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POSITIVE LINEAR OPERATORS IN C^*-ALGEBRAS

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Abstract. It is shown that every almost positive linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a Banach $*$-algebra $\mathcal{A}$ to a Banach $*$-algebra $\mathcal{B}$ is a positive linear operator when $h(rx) = rh(x)\,(r > 1)$ holds for all $x \in \mathcal{A}$, and that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a unital C^*-algebra $\mathcal{A}$ to a unital C^*-algebra $\mathcal{B}$ is a positive linear operator when $h(2^n u^* y) = h(2^n u)^* h(y)$ holds for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, by using the Hyers-Ulam-Rassias stability of functional equations.

Under a more weak condition than the condition as given above, we prove that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a unital C^*-algebra $\mathcal{A}$ to a unital C^*-algebra $\mathcal{B}$ is a positive linear operator. It is applied to investigate states, center states and center-valued traces.

1. Introduction and preliminaries

Ulam [24] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated.

Hyers [4] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon$$

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for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

Th. M. Rassias [19] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$, then inequality (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.*

Th. M. Rassias [20] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. Gajda [2] following the same approach as in Th. M. Rassias [19], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [2], as well as by Th. M. Rassias and Šemrl [22] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The counterexamples of Gajda [2], as well as of Th. M. Rassias and Šemrl [22] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [3], who among others studied the Hyers-Ulam-Rassias stability of functional equations. The inequality (1.1) that was introduced for the first time by Th. M. Rassias [19] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *Hyers-Ulam-Rassias stability* of functional equations (cf. the book of D. H. Hyers, G. Isac, and Th. M. Rassias [5]). Jun and Lee [10] proved the stability of Jensen's equation. Park [16] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Beginning around the year 1980, the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (cf. D. H. Hyers and Th. M. Rassias

[6], Th. M. Rassias [21] and the references therein). For further research developments in stability of functional equations the readers are referred to the works of K. Jun and H. Kim [8, 9], H. Kim [14], M. S. Moslehian [15], C. Park [18] and the references cited therein.

Johnson [7, Theorem 7.2] also investigated almost algebra $*$-homomorphisms between Banach $*$-algebras: Suppose that $\mathcal{U}$ and $\mathcal{B}$ are Banach $*$-algebras which satisfy the conditions of [7, Theorem 3.1]. Then for each positive ϵ and K there is a positive δ such that if $T \in L(\mathcal{U}, \mathcal{B})$ with $\|T\| < K$, $\|T^\vee\| < \delta$ and $\|T(x^*)^* - T(x)\| \le \delta\|x\|$ $(x \in \mathcal{U})$, then there is a $*$-homomorphism $T' : \mathcal{U} \to \mathcal{B}$ with $\|T - T'\| < \epsilon$. Here $L(\mathcal{U}, \mathcal{B})$ is the space of bounded linear maps from $\mathcal{U}$ into $\mathcal{B}$, and $T^\vee(x, y) = T(xy) - T(x)T(y)$ $(x, y \in \mathcal{U})$. See [7] for details.

By a positive linear operator $h : \mathcal{A} \to \mathcal{B}$ of a Banach $*$-algebra $\mathcal{A}$ to a Banach $*$-algebra $\mathcal{B}$, we mean a linear mapping with the property $h(x^*x) = h(x)^*h(x)$ for all $x \in \mathcal{A}$. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, then h maps positive elements of $\mathcal{A}$ into positive elements of $\mathcal{B}$. Recall that an element x in a C^*-algebra $\mathcal{A}$ is positive if and only if there exists $y \in A$ such that $x = y^*y$.

In Section 2, using the Hyers-Ulam-Rassias stability method, we prove that every almost positive linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a Banach $*$-algebra $\mathcal{A}$ to a Banach $*$-algebra $\mathcal{B}$ is a positive linear operator when $h(rx) = rh(x)$ holds for some $r > 1$ and all $x \in \mathcal{A}$. In Section 3, in Theorem 3.1, we prove that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a unital C^*-algebra $\mathcal{A}$ to a unital C^*-algebra $\mathcal{B}$ is a positive linear operator when $h(2^n u^* y) = h(2^n u)^* h(y)$ holds for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$. Then in Theorem 3.2, we weaken the conditions of Theorem 3.1 to obtain the same conclusion.

In Section 4, the results of Sections 2 and 3 are applied to investigate states, center states and center-valued traces.

2. Positive linear operators in Banach $*$-algebras

We investigate positive linear operators in Banach C^*-algebras associated with the Cauchy functional equation $f(x + y) = f(x) + f(y)$.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be (complex) Banach $*$-algebras, and $h : \mathcal{A} \to \mathcal{B}$ a mapping satisfying $h(rx) = rh(x)$ for some $r > 1$ and all $x \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^3 \to [0, \infty)$ such that*

$$\lim_{j\to\infty} r^{-j} \varphi(r^j x, r^j y, r^j z) = 0, \tag{2.1}$$

$$\|h(\mu x + \mu y + z^* z) - \mu h(x) - \mu h(y) - h(z)^* h(z)\| \le \varphi(x, y, z) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x, y, z \in \mathcal{A}$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a positive linear operator.

Proof. Since $h(0) = rh(0)$, $h(0) = 0$. Put $z = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (2.2). By (2.2) and the assumption that $h(rx) = rh(x)$ for all $x \in \mathcal{A}$,

$$\begin{aligned}\|h(x+y) - h(x) - h(y)\| &= r^{-n}\|h(r^n x + r^n y) - h(r^n x) - h(r^n y)\| \\ &\leq r^{-n}\varphi(r^n x, r^n y, 0),\end{aligned}$$

which tends to zero as $n \to \infty$ by (2.1). So

$$h(x+y) = h(x) + h(y) \tag{2.3}$$

for all $x, y \in \mathcal{A}$.

Put $y = z = 0$ in (2.2). By (2.2) and the assumption that $h(rx) = rh(x)$ for all $x \in \mathcal{A}$,

$$\|h(\mu x) - \mu h(x)\| = r^{-n}\|h(r^n \mu x) - \mu h(r^n x)\| \leq r^{-n}\varphi(r^n x, 0, 0),$$

which tends to zero as $n \to \infty$ by (2.1). So

$$h(\mu x) = \mu h(x) \tag{2.4}$$

for all $\mu \in \mathbb{T}^1$ and $x \in \mathcal{A}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [11, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. So by (2.3) and (2.4)

$$\begin{aligned}h(\lambda x) &= h\left(\frac{M}{3} \cdot 3\frac{\lambda}{M}x\right) = M \cdot h\left(\frac{1}{3} \cdot 3\frac{\lambda}{M}x\right) = \frac{M}{3}h\left(3\frac{\lambda}{M}x\right) \\ &= \frac{M}{3}h(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(h(\mu_1 x) + h(\mu_2 x) + h(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)h(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}h(x) \\ &= \lambda h(x)\end{aligned}$$

for all $x \in \mathcal{A}$. If $\lambda = 0$, then we also have $h(\lambda x) = h(0) = 0 = \lambda h(x)$ for all $x \in \mathcal{A}$. Hence

$$h(\zeta x + \eta y) = h(\zeta x) + h(\eta y) = \zeta h(x) + \eta h(y)$$

for all $\zeta, \eta \in \mathbb{C}$ and all $x, y \in \mathcal{A}$. So the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

Put $x = y = 0$ in (2.2). By (2.2) and the assumption that $h(rx) = rh(x)$ for all $x \in \mathcal{A}$,

$$\begin{aligned}\|h(z^*z) - h(z)^*h(z)\| &= r^{-2n}\|h(r^n z^* r^n z) - h(r^n z)^* h(r^n z)\| \\ &\leq r^{-2n}\varphi(0, 0, r^n z) \leq r^{-n}\varphi(0, 0, r^n z),\end{aligned}$$

which tends to zero as $n \to \infty$ by (2.1). So

$$h(z^*z) = h(z)^*h(z)$$

for all $z \in \mathcal{A}$. Thus $h : \mathcal{A} \to \mathcal{B}$ is positive.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a positive linear operator. $\square$

Corollary 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be (complex) Banach $*$-algebras, and $h : \mathcal{A} \to \mathcal{B}$ a mapping satisfying $h(rx) = rh(x)$ for some $r > 1$ and all $x \in \mathcal{A}$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y + z^* z) - \mu h(x) - \mu h(y) - h(z)^* h(z)\| \leq \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{A}$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a positive linear operator.

Proof. Define $\varphi(x, y, z) = \theta(||x||^p + ||y||^p + ||z||^p)$, and apply Theorem 2.1. □

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be (complex) Banach $*$-algebras, and $h : \mathcal{A} \to \mathcal{B}$ a mapping satisfying $h(rx) = rh(x)$ for some $r > 1$ and all $x \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^3 \to [0, \infty)$ satisfying (2.1) such that*

$$\|h(\mu x + \mu y + z^* z) - \mu h(x) - \mu h(y) - h(z)^* h(z)\| \leq \varphi(x, y, z) \tag{2.5}$$

for $\mu = 1, i$, and all $x, y, z \in \mathcal{A}$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a positive linear operator.

Proof. Put $z = 0$ and $\mu = 1$ in (2.5). By the same reasoning as in the proof of Theorem 2.1, the mapping $h : \mathcal{A} \to \mathcal{B}$ is additive. Thus $h(tx) = th(x)$ for any rational number t and all $x \in \mathcal{A}$. For every $t \in \mathbb{R}$, there exists a sequence of rational numbers $\{t_n\}$ such that $t = \lim_{n\to\infty} t_n$. Since the mapping $t \mapsto h(tx)$ is continuous for each fixed $x \in \mathcal{A}$, we conclude $h(tx) = th(x)$. Thus h is $\mathbb{R}$-linear.

Put $y = z = 0$ and $\mu = i$ in (2.5). By the same method as in the proof of Theorem 2.1, one can obtain that

$$h(ix) = ih(x)$$

for all $x \in \mathcal{A}$. For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$h(\lambda x) = h(sx + itx) = sh(x) + th(ix) = sh(x) + ith(x) = (s + it)h(x) = \lambda h(x)$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$h(\zeta x + \eta y) = h(\zeta x) + h(\eta y) = \zeta h(x) + \eta h(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $h : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 2.1. □

3. Positive linear operators in unital C^*-algebras

From now on, assume that $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras. Let e be a unit in $\mathcal{A}$ and let e' be a unit in $\mathcal{B}$. Let $\mathcal{U}(\mathcal{A}) := \{u \in \mathcal{A} \mid uu^* = u^*u = e\}$.

Theorem 3.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(2^n u^* y) = h(2^n u)^* h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty, \tag{3.1}$$

$$\|h(\mu x+\mu y)-\mu h(x)-\mu h(y)\| \le \varphi(x,y) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in \mathcal{A}$. *Assume that* $\lim_{n\to\infty} 2^{-n}h(2^n e) = e'$. *Then the mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a positive linear operator.*

Proof. Put $\mu = 1 \in \mathbb{T}^1$ in (3.2). It follows from Găvruta Theorem [3] that there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)| \le 2^{-1}\widetilde{\varphi}(x,x)$$

for all $x \in \mathcal{A}$. The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} 2^{-n}h(2^n x)$$

for all $x \in \mathcal{A}$.

Put $y = 0$ in (3.2). Then $\|h(\mu x) - \mu h(x)\| \le \varphi(x,0)$. So

$$2^{-n}\|h(2^n\mu x) - \mu h(2^n x)\| \le 2^{-n}\varphi(2^n x, 0),$$

which tends to zero as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence

$$H(\mu x) = \lim_{n\to\infty} 2^{-n}h(2^n\mu x) = \lim_{n\to\infty} 2^{-n}\mu h(2^n x) = \mu H(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

By the same method as in the proof of Theorem 2.1, one can show that the unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear operator.

Since $h(2^n u^* y) = h(2^n u)^* h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$,

$$H(u^*y) = \lim_{n\to\infty} 2^{-n}h(2^n u^* y) = \lim_{n\to\infty} 2^{-n}h(2^n u)^* h(y) = H(u)^* h(y) \tag{3.3}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (3.3),

$$2^n H(u^*y) = H(2^n u^* y) = H(u^*(2^n y)) = H(u)^* h(2^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(u^*y) = 2^{-n}H(u)^* h(2^n y) = H(u)^* \cdot 2^{-n}h(2^n y) \tag{3.4}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (3.4) as $n \to \infty$, we obtain

$$H(u^*y) = H(u)^* H(y) \tag{3.5}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [12, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$), it follows from (3.5) that

$$\begin{aligned} H(x^*y) &= H(\sum_{j=1}^m \overline{\lambda}_j u_j^* y) = \sum_{j=1}^m \overline{\lambda}_j H(u_j^* y) = \sum_{j=1}^m \overline{\lambda}_j H(u_j)^* H(y) \\ &= H(\sum_{j=1}^m \lambda_j u_j)^* H(y) = H(x)^* H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. In particular,

$$H(x^*x) = H(x)^* H(x)$$

for all $x \in \mathcal{A}$. So $H : \mathcal{A} \to \mathcal{B}$ is a positive linear operator.

By (3.3) and (3.5),

$$H(e)^* H(y) = H(e^* y) = H(e)^* h(y)$$

for all $y \in \mathcal{A}$. Since $H(e) = \lim_{n\to\infty} 2^{-n} h(2^n e) = e'$,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a positive linear operator. □

We investigate positive linear operators in unital C^*-algebras associated with the Cauchy functional equation.

Theorem 3.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^3 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y, 2^j z) < \infty, \tag{3.6}$$

$$\|h(\mu x + \mu y + 2^n u^* z) - \mu h(x) - \mu h(y) - h(2^n u)^* h(z)\| \le \varphi(x, y, z) \tag{3.7}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, all $n = 0, 1, 2, \ldots$, and all $x, y, z \in \mathcal{A}$. Assume that $\lim_{n\to\infty} 2^{-n} h(2^n e) = e'$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a positive linear operator.

Proof. Put $z = 0$ and $\mu = 1$ in (3.7). It follows from Găvruta Theorem [3] that there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \le 2^{-1} \widetilde{\varphi}(x, x, 0)$$

for all $x \in \mathcal{A}$. The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} 2^{-n} h(2^n x)$$

for all $x \in \mathcal{A}$.

Put $x = y = 0$ in (3.7). Then

$$\|h(2^n u^* z) - h(2^n u)^* h(z)\| \le \varphi(0, 0, z)$$

for all $z \in \mathcal{A}$. So

$$2^{-n} \|h(2^n u^* z) - h(2^n u)^* h(z)\| \le 2^{-n} \varphi(0, 0, z),$$

which tends to zero as $n \to \infty$. Replacing z by y, one can obtain

$$H(u^* y) = \lim_{n\to\infty} 2^{-n} h(2^n u^* y) = \lim_{n\to\infty} 2^{-n} h(2^n u)^* h(y) = H(u)^* h(y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$.

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

4. Positive linear functionals on C^*-algebras

States and traces are so important in the theory of C^*-algebras and von Neumann algebras (see [1, 12, 13, 17, 23]).

By a *state* on $\mathcal{A}$ we mean a positive linear functional $\rho : \mathcal{A} \to \mathbb{C}$ such that $\rho(e) = 1$. We describe ρ as a *tracial state* if, in addition, $\rho(xy) = \rho(yx)$ for all $x, y \in \mathcal{A}$. A state ρ is called *pure* if every positive linear functional ψ on $\mathcal{A}$, majorized by ρ in the sense that $\psi(x^*x) \le \rho(x^*x)$, is of the form $\lambda\rho$, $0 \le \lambda \le 1$.

Theorem 4.1. *Let $h : \mathcal{A} \to \mathbb{C}$ be a function with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^3 \to [0, \infty)$ satisfying (3.6) such that*

$$|h(\mu x + \mu y + 2^n u^* z) - \mu h(x) - \mu h(y) - \overline{h(2^n u)} f(z)| \le \varphi(x, y, z)$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, all $n = 0, 1, 2, \ldots$, and all $x, y, z \in \mathcal{A}$. Assume that $\lim_{n\to\infty} 2^{-n} h(2^n e) = 1$. Then the function $h : \mathcal{A} \to \mathbb{C}$ is a pure and tracial state.

Proof. According to the proof of Theorem 3.2, the mapping $h : \mathcal{A} \to \mathbb{C}$ is a positive linear operator such that

$$h(x^* y) = \overline{h(x)} h(y) \tag{4.1}$$

for all $x, y \in \mathcal{A}$. In particular, $h(e) = |h(e)|^2$. This implies $h(e) = 1$, so h is a state. Furthermore,

$$h(x^*) = h(x^* e) = \overline{h(x)} h(e) = \overline{h(x)}$$

for all $x \in \mathcal{A}$. Thus (4.1) yields

$$h(xy) = \overline{h(x^*)} h(y) = h(x) h(y)$$

for all $x, y \in \mathcal{A}$. Then we have

$$h(xy) = h(x)h(y) = h(y)h(x) = h(yx),$$

so h is a tracial state. By [12, Proposition 4.4.1] and the remark following the proposition, h is a pure state. $\square$

By a *center state* on $\mathcal{A}$ we mean a positive linear operator $\rho : \mathcal{A} \to \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{A}$, such that $\rho(c) = c$ and $\rho(cx) = c\rho(x)$ for all $c \in \mathcal{Z}$ and all $x \in \mathcal{A}$. By a *center-valued trace* on $\mathcal{A}$ we mean a positive linear operator $\tau : \mathcal{A} \to \mathcal{Z}$ such that $\tau(c) = c$ and $\tau(xy) = \tau(yx)$ for all $c \in \mathcal{Z}$ and all $x, y \in \mathcal{A}$.

Theorem 4.2. *Let $h : \mathcal{A} \to \mathcal{Z}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^3 \to [0, \infty)$ satisfying (3.6) such that*

$$\|h(wx + wy + 2^n u^* z) - wh(x) - wh(y) - h(2^n u)^* h(z)\| \le \varphi(x, y, z)$$

for all $w \in \mathcal{U}(\mathcal{Z})$, all $u \in \mathcal{U}(\mathcal{A})$, all $n = 0, 1, 2, \ldots$, and all $x, y, z \in \mathcal{A}$. Assume that $\lim_{n\to\infty} 2^{-n} h(2^n e) = e$. Then the mapping $h : \mathcal{A} \to \mathcal{Z}$ is a center state and center-valued trace.

Proof. For $w = \mu e$, with $\mu \in \mathbb{T}^1$, we get the condition (3.7). So Theorem 3.2 implies that h is a positive linear operator. Thus

$$\|h(wx) - wh(x)\| = 2^{-n}\|h(2^n wx) - wh(2^n x)\| \le 2^{-n}\varphi(2^n x, 0, 0),$$

which tends to zero as $n \to \infty$. Therefore, $h(wx) = wh(x)$ for all $w \in \mathcal{U}(\mathcal{Z})$ and all $x \in \mathcal{A}$. This yields $h(cx) = ch(x)$ for all $c \in \mathcal{Z}$ and all $x \in \mathcal{A}$. Since $h(e) = e$,

$$h(c) = h(ce) = ch(e) = c$$

for all $c \in \mathcal{Z}$. Hence h is a center state.

For $x = y = 0$, we get

$$\|h(u^*z) - h(u)^*h(z)\| = 2^{-n}\|h(2^n u^*z) - h(2^n u)^*h(z)\| \le 2^{-n}\varphi(0, 0, z),$$

which tends to zero as $n \to \infty$. Hence $H(u^*z) = H(u)^*h(z)$ for all $u \in \mathcal{U}(\mathcal{Z})$ and all $z \in \mathcal{A}$. In particular,

$$h(u^*) = h(u^*e) = h(u)^*h(e) = h(u)^*.$$

Therefore,

$$h(uz) = h(u^*)^*h(z) = h(u)h(z)$$

for all $u \in \mathcal{U}(\mathcal{Z})$ and all $z \in \mathcal{A}$. This yields

$$h(xy) = h(x)h(y)$$

for all $x, y \in \mathcal{A}$. Hence

$$h(xy) = h(x)h(y) = h(y)h(x) = h(yx)$$

for all $x, y \in \mathcal{A}$, so h is a center-valued trace. □

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