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OVERRINGS OF THE KRONECKER FUNCTION RING Kr(D, $*$) OF A PRÜFER $*$-MULTIPLICATION DOMAIN D

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Abstract. Let $*$ be an *e.a.b.* star operation on an integrally closed domain D, and let $Kr(D,*)$ be the Kronecker function ring of D. We show that if D is a P$*$MD, then the mapping $D_\alpha \mapsto Kr(D_\alpha, v)$ is a bijection from the set $\{D_\alpha\}$ of $*$-linked overrings of D into the set of overrings of $Kr(D,v)$. This is a generalization of [5, Proposition 32.19] that if D is a Prüfer domain, then the mapping $D_\alpha \mapsto Kr(D_\alpha, b)$ is a one-to-one mapping from the set $\{D_\alpha\}$ of overrings of D onto the set of overrings of $Kr(D,b)$.

1. Introduction

Let D be an integral domain with quotient field K, and let $\mathbf{F}(D)$ (resp., $\mathbf{f}(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D; so $\mathbf{f}(D) \subseteq \mathbf{F}(D)$. An *overring* of D means a ring between D and K. Let X be an indeterminate over D. For a polynomial $f \in K[X]$, the *content* of f, denoted by $c_D(f)$, is the fractional ideal of D generated by the coefficients of f.

Let $*$ be a star operation on D (see the next page for definitions related to star operations). It is well known that if D is a Prüfer domain, then the mapping $D_\alpha \mapsto Kr(D_\alpha, v)$ is a one-to-one mapping from the set $\{D_\alpha\}$ of overrings of D onto the set of overrings of $Kr(D,d)$ [5, Proposition 32.19]. The purpose of this paper is to extend this result to Prüfer $*$-multiplication domains (P$*$MDs). Precisely, we show that if D is a P$*$MD, then T is an overring of $Kr(D,*)$ if and only if $T \cap K$ is $*$-linked over D, $T \cap K$ is a PvMD and $T = Kr(T \cap K, v)$. As a corollary, we have that if D is a P$*$MD, then the mapping $D_\alpha \mapsto Kr(D_\alpha, v)$ is a bijection from the set $\{D_\alpha\}$ of $*$-linked overrings of D into the set of overrings of $Kr(D,*)$. This also recovers the result of [5, Proposition 32.19] (here $* = d$). We finally give a PvMD that has infinitely many Kronecker function rings.

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We next review some notations and definitions related to star operations. A mapping $* : \mathbf{F}(D) \to \mathbf{F}(D)$, $I \mapsto I^*$, is called a *star operation on* D if the following three conditions are satisfied for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$;

(i) $(aD)^* = aD$ and $(aI)^* = aI^*$,

(ii) $I \subseteq I^*$ and if $I \subseteq J$ then $I^* \subseteq J^*$, and

(iii) $(I^*)^* = I^*$.

Given a star operation $*$ on D, we can construct a new star operation $*_f$ on D as follows;

$$I^{*_f} = \cup\{J^* | J \subseteq I \text{ and } J \in \mathbf{f}(D)\} \text{ for all } I \in \mathbf{F}(D).$$

The simplest example of star operations is the d-operation, which is the identity function on $\mathbf{F}(D)$, i.e., $I^d = I$ for all $I \in \mathbf{F}(D)$. Other well-known star operations are the v- and t-operations. The v-operation is defined by $I^v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in K | xI \subseteq D\}$, and the t-operation is given by $t = v_f$. We say that $*$ is of *finite character* if $*_f = *$. Obviously, $(*_f)_f = *_f$ and $d_f = d$; so $*_f$, d and t are of finite character. If $*_1$ and $*_2$ are star operations on D, we mean by $*_1 \leq *_2$ that $I^{*_1} \subseteq I^{*_2}$ for all $I \in \mathbf{F}(D)$. It is well known that $*_f \leq *$, $d \leq * \leq v$ and $d \leq *_f \leq t$ for any star operation $*$ on D.

An $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I^* = I$, while a $*$-ideal is called a *maximal* $*$-*ideal* if it is maximal among proper integral $*$-ideals of D. Let $*$-Max(D) be the set of maximal $*$-ideals of D. We know that if $* = *_f$, then $*$-Max$(D) \neq \emptyset$ when D is not a field; each maximal $*$-ideal is a prime ideal; each proper integral $*$-ideal is contained in a maximal $*$-ideal; and each prime ideal minimal over an integral $*$-ideal is a $*$-ideal. An $I \in \mathbf{F}(D)$ is said to be $*$-*invertible* if $(II^{-1})^* = D$, and D is called a *Prüfer* $*$-*multiplication domain* (P$*$MD) if each $I \in \mathbf{f}(D)$ is $*_f$-invertible. Hence Prüfer domains are just the PdMDs. An overring R of D is $*$-*linked over* D if $I^* = D$ implies $(IR)^v = R$ for all $I \in \mathbf{f}(D)$; equivalently, if $(Q \cap D)^{*_f} \subsetneq D$ for all $Q \in t$-Max(R) [1, Proposition 3.2].

A star operation $*$ on D is said to be *endlich arithmetisch brauchbar* (*e.a.b.*) if, for all $A, B, C \in \mathbf{f}(D)$, $(AB)^* \subseteq (AC)^*$ implies $B^* \subseteq C^*$. It is obvious that $*$ is *e.a.b.* if and only if $*_f$ is *e.a.b.* and that if D is a P$*$MD, then $*$ is an *e.a.b.* star operation. Note that if D admits an *e.a.b.* star operation, then D is integrally closed [5, Corollary 32.8]. Conversely, if D is integrally closed, let $I^b = \cap\{IV | V$ is a valuation overring of $D\}$ for each $I \in \mathbf{F}(D)$; then the mapping $b : \mathbf{F}(D) \to \mathbf{F}(D)$, given by $I \mapsto I^b$, is an *e.a.b.* star operation of finite character on D [5, Theorem 32.5]. Let $*$ be an *e.a.b.* star-operation on D, and define

$$Kr(D, *) = \{\tfrac{f}{g} | f, g \in D[X],\ g \neq 0 \text{ and } c_D(f) \subseteq c_D(g)^*\}.$$

Then $Kr(D, *)$ is a Bezout domain with quotient field $K(X)$, $Kr(D, *) = Kr(D, *_f)$, $I^* = IKr(D, *) \cap K$ for all $I \in \mathbf{f}(D)$ and $fKr(D, *) = c_D(f)Kr(D, *)$ for all $f \in K[X]$ [5, Theorem 32.7 and its proof]. In particular, if $*_1$ and $*_2$ are *e.a.b.* star operations on D, then $Kr(D, *_1) \subseteq Kr(D, *_2)$ if and only if

$(*_1)_f \leq (*_2)_f$ (cf. Lemma 1(3)); hence $Kr(D,b) \subseteq Kr(D,*)$ for any *e.a.b.* star operation $*$ on D [5, Corollary 32.14]. The $Kr(D,*)$ is called *the Kronecker function ring of D with respect to the star-operation* $*$. For more on Kronecker function rings, see [5, Section 32] or Fontana-Loper's interesting survey article [4].

2. Main results

Throughout D is an integral domain with quotient field K, $*$ is a star operation on D, X is an indeterminate over D and $N_* = \{f \in D[X] | c_D(f)^* = D\}$.

We begin this section with a simple lemma, which is essential in the proof of the main result (Theorem 3) of this paper.

Lemma 1. *Let T be an overring of $D[X]$ such that $T \cap K = D$, and let $I^\star = IT \cap K$ for all $I \in \mathbf{F}(D)$.*

(1) *The mapping $\star : \mathbf{F}(D) \to \mathbf{F}(D)$, given by $I \mapsto I^\star$, is a star operation of finite character on D.*
(2) *If T is a Prüfer domain, then $\star$ is an e.a.b. star operation.*
(3) *If $*$ is an e.a.b. star operation on D and if $T = Kr(D,*)$, then $*_f = \star$.*

Proof. (1) Let $0 \neq a \in K$ and $A, B \in \mathbf{F}(D)$. Then (i) $(aD)^\star = (aD)T \cap K = aT \cap K = a(T \cap K) = aD$ and $(aA)^\star = (aAT) \cap K = a(AT \cap K) = aA^\star$, (ii) $A \subseteq AT \cap K = A^\star$ and if $A \subseteq B$ then $A^\star = AT \cap K \subseteq BT \cap K = B^\star$, and (iii) since $A^\star \subseteq AT$, we have $A^\star T = AT$; hence $(A^\star)^\star = A^\star T \cap K = AT \cap K = A^\star$. Thus $\star$ is a star operation on D.

Next, let $A \in \mathbf{F}(D)$. Choose $u \in A^\star$, then $u \in AT$, and hence $u = \sum_{i=1}^n a_i t_i$ for some $a_i \in A$ and $t_i \in T$. Set $I = (a_1, \ldots, a_n)D$, then $I \in \mathbf{f}(D)$ and $I \subseteq A$; so $u \in IT \cap K = I^\star = I^{\star_f} \subseteq A^{\star_f}$. Hence $A^\star \subseteq A^{\star_f}$, and thus $A^\star = A^{\star_f}$. Thus $\star$ is of finite character.

(2) If $A, B, C \in \mathbf{f}(D)$ such that $(AB)^\star \subseteq (AC)^\star$, then $(AT)(BT) = (AB)T = (AB)^\star T \subseteq (AC)^\star T = (AT)(CT)$. Note that AT is finitely generated; so AT is invertible, and hence $BT \subseteq CT$. Thus $B^\star = BT \cap K \subseteq CT \cap K = C^\star$.

(3) If $J \in \mathbf{f}(D)$, then $J^* = JKr(D,*) \cap K = JT \cap K = J^\star$, and hence $I^{*_f} = \cup\{J^* | J \subseteq I \text{ and } J \in \mathbf{f}(D)\} = \cup\{J^\star | J \subseteq I \text{ and } J \in \mathbf{f}(D)\} = I^\star$ for each $I \in \mathbf{F}(D)$. Thus $*_f = \star$. □

Let $I \in \mathbf{f}(D)$. It is well known that if P is a prime ideal of D, then $(ID_P)^{-1} = I^{-1}D_P$ [5, Theorem 4.4] and ID_P is invertible if and only if ID_P is principal [5, Proposition 7.4]. Also, it is clear that I is $*_f$-invertible if and only if $II^{-1} \nsubseteq P$ for all $P \in *_f\text{-Max}(D)$. Thus D is a P∗MD if and only if D_P is a valuation domain for all $P \in *_f\text{-Max}(D)$ [6, Theorem 1.1]. We next review some more characterizations of P∗MDs.

Lemma 2. *The following statements are equivalent for an integral domain D.*

(1) *D is a P∗MD.*
(2) *$D[X]_{N_*}$ is a Prüfer domain.*

(3) *D is a PvMD and $*_f = t$.*
(4) *$*$ is e.a.b. and $Kr(D,*) = D[X]_{N_*}$.*
(5) *Each $*$-linked overring of D is a PvMD.*

Proof. For the proof, see [3, Theorem 3.1, Remark 3.1 and Proposition 3.4] and [2, Theorem 3.7]. □

Let R be an overring of a Prüfer domain D. If Q is a prime ideal of R, then $D_{Q\cap D} \subseteq R_Q$ and $D_{Q\cap D}$ is a valuation domain; hence R_Q is a valuation domain. Note that Prüfer domains are PdMDs and d_f-Max(D) is the set of maximal ideals of R. Thus R is a Prüfer domain by the remark before Lemma 2 (or [5, Theorem 26.1]). We are now ready to prove the main result of this paper.

Theorem 3. *If D is a P$*$MD, then T is an overring of $Kr(D,*)$ if and only if $T \cap K$ is $*$-linked over D, $T \cap K$ is a PvMD and $T = Kr(T \cap K, v)$.*

Proof. Set $T \cap K = R$ and $N_v(R) = \{f \in R[X] | c_R(f)^v = R\}$, and note that R is $*$-linked over D if and only if $N_* \subseteq N_v(R)$. Next, note that $D[X]_{N_*} = Kr(D,*)$ and each $*$-linked overring of D is a PvMD by Lemma 2.

($\Rightarrow$) Let $\star$ be the star operation of finite character on R defined by $A^\star = AT \cap K$ for all $A \in \mathbf{F}(R)$ (cf. Lemma 1).

We first show that R is $*$-linked over D; hence R is a PvMD and $D[X]_{N_*} \subseteq R[X]_{N_v(R)} = Kr(R,v)$ by Lemma 2. To prove this, let $I \in \mathbf{f}(D)$ with $I^* = D$. Then $ID[X]_{N_*} = D[X]_{N_*}$, and hence $T = IT = (IR)T$. Hence $(IR)^\star = (IR)T \cap K = T \cap K = R$, and since $(IR)^\star \subseteq (IR)^v \subseteq R$, we have $(IR)^v = R$.

Next, since T is a Prüfer domain, $\star$ is an *e.a.b.* star operation (Lemma 1(2)). Also, if $f \in K[X]$, then $fKr(D,*) = c_D(f)Kr(D,*)$, and hence $fT = c_D(f)T = c_R(f)T$. Now, if $0 \neq f, g \in R[X]$, then $\frac{f}{g} \in Kr(R,\star) \Leftrightarrow c_R(f)^\star \subseteq c_R(g)^\star \Leftrightarrow fT = c_R(f)T \subseteq c_R(g)T = gT \Leftrightarrow \frac{f}{g} \in T$. Thus $Kr(R,\star) = T$.

If $f \in N_*$, then $D[X]_{N_*} = fD[X]_{N_*} = fKr(D,*) = c_D(f)Kr(D,*)$. Hence $T = fT = c_D(f)T = c_R(f)T$, and thus $c_R(f)^\star = c_R(f)T \cap K = T \cap K = R$. So if we set $N_\star(R) = \{f \in R[X] | c_R(f)^\star = R\}$, then $N_* \subseteq N_\star(R)$, and hence $D[X]_{N_*} \subseteq R[X]_{N_\star(R)}$. Note that $D[X]_{N_*}$ is a Prüfer domain; hence $R[X]_{N_\star(R)}$ is a Prüfer domain, and so $\star_f = t$ by Lemma 2. Thus $T = Kr(R,\star) = Kr(R,v)$.

($\Leftarrow$) Since R is $*$-linked over D, we have $N_* \subseteq N_v(R)$, and hence $D[X]_{N_*} \subseteq R[X]_{N_v(R)}$. Note that $Kr(R,v) = R[X]_{N_v(R)}$ by Lemma 2. Thus T is an overring of $Kr(D,*) = D[X]_{N_*}$. □

Corollary 4. *Let D be a P$*$MD, and let $\{D_\alpha\}$ be the set of $*$-linked overrings of D. The mapping $D_\alpha \mapsto Kr(D_\alpha, v)$ is a bijection from the set $\{D_\alpha\}$ into the set of overrings of $Kr(D,*)$.*

Proof. Let φ be the mapping $D_\alpha \mapsto Kr(D_\alpha, v)$. First, note that each D_α is a PvMD and v is *e.a.b.* on D_α by Lemma 2. So each $Kr(D_\alpha, v)$ is an overring of $Kr(D,*)$ by Theorem 3. Thus φ is well-defined.

Next, if T is an overring of $Kr(D,*)$, then $T \cap K$ is $*$-linked over D and $T = Kr(T \cap K, v)$ by Theorem 3. Thus φ is surjective. Finally, if $\varphi(D_{\alpha_1}) = \varphi(D_{\alpha_1})$, then $Kr(D_{\alpha_1}, v) = Kr(D_{\alpha_2}, v)$, and thus $D_{\alpha_1} = Kr(D_{\alpha_1}, v) \cap K = Kr(D_{\alpha_2}, v) \cap K = D_{\alpha_2}$. Thus φ is injective. □

It is well known that if a nonzero finitely generated ideal I of D is invertible, then $I^v = I$. Hence if D is a Prüfer domain, then $d = t = b$, and thus $Kr(D,v) = Kr(D,b) = Kr(D,d)$. Thus a Prüfer domain D has a unique Kronecker function ring. Clearly, each overring of a Prüfer domain D is d-linked over D; hence by Corollary 4, we have:

Corollary 5 ([5, Proposition 32.19])**.** *Let D be a Prüfer domain, and let $\{D_\alpha\}$ be the set of overrings of D. The mapping $D_\alpha \mapsto Kr(D_\alpha, b)$ is a one-to-one mapping from the set $\{D_\alpha\}$ onto the set of overrings of $Kr(D,b)$.*

The next result is a special case of Corollary 4.

Corollary 6. *Let D be a PvMD, and let $\{D_\alpha\}$ be the set of t-linked overrings of D. The mapping $D_\alpha \mapsto Kr(D_\alpha, v)$ is a bijection from the set $\{D_\alpha\}$ into the set of overrings of $Kr(D,*)$.*

As we noted in the remark before Corollary 5, the $Kr(D_\alpha, b)$ of Corollary 5 is the unique Kronecker function ring of D_α. However, if D is a P$*$MD, then D is a PvMD, and hence v is an *e.a.b.* star operation on D and $Kr(D,*') \subseteq Kr(D,v)$ for any *e.a.b.* star operation $*'$ on D. Hence the $Kr(D_\alpha, v)$ of Corollary 4 is the maximal Kronecker function ring of D_α under inclusion.

We end this paper with an example to show that the $Kr(D_\alpha, v)$ of Corollary 4 need not be a unique Kronecker function ring of D_α.

Lemma 7 ([2, Lemma 3.1])**.** *Let $\{V_\beta\}$ be the set of $*$-linked valuation overrings of an integrally closed domain D, and let $A^{*_c} = \cap_\beta AV_\beta$ for each $A \in \mathbf{F}(D)$. Then $*_c$ is an e.a.b. star operation of finite character on D and $*_c$-*$\mathrm{Max}(D) = *_f$-$\mathrm{Max}(D)$.

Example 8. Let F be a field, $\{X_i | i \in \mathbb{N}\}$ be a set of indeterminates over F, $D = F[\{X_i\}]$ be the polynomial ring, and $\mathcal{P}_i$ be the set of prime ideals P of D with $\mathrm{ht}(P) = i$. Let $I^{*_i} = \cap_{P \in \mathcal{P}_i} ID_P$ for all $I \in \mathbf{F}(D)$, then

(1) D is a UFD, and hence D is a PvMD.
(2) $*_i$ is a star operation on D and $(*_i)_f$-$\mathrm{Max}(D) = \mathcal{P}_i$.
(3) $(*_i)_c$-$\mathrm{Max}(D) = (*_i)_f$-$\mathrm{Max}(D) = \mathcal{P}_i$.
(4) $t = (*_1)_c \gneq (*_2)_c \gneq (*_3)_c \gneq \cdots\cdots \gneq b$.
(5) $Kr(D,v) = Kr(D,(*_1)_c) \supsetneq Kr(D,(*_2)_c) \supsetneq Kr(D,(*_3)_c) \supsetneq \cdots \supsetneq Kr(D,b)$, and thus D has infinitely many Kronecker function rings.

Proof. (1) Clear. (2) See [1, Example 4.5]. (3) This is an immediate consequence of (2) and Lemma 7. (4) Obviously, $*_i \gneq *_{i+1}$; hence each $*_i$-linked overring of D is $*_{i+1}$-linked over D, and thus $(*_i)_c \geq (*_{i+1})_c$. However, since $(*_i)_c$-$\mathrm{Max}(D) \neq (*_{i+1})_c$-$\mathrm{Max}(D)$ by (3), we have $(*_i)_c \neq (*_{i+1})_c$. Thus

$(*_i)_c \gneq (*_{i+1})_c$. (5) By (4), $Kr(D, (*_i)_c) \supseteq Kr(D, (*_{i+1})_c)$. If $Kr(D, (*_i)_c) = Kr(D, (*_{i+1})_c)$, then $(*_i)_c = ((*_i)_c)_f = ((*_{i+1})_c)_f = (*_{i+1})_c$ by Lemma 1(3), which is contrary to (4). Thus $Kr(D, (*_i)_c) \supsetneq Kr(D, (*_{i+1})_c)$. □

References

[1] G. W. Chang, *$*$-Noetherian domains and the ring $D[X]_{N_*}$*, J. Algebra **297** (2006), no. 1, 216–233.

[2] ———, *Prüfer $*$-multiplication domains, Nagata rings, and Kronecker function rings*, J. Algebra **319** (2008), no. 1, 309–319.

[3] M. Fontana, P. Jara, and E. Santos, *Prüfer $*$-multiplication domains and semistar operations*, J. Algebra Appl. **2** (2003), no. 1, 21–50.

[4] M. Fontana and K. A. Loper, *An historical overview of Kronecker function rings, Nagata rings, and related star and semistar operations*, Multiplicative ideal theory in commutative algebra, 169–187, Springer, New York, 2006.

[5] R. Gilmer, *Multiplicative Ideal Theory*, Pure and Applied Mathematics, No. 12. Marcel Dekker, Inc., New York, 1972.

[6] E. G. Houston, S. B. Malik, and J. L. Mott, *Characterizations of $*$-multiplication domains*, Canad. Math. Bull. **27** (1984), no. 1, 48–52.

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