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THE DISJOINT CURVE PROPERTY AND BRIDGE SURFACES

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Abstract. We show that every bridge surface of certain types of $(1, 1)$ prime knot has the disjoint curve property. Also we determine when a bridge surface of a pretzel knot of type $(-2, 3, n)$ has the disjoint curve property.

1. Introduction

Let M denote a compact orientable 3-manifold and let $(W_1, W_2; H)$ be a genus g Heegaard splitting of M. In 1960s, W. Haken [4] introduced a condition of Heegaard splittings which is called the reducibility. A splitting $(W_1, W_2; H)$ is said to be *reducible* if there are essential disks $D_i \subset W_i$ $(i = 1, 2)$ with $\partial D_1 = \partial D_2$. Otherwise, $(W_1, W_2; H)$ is said to be *irreducible*. He proved that all splittings of a reducible manifold are themselves reducible. Strongly irreducibility was introduced by A. Casson and McA. Gordon [3] as a generalization of irreducibility. A splitting $(W_1, W_2; H)$ is said to be *weakly reducible* if there are essential disks $D_i \subset W_i$ $(i = 1, 2)$ with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $(W_1, W_2; H)$ is said to be *strongly irreducible*. They showed that all splittings of a non-Haken manifold are either reducible or strongly irreducible. A. Thompson [17] introduced the notion of disjoint curve property as a further generalization of reducibility. A splitting $(W_1, W_2; H)$ admits the *disjoint curve property* if there are essential disks $D_i \subset W_i$ $(i = 1, 2)$ and an essential loop $c \subset H$ with $(\partial D_1 \cup \partial D_2) \cap c = \emptyset$. A splitting is *full* if it does not have the disjoint curve property. A. Thompson proved that all splittings of a toroidal 3-manifold have the disjoint curve property in [17]. And J. Hempel has shown that each splitting of a Seifert fibred space has the disjoint curve property using the classification of splittings of Seifert fibered spaces in [8]. Thus, in any 3-manifold which is reducible, toroidal, or a Seifert fibred space, all Heegaard splittings have the disjoint curve property.

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However, it is certainly not the case that all splittings of all manifolds have the disjoint curve property. J. Hempel in [8] adapts an argument of T. Kobayashi [11] to produce examples of splittings which are full and are in fact arbitrarily far from having the disjoint curve property.

In this paper, we show that every bridge surface of certain types of $(1,1)$-prime knot has the disjoint curve property. T. Saito has shown that a bridge surface of $(1,1)$-hyperbolic knot is full, except for certain type of knots in [15]. Also we determine that every bridge surface of the pretzel knot of type $(-2,3,n)$ has the disjoint curve property.

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2. Preliminaries

In this section, we recall standard notation and review several notions from the theory of g-genus n-bridge Heegaard splittings. We would like to refer [5] for more detail about relationship between definitions.

Let W be a 3-manifold with non-empty boundary ∂W and $K = \{k_1, k_2, \ldots, k_n\}$ be a set of disjoint arcs properly embedded in W, that is, $k_i \cap \partial W = \partial k_i$ for every $1 \leq i \leq n$.

Definition 2.1. K is *trivial* in (W, K) if there is a set $\{D_1, \ldots, D_n\}$ of disjoint discs embedded in W such that $D_i \cap (\cup k_i) = \partial D_i \cap k_i = k_i$ and so that $D_i \cap \partial W$ is the arc $cl(\partial D_i - k_i)$. We call D_i a cancelling disc of k_i.

When W is a ball and K is trivial, the pair (W, K) is called a *trivial n-string tangle.*

Let T be a properly embedded 1-manifold in W, that is, $T \cap \partial W = \partial T$ and let F be a 2-manifold properly embedded in W. Suppose that F is transverse to T. In particular $\partial F \cap T = \emptyset$.

Definition 2.2. F is *T-compressible* if there is a disc D embedded in W such that D is disjoint from T, that is $D \cap F = \partial D$ and that ∂D dose not bound a disc disjoint from T on F. Such D is called a T-compressing disc of F. We call F a *T-incompressible* if it is not T-compressible.

Definition 2.3. A 2-manifold F is *meridionally compressible* in (W, T) if there is a disc D embedded in W such that $intD$ intersects T transversely in a single point, that $D \cap F = \partial D \cap (F - T) = \partial D$ and that ∂D in F does not bound a disc whose interior intersects T transversely in a single point. Such D is called a meridionally compressing disc of F. F is *meridionally incompressible* if it is not meridionally compressible.

We define a T-compressible 2-submanifold and a meridionally compressible 2-submanifold of ∂W similarly. Assume that either F is a 2-manifold properly embedded in W such that F is transverse to T, or F is a 2-submanifold of ∂W

with $\partial F \cap T = \emptyset$. A simple loop l on F is said to be *T-essential* if it is disjoint from T and if it does not bound a disc which intersects T transversely in zero or one point.

Let M be a closed orientable 3-manifold and L a link in M. Let H be a genus g Heegaard splitting surface of M, that is, H divides M into two handlebodies W_1 and W_2 of genus g. Suppose that H is transverse to L.

Definition 2.4. H is a *g-genus n-bridge splitting*, (g, n)-splitting, of (M, L) if L intersects W_i in a set of trivial n arcs for $i = 1$,2.
A link L is called a *g-genus n-bridge link*, (g, n)-link, if it admits a g-genus n-bridge splitting. A link in S^3 is simply called an *n-bridge link* in S^3 if it has a 0-genus n-bridge spitting.

Let K be a knot in a closed 3-manifold M. Let $(M, K) = (W_1, k_1) \cup_H (W_2, k_2)$ be a bridge splitting of a 3-manifold (M, K).

Definition 2.5. H is *K-reducible* if W_1 and W_2 contain K-compressing or meridionally compressing discs D_1 and D_2 of H respectively such that $\partial D_1 = \partial D_2$ in H. H is *K-irreducible* if it is not K-reducible.

Note that if H is K-reducible, then K is the trivial knot bounding a disc composed of two cancelling discs as shown in [6].

Definition 2.6. H is *weakly K-reducible* if W_1 and W_2 contain K-compressing or meridionally compressing discs D_1 and D_2 of H respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$. H is *strongly K-irreducible* if it is not weakly K-reducible.

Definition 2.7. H has the *disjoint curve property* if there exist essential simple closed curves c, ∂D_1, and ∂D_2 on H such that $c \cap (\partial D_1 \cup \partial D_2) = \emptyset$ and D_1 and D_2 are K-compressing or meridionally compressing discs in W_1 and W_2 respectively.

By $E(K)$, we mean the exterior of K in M, i.e., $E(K) = cl(M - N(K))$, where $N(K)$ is a regular neighborhood of K in M.

Definition 2.8. (1) K is a *trivial knot* if it bounds a disc imbedded in M.

(2) K is a *core knot* if K is non-trivial and M admits a genus one Heegaard splitting $(V_1, V_2; P)$ such that K is isotopic to the core of V_i for $i = 1$ or 2.
(3) K is a *2-bridge knot* if there is a genus zero Heegaard splitting $(B_1, B_2 : P_0)$ of S^3 such that $(B_i, B_i \cap K)$ $(i = 1, 2)$ is a 2-string trivial tangle.
(4) K is *split* if M contains a sphere S which decomposes M into a punctured lens space and a ball containing K in its interior. This sphere S is called a splitting sphere.
(5) K is a *composite knot* if M contains a 2-sphere S which intersects K transversely in 2 points and $S \cap E(K)$ is ∂-incompressible in $E(K)$. We call this 2-sphere S a decomposing sphere. A knot is said to be *prime* if it is not composite.

It is proved in [7] that if H is weakly K-reducible, then K is the trivial knot or a 2-bridge knot when $M = S^3$, and K is a core knot or a composite knot of a core knot and a 2-bridge knot when M is a lens space.

Definition 2.9. A knot K is called a *torus knot* if K is isotopic to a simple loop on a genus one Heegaard surface of M and is not a core knot.
The torus knot $T_{p,q}$ of type (p, q) is the knot which wraps around the standard solid torus p times in the longitudinal direction, and q times in the meridional direction.

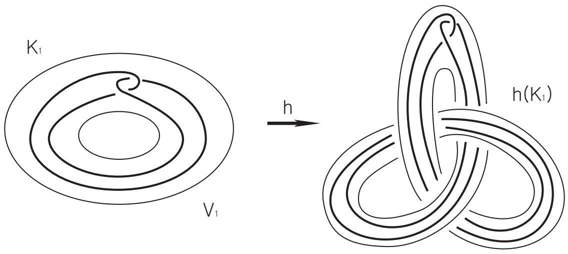

Figure 1. Satellite knot

Let V_1 be an unknotted solid torus in S^3 and K_1 be a knot in V_1. Let V_2 be a tubular neighborhood of a non-trivial knot K_2. For a homeomorphism $h : V_1 \to V_2$, the knot $K = h(K_1)$ is called a satellite knot and K_2 its companion.

Definition 2.10. A knot K is said to be *satellite* if $E(K)$ contains an incompressible torus T which is not parallel to $\partial E(K)$.

Definition 2.11. A *hyperbolic knot* is a knot whose complement can be endowed with a metric of constant curvature -1.

The seminal work of W. Thurston demonstrates that every knot in S^3 is either a torus knot, a satellite knot or a hyperbolic knot. These three categories are mutually exclusive.

3. Heegaard splitting

In Section 3, we may consider 3-manifolds with nonempty boundaries such as knot complements. In those cases, we use the notion of Heegaard splitting of compression body.

J. Hempel has shown that if a closed orientable 3-manifold M is reducible, Seifert fibered or toroidal, then any splitting of M has the disjoint curve property in [8].

J. Hempel's result is generalized to the Proposition 3.1 by T. Saito [16]. He introduced a notion called the disjoint (A, D)-pair property. Here, a Heegaard splitting $(W_1, W_2; H)$ admits the *disjoint* (A, D)*-pair property* if there are an essential annulus A_i normally embedded in W_i and an essential disk D_j in W_j, $((i, j) = (1, 2)$ or $(2, 1))$ such that $\partial A_i \cap \partial D_j = \emptyset$.

Proposition 3.1 ([16]). *Let M be a compact orientable 3-manifold. If M is reducible, Seifert fibered or toroidal, then any genus $g \geq 2$ Heegaard splitting of M admits the disjoint (A, D)-pair property. Moreover, if a Heegaard splitting admits the disjoint (A, D)-pair property, then it admits the disjoint curve property.*

We can deduce the following corollary from Proposition 3.1.

Corollary 3.2. *If K is a torus knot in S^3, then every genus $g \geq 2$ Heegaard splitting of $E(K)$ has the disjoint curve property.*

Proof. Since K is a torus knot, $E(K)$ is a Seifert fibered manifold. Let $E(K) = W_1 \cup_H W_2$ for $g \geq 2$. By Proposition 3.1, H has the disjoint curve property. □

A 3-manifold is *toroidal* if it contains an essential torus, namely an incompressible torus which is not parallel to a boundary component.

Corollary 3.3. *If K is a satellite knot in S^3, then every genus $g \geq 2$ Heegaard splitting of $E(K)$ has the disjoint curve property.*

Proof. Since K is a satellite knot, $E(K)$ is toroidal. Let $E(K) = W_1 \cup_H W_2$ for $g \geq 2$. By Proposition 3.1, H has the disjoint curve property. □

4. Bridge surface

The next two propositions guarantee the existence of cancelling discs satisfying certain conditions for a Heegaard surface which is in (1,1) position with respect to the given knot.

Proposition 4.1 ([13, Theorem 3]). *Let L be a lens space and K a 1-bridge knot in L, and let $(W_1, W_2; H)$ be a Heegaard splitting of genus one of L which gives a 1-bridge representation of K i.e., $\alpha_i = W_i \cap K$ is a single trivial arc in W_i $(i = 1, 2)$.*

Suppose that K is a non-trivial torus knot and is not a core of L. Then for $i = 1, 2$, there exists a disk $\triangle_i$ in W_i such that $\partial W_i \cap \triangle_i = \beta_i$ is an arc in ∂W_i, $\partial \triangle_i = \alpha_i \cup \beta_i$ and $\beta_1 \cap \beta_2 = \partial\beta_1 = \partial\beta_2$.

Proposition 4.2 ([7, Theorem III]). *Let M be the S^3 or a lens space (not homeomorphic to $S^2 \times S^1$). Let H be a genus 1 Heegaard surface of M. This surface H divides M into two solid tori W_1 and W_2. Suppose a knot K is in 1-genus 1-bridge position with respect to H and H is neither K-reducible nor weakly K-reducible.*

If K is a satellite knot, then there is an annulus Z on H such that there is a cancelling disc C_i of t_i (trivial arc in V_i) with $(\partial C_i \cap H) \subset Z$ for $i = 1, 2$. Moreover, the incompressible torus is isotopic to $\partial N(C_1 \cup Z \cup C_2)$ in $E(K)$.

Using Propositions 4.1 and 4.2, we can deduce the disjoint curve property for bridge surfaces of non trivial torus knots and satellite knots.

Theorem 4.3. *Let M be the S^3 or a lens space and K be a 1-bridge knot in M. If K is a non-trivial torus knot in M, then the bridge surface H has the disjoint curve property.*

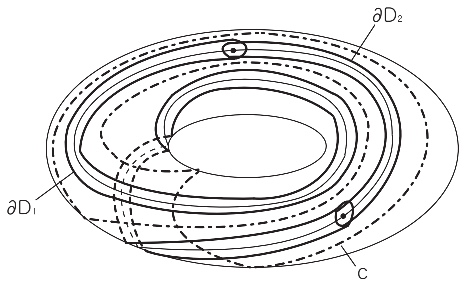

FIGURE 2. Essential curves satisfying the disjoint curve property for a torus knot

Proof. Any 1-bridge representation of a torus knot in a lens space is trivial. Let $(M, K) = (W_1, \alpha_1) \cup_H (W_2, \alpha_2)$ and C_i be a cancelling disc of α_i. By Proposition 4.1, $H \cap C_i = \beta_i$ is an arc in H. Then $A = N(\beta_1 \cup \beta_2)$ is an annulus in H. Now the disc $D_i = cl(\partial N(C_i, W_i) - H)$ is a properly embedded and K-compressing disc in W_i for $i = 1, 2$. $cl(H - \text{int}(A))$ is an annulus in H. Moreover one can find an essential curve c in $cl(H - \text{int}(A))$ such that $\partial D_1 \cap c = \emptyset$ and $\partial D_2 \cap c = \emptyset$. Therefore H has the disjoint curve property. We describe explicitly how to get essential curves satisfying the disjoint curve property for a torus knot in Figure 2. □

Theorem 4.4. *Let M be the S^3 or a lens space (not homeomorphic to $S^2 \times S^1$). Let H be a genus 1 Heegaard surface of M. This surface H divides M into two solid tori W_1 and W_2. Suppose a knot K is in 1-genus 1-bridge position with respect to H. If K is a satellite knot, then H has the disjoint curve property.*

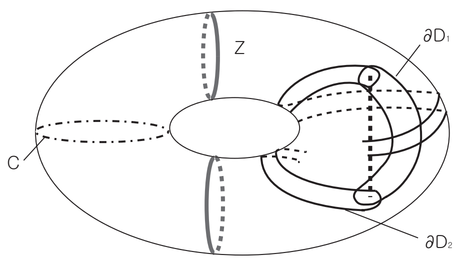

FIGURE 3

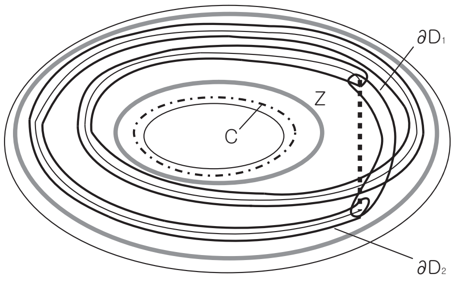

FIGURE 4

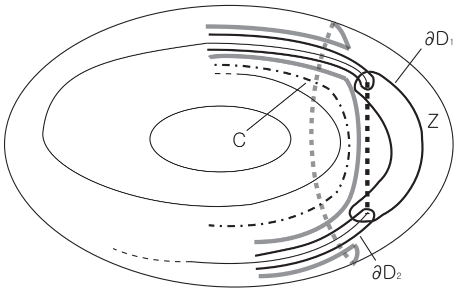

FIGURE 5

Proof. If H is K-reducible, then K is a trivial knot by [15]. If H is weakly K-reducible, then M is $S^2 \times S^1$ and K is a core knot by [15]. By assumption, we may assume that H is neither K-reducible nor weakly K-reducible. Let $(M, K) = (W_1, t_1) \cup_H (W_2, t_2)$, and C_i be a cancelling disc of t_i. By Proposition 4.2, there exists an annulus Z in H such that there is a cancelling disc C_i of t_i with $(\partial C_i \cap H) \subset Z$ for $i = 1, 2$. If we take $D_i = cl(\partial N(C_i, W_i) - H)$, then it is a properly embedded and K-compressing disc in W_i for $i = 1, 2$. Because $cl(H - \mathrm{int}(Z))$ is an annulus, one can find an essential curve c in $cl(H - \mathrm{int}(Z))$ such that $\partial D_1 \cap c = \emptyset$ and $\partial D_2 \cap c = \emptyset$. We describe explicitly how to get essential curves which satisfy the disjoint curve property for the case when $\partial C_2 \cap H$ is meridinal in Figure 3, the case when $\partial C_2 \cap H$ is logitudinal in Figure 4 and the case when $\partial C_2 \cap H$ is neither meridinal nor logitudinal in Figure 5. The dotted line in each figure indicates the knot in a splitted solid torus. Therefore H has the disjoint curve property. □

Now we consider hyperbolic knots.

A knot K in an orientable closed 3-manifold M is called a $(1, 1)$-*knot* if $(M, K) = (W_1, k_1) \cup_H (W_2, k_2)$. To define the distance of a $(1, 1)$-splitting, we use the twice punctured torus $\Sigma = H - K$. For notation and definition, we refer to [15].

For $i = 1$ or 2, let $\mathcal{K}(W_i)$ be the maximal subcomplex of $C(\Sigma)$ consisting of simplexes $\langle [c_0], [c_1], \ldots, [c_k] \rangle$ such that an essential loop representing $[c_j]$ $(j = 0, 1, \ldots, k)$ bounds a disk in $W_i - k_i$.

Definition. We define the distance of a $(1,1)$-splitting $(W_1, W_2; H)$ by
$d(W_1, W_2) = \min\{d([x],[y]) | \ [x] : \text{a vertex in } \mathcal{K}(W_1), \ [y] : \text{a vertex in } \mathcal{K}(W_2)\}$.

For a pair α (≥ 4) and β of co-prime integers and an element $\gamma \in \mathbb{Q} \cup \{1/0\}$, $K(\alpha, \beta; \gamma)$ denotes the knot K_2 in $K_1(\gamma)$, where $K_1 \cup K_2$ is the 2-bridge link of type (α, β) (cf. Chapter 10 of [14]) and $K_1(\gamma)$ is the manifold obtained by γ-surgery on K_1. By definition, every $K(\alpha, \beta; \gamma)$ is a $(1,1)$-knot.

Proposition 4.5 ([15])**.** *Let K be a $(1,1)$-knot in M. Suppose that (M, K) is not equivalent to $K(\alpha, \beta; \gamma)$ for any α, β and γ, and that the bridge index of K is at least three if $M \cong S^3$. Then K is a hyperbolic knot if and only if it has a $(1,1)$-splitting with distance ≥ 3.*

By Proposition 4.5, one can see that a $(1,1)$-knot is hyperbolic if a $(1,1)$-splitting does not have the disjoint curve property and a $(1,1)$-splitting with distance neither 0 nor 1.

Let $L(p_1, p_2, \ldots, p_n)$ be an n-pretzel link in S^3 where $p_i \in \mathbb{Z}$ represents the number of half twists. In particular, if $n = 3$, it is called a classical pretzel link, denoted by $L(p, q, r)$. For notation and definition for pretzel knots, we refer to [10].

If n is odd, then an n-pretzel link $L(p_1, p_2, \ldots, p_n)$ is a knot if and only if none of two p_i's are even. If n is even, then $L(p_1, p_2, \ldots, p_n)$ is a knot if and only if one of the p_i's is even. A pretzel knot is denoted by $K(p_1, p_2, \ldots, p_n)$.

A link L is almost alternating if it is not alternating and there is a diagram D_L of L such that one crossing change makes the diagram alternating; we call D_L an almost alternating diagram. By [10], classical pretzel links are prime and either alternating or almost alternating. W. Menasco has shown that prime alternating knots are either hyperbolic or torus knots [12]. It has been generalized by C. Adams that prime almost alternating knots are either hyperbolic or torus knots [2]. It is known that no satellite knot is an almost alternating knot [9]. By the following theorem [10], we can classify classical pretzel knots completely into hyperbolic or torus knots.

Proposition 4.6 ([10])**.** *The following are the only nontrivial pretzel knots which are torus knots.*

(1) $K(p, \pm 1, \mp 1)$ *are unknots for all* p.
(2) $K(\pm 1, \pm 1, \pm 1)$ *are* $(2, \pm 3)$ *torus knots.*
(3) $K(\pm 2, \mp 1, \pm r)$ *are* $(2, \pm r \mp 2)$ *torus knots.*
(4) $K(\mp 2, \pm 3, \pm 3)$*,* $K(\mp 2, \pm 3, \pm 5)$ *are* $(3, \pm 4)$*,* $(3, \pm 5)$ *torus knots, respectively.*

Now we consider a pretzel knot of type $(-2, 3, n)$. Every $K(-2, 3, n)$ pretzel knot, $n \in \mathbb{Z}$, can be described by the diagram as in Figure 6.

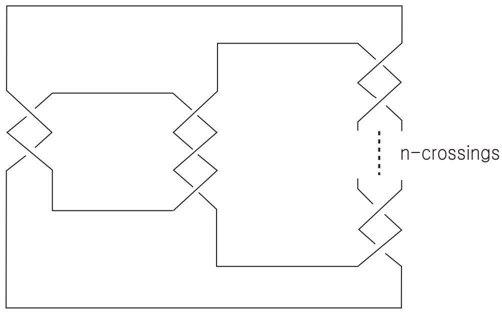

FIGURE 6. $(-2, 3, n)$ pretzel knot diagram

Theorem 4.7. *For pretzel knots of type $(-2, 3, n)$, the following holds.*

(1) *If $n = 1, 3, 5$, then the genus one 1-bridge surface which has the disjoint curve property.*
(2) *If $n \neq 1, 3, 5$, then there is a genus two 1-bridge surface which has the disjoint curve property.*

Proof. If $n = 1, 3, 5$, then $K(-2, 3, n)$ is a nontrivial torus knot by Proposition 4.6. According to Theorem 4.3, genus one 1-bridge surface has the disjoint curve property.

If $n \neq 1, 3, 5$, then each of these knots has a hyperbolic structure on its complement. Each pretzel knot of type $K(-2, 3, n)$ admits $(2, 1)$-splitting as follows. The given knot can be realized as a curve on the surface of a double torus. S^3 has a genus two Heegaard splitting. Namely, S^3 can be thought of as two double tori whose boundaries have been identified. Actually, S^3 has a unique Heegaard splitting of each genus.

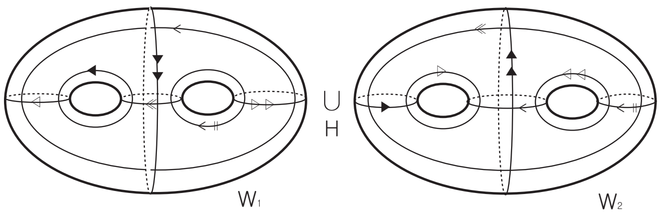

FIGURE 7. (2,1)-splitting

Let $(W_1, W_2; H)$ be a Heegaard splitting of genus 2 of S^3 and the knot can be divided into two arcs namely, $K(-2, 3, n) = K_1 \cup K_2$. Let t_i be a trivial arc in W_i obtained from K_i by pushing interior of K_i into intW_i for $i = 1, 2$. Then $(M, K) = (W_1, t_1) \cup_H (W_2, t_2)$ is a $(2, 1)$-splitting, and C_i is a cancelling disc of t_i (see Figure 7 for details).

If we take $D_i = cl(\partial N(C_i, W_i) - H)$, then it is a properly embedded K-compressing disc in W_i for $i = 1, 2$.

Because $\partial(N(C_1, H) \cup N(C_2, H))$ is an annulus in H, one can find an essential curve c in $cl(H - \partial(N(C_1, H) \cup N(C_2, H)))$ such that $\partial D_1 \cap c = \emptyset$ and $\partial D_2 \cap c = \emptyset$. Therefore H has the disjoint curve property. This completes the proof of Theorem 4.7. □

Example. We now describe how to construct $(2, 1)$-splitting and how to find 3 essential circles which satisfy the disjoint curve property for the $(-2, 3, 7)$-pretzel knot in Figure 8 and Figure 9.

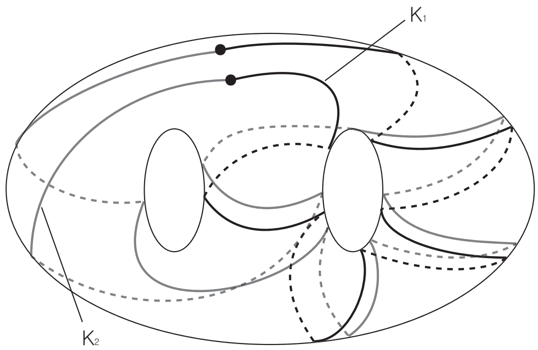

FIGURE 8. (2,1)-splitting for a (-2,3,7) pretzel knot

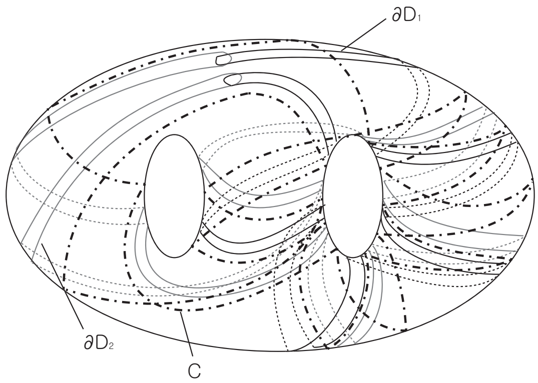

FIGURE 9. Essential curves satisfying the disjoint curve property for a (-2,3,7) pretzel knot

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