose a specific commensurate fraction to 3, '15/5' in this case, should be performed on her sense of logical necessity under the goal of making possible to subtract 7/5 from 3.

Another important feature of this problem, which should be taken into consideration all through this paper, is when it requires for a student to deal with a known or an unknown improper fractional quantity. A student who does not consider 9/7 as both a mixed number and as a number in its own right has not constructed improper fractions (Hackenberg, 2007b). Improper fractions make little sense to students who have only constructed a partitive fractional scheme. Being able to generate any fraction by a whole number iteration of a part of it (e.g. to make 7/3 inches from 1/3 inch iterated seven times), and still maintain its relationship to the whole are necessary for the meaning of improper fractions (Hackenberg, 2005). Steffe (2002) emphasize the splitting operation for the construction of improper fractions and Olive and Steffe (2002) view that coordinating three levels of units is involved in the construction of improper fractions. Therefore, it may well be argued that reasoning with a three-levels-of-units structure is essential for students to solve whichever form of linear equations involving improper fractions.

Prototype 2: *If I joined my string to a 2/3 inch-long string, the total length would be 3/4 of an inch. How long is my original string?*

It seems clear that a student who can solve this problem must have constructed *fraction adding scheme.* Even though this is a subtraction problem of 2/3 from 3/4 where reversible reasoning is required, from our point of view, there would not be much difficulty in finding the solution if a student has a fractional adding scheme. However, students' construction of a fractional adding scheme is not a simple process because bringing forth a fraction composition scheme and a commensurate fraction scheme is seemingly crucial in the construction of a fraction adding scheme (Steffe, 2003). Fraction composition scheme is the integration of an iterative fractional scheme with reversible partitioning operations and distributive strategies (Olive, 1999). Olive showed in his teaching experiment that a GNS was necessary, and the distributive strategies associated with a GNS and the ability of students to reverse their partitioning operations were key contributors to the students' construction of the fractional composition scheme. Another aspect to be noticed for the construction of the fraction adding scheme is to bring forth *common partitioning operations*, whose purpose is to partition a bar only once so that both unit fractions can be pulled from the bar. The common partitioning operations enable students to produce a partition that would transform the fractions that are involved in a sum into commensurate fractions that are multiples of the same unit fraction (Steffe, 2003). Therefore, in order to subtract 2/3 from 3/4, a student should be able to divide a whole into twelve for finding of a common measurement unit, 1/12 of 2/3 and 3/4. She might utilize her number sequence scheme of whole numbers to coordinate and compare her number sequence for 3s and 4s until common partitioning is accomplished. After transforming 3/4 into 9/12 and 2/3 into 8/12 through common partitioning operations, she will finally get the solution 1/12 by subtracting 8/12 from 9/12.

The Form of $ax=b$

Hackenberg (2005) clarifies that the equation ax=b is essentially a statement of division and considering its construction and solution requires understanding how a student produces division, which entails understanding the student's multiplying scheme and multiplicative reasoning. Further, she argues that any statement of division inherently involves reasoning with fractions although fractions may appear implicitly or be disguised. Therefore, both multiplicative and fractional reasoning are foundational in studying how students construct and solve linear equations like $ax=b$.

Prototype 3: *Twenty-eight liters of water is four times as much as the amount that you have in your bucket. How much do you have?*

When a and b are whole numbers and the whole number relationship with the unknown does divide the known whole number quantity, ENS students might have difficulty to solve it. To solve this problem, a student's concept of 28 should be able to symbolize the structure of composite units. That is, 28 can be viewed as a unit of four units of an unknown quantity. Steffe (1992) claims that "using a unit to re-process the elements of a unit of units of units is essential in establishing the composite unit as being iterable" (p. 296). Therefore, we are suspicious that a student who is yet to construct a three-levels-of-units structure could solve this problem. ENS students can solve a problem like sharing 28 items among four people by estimating an unknown quantity, where they try it to see if it works, and so on until they find one that does work. However, in order for an ENS student to solve the above problem, she at least needs to construct a *composite units partitioning scheme*, which characterizes a GNS (Steffe, 1992). In other words, the student should go through the process of establishing the operations to feed the results to the situation of the scheme, that is, the completeness of inversion between a unit-segmenting operation and a units-coordinating operation. This is the precise process that interiorizes the involved sequence of composite units (Steffe, 1992). However, taking the results of counting composite units as belonging to the items that one intends to count is what an ENS student has yet to construct.

Prototype 4: *If one liter of juice is five times as much as mine. How much liter of juice do I have now?*

This problem seems to necessitate for a student to be able to bring forth the splitting operation. Splitting operation is the composition of partition and iterating (Steffe, 2002) and it would need to happen simultaneously rather than sequentially. This multiplicative reasoning with fractions, and reversing that reasoning are fundamental in constructing schemes and operations that sustain solving basic linear equations of the $ax=b$ (Hackenberg, 2005). Although a partitive fractional scheme permits students to establish meaning for unit fractional number words like "one tenth" as one out of ten little pieces that could be iterated ten times to produce a partitioned segment of length equal to the original (Steffe, 2003), the student who have constructed the partitive fractional scheme is not expected to flexibly deal with this problem situation. Rather, a student should go further from the status and construct an iterative fractional scheme upon the emergence of the splitting operation. For example, for students with iterative fractional schemes, prior to activity, one-eighth implies a whole consisting of one-eighth iterated eight times. In contrast, even though students who have constructed partitive fractional schemes can iterate one-fifth three times to make three-fifths, one-fifth is not yet an iterable unit for them because one-fifth does not have this a priori multiplicative relationship to the whole (Hackenberg, 2007b). Therefore, for this problem, a student should be aware that 'my water' implies one-fifth of a whole consisting of one-fifth iterated five times prior to activity.

Prototype 5: *If 1/3 meter of string is five times as long as my string. How much string do I have now?*

This problem can be distinct from the prototype 4 in that, a student must have constructed a unit fractional composition scheme, which is based on recursive partitioning.

> The goal of fractional composition scheme is to find how much a fraction is of a fractional whole, and the situation is the result of taking a fractional part out of a fractional part of the whole, hence the name composition. The activity of the scheme is the reverse of the operations that produced the fraction of a fraction, with the important addition of the subscheme, recursive partitioning. The result of the scheme is the fractional part of the whole constituted by the fraction of a fraction (Steffe, 2004, p. 140).

Therefore, the unit fractional composition requires for a student to have constructed a three-levels-of-units structure with *connected numbers*, "a number sequence whose countable items are the elements of a connected but segmented continuous unit" (Steffe, 2009a, p. 12). Prior to activity, a student should be able to disembed 1/3 of a bar, if the activity is conducted with JavaBars, from a hypothetical whole bar and take into account the result of taking 1/5 of the disembedded 1/3 as 1/15 of a hypothetical whole bar.

Prototype 6: *Peppermint Stick Problem: A 7-inch peppermint stick is three times longer than another stick. How long is the other stick? (Hackenberg, 2005, p. 62)*

Although the known fractional quantity and the fractional relationship with the unknown are whole numbers, being able to construct and use fractional quantities and unit fractional relationship is indispensable because the whole number relationship with the unknown does not divide the known whole number quantity. This problem can be viewed as similar to the prototype 5 in that it requires students' reversible fractional scheme based on a three-levels-of-units structure, but be distinguished in that reasoning with distributive activity explicitly emerges. If a student with distributive reasoning forms a goal of a distributive partitioning scheme, say, sharing four identical candy bars equally among five people, the student can partition each candy bar into five parts, distribute one part from each of the four candy bars to each of the five people with understanding that the share of one person can be replicated five times to produce the whole of the four candy bars. The student also knows that 4/5 of one candy bar is identical to 1/5 of all of the candy bars. Therefore, with the peppermint stick problem, a student with the distributive partitioning scheme sets a goal to take 1/3 of the 7-inch peppermint stick. Then the student might be able to partition each one-inch stick into three parts, take one part from each of the seven one-inch peppermint sticks, and unite them for the amount of another stick. Further, the student is explicitly aware that the result of his or her activity amounts to 7/3 of an inch, which is the answer of the problem.

Prototype 7: *In the science class, two teams made racecars. The Lizards' racecar travels 2/3 of a meter. That's 3/4 of how far the Cobras' car travels. Can you make the distance the Cobras' car traveled and tell how far it went? (Hackenberg, 2007a, p. 15)*

Hackenberg (2005) asserts that this type of problems would be complex because they involve a known fractional quantity and a fractional relationship, where the numerator of the fractional relationship does not divide the numerator of the fractional quantity. When Hackenberg (2007a) presented this problem to Deborah, a sixth grade student in her teaching experiment, she solved it by partitioning each of the 2/3 of a meter into six equal parts, pulling out one of those small parts, and making a 16-part bar (see Figure 2).

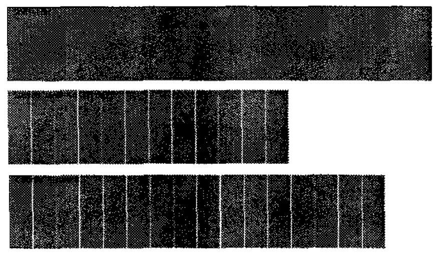

Figure 2. Deborah's solution of the Race Car Problem (from Hackenberg, 2007a)

Hackenberg claimed that for Deborah, 2/3 of a meter from her hypothetical one meter was a unit of two units each containing six units, and simultaneously a unit of three units each containing four units for finding 1/3 of 2/3 of a meter. Another noticeable event with Deborah was that she seemed to use reciprocal relationships in her solutions of other similar problems. Hackenberg explained that Deborah's use of reciprocal relationships might be attributed to her abstraction of a fraction as a mathematical concept, which is a program of operations abstracted from the experience of using particular schemes that includes an awareness of how the schemes are composed and an ability to operate with this awareness. She also claimed that the central difference between two students, who could reason with distribution and the other two students, who could not, was whether they hold a three-levels-of-units structure in mind and operate further with it (Hackenberg, 2007a).

However, we are suspicious of Hackenberg's view that solving this type of problems stands in for students' complex construction, the *rational numbers of arithmetic* (RNA) - being able to relate any two fractions multiplicatively (Steffe, 2002). Olive (1999) claims that a fraction scheme would need to include fractions as measurement units to be regarded as a scheme for generating the RNA. However, through the ontogenetic analysis based on our first-order models and on others' work of students' second-order models in the context of linear equations of the form $x+a=b$ and $ax=b$, it is doubtful that *fraction segmenting operations* need to be constructed for the solutions. That is, in order for students to solve the linear equations of the form $ax=b$, they are likely to conduct, so called, partitive division rather than quotitive division in mathematician's terms. Thus, it would be argued that the RNA be enough for students' solving linear equations, but not all schemes and operations of RNA might be necessary for dealing with linear equations.

Problem situation of 2/3x=1and 2/3x=y

There was an interesting sequence of events in our teaching experiment with Damon, a seventh grade student. Although he was regarded as a three-levels-of-units student at the beginning of the semester, he could not solve a relatively simple linear equation problem (You have your bar, with JavaBars, on the screen. If 2/3 of my bar is your bar, can you make my bar?) until the end of the teaching experiment. He kept trying to divide his bar into three rather two whenever the teacher posed this question, whereas his partner, Carol, who also had a three-levels-of-units structure, easily solved the problem. His disconcertion was out of our expectation. On the final day of the teaching experiment, when the teacher posed the same question, he thought that he was still asking a linear equation problem of 2/3x=1. However, Damon's response in the last episode revealed that he assimilated the problem situation as 2/3x=y rather than 2/3x=1.

('T' stands for teacher and 'D' stands for Damon.)

T: Make a bar. (Damon makes a square-shaped bar on the screen.) That's your bar. So 2/3 of my bar is your bar, can you make my bar?

D: So··· (divides his bar into three pieces) This is 2/3 of your bar···

T: Can you explain my question? This is···

D: That is basically··· My bar is 2/3 of yours. So, I have to find out how much of your bar.

T: Right.

D: So··· (thirty seconds after) hold up. Mine is 2/3 of yours (clears all marks on his bar) but how much is my bar? Is mine is like one?

T: How much is my bar?

D: How can I find yours when I don't know what mine equals. So mine is just like equals one bar?

T: Yes, yes, you can do.

D: Okay, that will be··· If I have one bar, then divided by two, a half (divides his bar into two pieces and pulls out one out of two pieces three times.) That's your bar (pointing out three pieces of one half bar).

Once he realized that he could identify his bar as one, he easily constructed 3/2 of his bar by dividing his bar into two pieces. Since there was no more chance to investigate with Damon, it is hard to conjecture what constraint did interfere with his reversible reasoning. However, his predisposition to numeric calculation in dealing with his constructed mathematical situation shown through all teaching episodes might prevent him from constructing his algebraic reasoning based on his quantitative reasoning. Another possible explanation is that he constructed his problem situation as involving two unknown quantities, rather than one known quantity. In other words, after Damon was allowed to consider his bar as one (a known quantity), he immediately made the other bar (an unknown bar) for his solution. When his bar was unknown, that is, when he was to operate on an unknown quantity for constructing another unknown quantity, he appeared at a loss for a while. This interpretation does make sense because the first step for all linear equation problems presented was the operation on or with an already known quantity. Assimilating two unknowns in the problem situation might prohibit Damon from assigning a specific number to the outcome although the relationship between two quantities was being indicated. Therefore, Damon's first construction of the problem situation can be regarded as the structure of $2/3x=y$, not as that of $2/3x=1$. The situation of involving two unknowns seems to require more advanced algebraic thinking in that a particular number to operate on disappears and the relationship between two unknowns remains by itself.

V. Final Comments

By means of employing students' whole number and fraction schemes and operations, the ontogenetic analysis for solving linear equations of the form $x+a=b$ and $ax=b$ was offered. In regard to the form of $x+a=b$, two prototypes were presented both of which requires students to have constructed a commensurate fraction scheme. We also maintain that more advanced mathematical knowledge, such as fraction composition and fraction addition might be necessary for students to solve the same form of linear equations as fractional constants appear in the problems. In case of linear equation problems in the form of ax=b, five different prototypes were suggested in order to distinguish required students' mathematical knowledge for each of the problem types. The analysis indicates that a distinct level of whole number and fraction knowledge seems to be necessary at each five different prototypes of problems depending on the diverse relationship with unknown quantity and the amount of known quantity. Especially, it demonstrates that the highest level of mathematical knowledge might be demanded when the problem involves a known fractional quantity and a fractional relationship, where the numerator of the fractional relationship does not divide the numerator of the fractional quantity. Additionally, the last excerpt shows that students can assimilate a linear equation problem as involving two unknowns or as one unknown and one known based on the way in which the problem was posed irrespective of the teacher's intention. The proposed seven prototypes of linear equation problems based on our theoretical hypotheses will shed light on the research on learning and teaching linear equations in school. The proposed hypotheses, however, are (and should be) subject to revisions until the researchers' model is not countermanded by further observations. It also should be acknowledged that the range of our analysis is restricted to the cases that unknown quantities in linear equations are positive integers or positive fractions. Dealing with other numbers such as negative numbers and irrational numbers is beyond the scope of this paper.

As an implication in mathematics instruction, the results of this analysis call for teacher's attentive awareness of students' prior knowledge, specifically related to whole numbers and fractions, for teaching linear equations. That is, teachers must choose appropriate linear equation tasks that fit to their students' present mathematical ability even though the problems seem to be identical in the representational forms. Further, this study suggests that focusing on manipulation of literal symbols to get the answer of linear equations regardless of various types of linear equations might impede their conceptual understanding of variables as unknowns. Although the attempt to express them in symbols is part of building conceptual operations, symbolic operations can become the focus of instruction once students have developed coherent and stable meaning that they may express symbolically (Thompson & Saldanha, 2003).

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