

RICCI CURVATURE OF SUBMANIFOLDS OF AN \mathcal{S} -SPACE FORM

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ABSTRACT. Involving the Ricci curvature and the squared mean curvature, we obtain a basic inequality for a submanifold of an \mathcal{S} -space form tangent to structure vector fields. Equality cases are also discussed. As applications we find corresponding results for almost semi-invariant submanifolds, θ -slant submanifolds, anti-invariant submanifold and invariant submanifolds. A necessary and sufficient condition for a totally umbilical invariant submanifold of an \mathcal{S} -space form to be Einstein is obtained. The inequalities for scalar curvature and a Riemannian invariant Θ_k of different kind of submanifolds of a \mathcal{S} -space form $\bar{M}(c)$ are obtained.

1. Introduction

One of the basic interests in the submanifold theory is to establish simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold. The main intrinsic invariants include the classical curvature invariants namely the Ricci curvature and the scalar curvature. For a unit vector X in an n -dimensional submanifold M of a real space form $R^m(c)$, B. Y. Chen [13] proved the following basic inequality

$$(1) \quad \|H\|^2 \geq \frac{4}{n^2} \{\text{Ric}(X) - (n-1)c\}$$

involving the Ricci curvature Ric and the squared mean curvature $\|H\|^2$ of the submanifold. The inequality (1) drew attention of several authors and they established same kind of inequalities for different kind of submanifolds in ambient manifolds possessing different kind of structures. The submanifolds include mainly invariant, anti-invariant and slant submanifolds, while ambient manifolds include mainly real, complex and Sasakian space forms.

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On the other hand, the concept of framed metric structure unifies the concepts of almost Hermitian and almost contact metric structures. In particular, an \mathcal{S} -structure generalizes Kaehler and Sasakian structure. In [2], D. Blair discusses principal toroidal bundles and generalizes the Hopf fibration to give a canonical example of an \mathcal{S} -manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. An \mathcal{S} -manifold of constant f -sectional curvature c is called an \mathcal{S} -space form $\widetilde{M}(c)$ [7], which generalizes the complex space form and Sasakian space form.

Motivated by the result of Chen in [13], recently in [19], a general basic inequality involving the Ricci curvature and the squared mean curvature of a submanifold in any Riemannian manifold was established and its several applications were presented. Using this inequality, in a previous paper [22], a basic inequality for integral submanifolds of an \mathcal{S} -space form $\widetilde{M}(c)$ was obtained and applied this to recover the already known inequalities for totally real submanifolds in complex space forms and C -totally real submanifolds in Sasakian space forms. This paper is the continuation of the study started in [22]. The paper is organized as follows. In Section 2, we present a brief account of Ricci curvature and scalar curvature in a Riemannian manifold. We also give basic formulas and definitions for a submanifold. Then we recall the result of [19] giving a general basic inequality involving the Ricci curvature and the squared mean curvature of a submanifold in any Riemannian manifold. Section 3 includes basic preliminaries about framed metric manifolds and its submanifolds. The emphasis is given on \mathcal{S} -manifolds and \mathcal{S} -space forms. The submanifolds include almost semi-invariant submanifolds [33], which in particular cases, reduce to θ -slant, invariant and anti-invariant submanifolds. In Section 4, we obtain a basic inequality involving the Ricci curvature and the squared mean curvature of submanifolds of an \mathcal{S} -space form $\widetilde{M}(c)$ tangent to the structure vector fields, and present some of its consequences. In Section 5, we obtain a necessary and sufficient condition for a totally umbilical invariant submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$ to be Einstein. In Section 6, the inequalities for scalar curvature and a Riemannian invariant Θ_k of different kind of submanifolds of a \mathcal{S} -space form are obtained.

2. Ricci curvature of submanifolds

Let M be an n -dimensional Riemannian manifold. Let $\{e_1, \dots, e_k\}$, $2 \leq k \leq n$, be an orthonormal basis of a k -plane section Π_k of $T_p M$. If $k = n$, then $\Pi_n = T_p M$; and if $k = 2$, then Π_2 is a plane section of $T_p M$. For a fixed $i \in \{1, \dots, k\}$, a k -Ricci curvature of Π_k at e_i , denoted $\text{Ric}_{\Pi_k}(e_i)$, is defined by [13]

$$(2) \quad \text{Ric}_{\Pi_k}(e_i) = \sum_{j \neq i}^k K_{ij},$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j . An n -Ricci curvature $\text{Ric}_{T_p M}(e_i)$ is the usual Ricci curvature of e_i , denoted $\text{Ric}(e_i)$. Thus for any orthonormal basis $\{e_1, \dots, e_n\}$ for $T_p M$ and for a fixed $i \in \{1, \dots, n\}$, we have

$$\text{Ric}_{T_p M}(e_i) \equiv \text{Ric}(e_i) = \sum_{j \neq i}^n K_{ij}.$$

The scalar curvature $\tau(\Pi_k)$ of the k -plane section Π_k is given by

$$(3) \quad \tau(\Pi_k) = \sum_{1 \leq i < j \leq k} K_{ij}.$$

Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of Π_k at p under the exponential map at p . We define the *normalized scalar curvature* $\tau_N(\Pi_k)$ of Π_k by [20]

$$(4) \quad \tau_N(\Pi_k) = \frac{2\tau(\Pi_k)}{k(k-1)}.$$

The *normalized scalar curvature* at p is defined as [12]

$$(5) \quad \tau_N(p) = \frac{2\tau(p)}{n(n-1)}.$$

Then, we see that $\tau_N(p) = \tau_N(T_p M)$. The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space $T_p M$ of M at p , that is, $\tau(p) = \tau(T_p M)$. If Π_2 is a plane section and $\{e_1, e_2\}$ is any orthonormal basis for Π_2 , then

$$\text{Ric}_{\Pi_2}(e_1) = \text{Ric}_{\Pi_2}(e_2) = \tau(\Pi_2) = \tau_N(\Pi_2) = K_{12}.$$

Let M be an n -dimensional submanifold of an m -dimensional Riemannian manifold \widetilde{M} equipped with a Riemannian metric \widetilde{g} . We use the inner product notation $\langle \cdot, \cdot \rangle$ for both the metrics \widetilde{g} of \widetilde{M} and the induced metric g on the submanifold M . The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \quad \text{and} \quad \widetilde{\nabla}_X N = -A_N X + \nabla_X^\perp N$$

for all $X, Y \in TM$ and $N \in T^\perp M$, where $\widetilde{\nabla}$, ∇ and ∇^\perp are respectively the Riemannian, induced Riemannian and induced normal connections in \widetilde{M} , M and the normal bundle $T^\perp M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $\langle \sigma(X, Y), N \rangle = \langle A_N X, Y \rangle$. The equation of Gauss is given by

$$(6) \quad \begin{aligned} R(X, Y, Z, W) &= \widetilde{R}(X, Y, Z, W) + \langle \sigma(X, W), \sigma(Y, Z) \rangle \\ &\quad - \langle \sigma(X, Z), \sigma(Y, W) \rangle \end{aligned}$$

for all $X, Y, Z, W \in TM$, where \widetilde{R} and R are the Riemann curvature tensors of \widetilde{M} and M respectively.

The mean curvature vector H is given by $H = \frac{1}{n}\text{trace}(\sigma)$. The submanifold M is *totally geodesic* in \widetilde{M} if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X, Y) = g(X, Y)H$ for all $X, Y \in TM$, then M is *totally umbilical*.

The *relative null space* of M at p is defined by [13]

$$\mathcal{N}_p = \{X \in T_p M \mid \sigma(X, Y) = 0 \text{ for all } Y \in T_p M\},$$

which is also known as the *kernel of the second fundamental form* at p [14].

Now, let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_p M$ and e_r belongs to an orthonormal basis $\{e_{n+1}, \dots, e_m\}$ of the normal space $T_p^\perp M$. We put

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^n \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle.$$

Let K_{ij} and \widetilde{K}_{ij} denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold \widetilde{M} respectively. Thus, we can say that K_{ij} and \widetilde{K}_{ij} are the “intrinsic” and “extrinsic” sectional curvature of the $\text{Span}\{e_i, e_j\}$ at p . In view of (6), we get

$$(7) \quad K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^m (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2).$$

From (7) it follows that

$$(8) \quad 2\tau(p) = 2\widetilde{\tau}(T_p M) + n^2 \|H\|^2 - \|\sigma\|^2,$$

where $\widetilde{\tau}(T_p M)$ denotes the scalar curvature of the n -plane section $T_p M$ in the ambient manifold \widetilde{M} . Thus, we can say that $\tau(p)$ and $\widetilde{\tau}(T_p M)$ are the “intrinsic” and “extrinsic” scalar curvature of the submanifold at p respectively.

We denote the set of unit vectors in $T_p M$ by $T_p^1 M$; thus

$$T_p^1 M = \{X \in T_p M \mid \langle X, X \rangle = 1\}.$$

Now, we recall the following result (cf. [19, Theorem 3.1], [17, Theorem 6.1]).

Theorem 2.1. *Let M be an n -dimensional submanifold of a Riemannian manifold \widetilde{M} . Then the following statements are true.*

(a) *For $X \in T_p^1 M$ we have*

$$(9) \quad \text{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + \widetilde{\text{Ric}}_{(T_p M)}(X),$$

where $\widetilde{\text{Ric}}_{(T_p M)}(X)$ is the n -Ricci curvature of $T_p M$ at $X \in T_p^1 M$ with respect to the ambient manifold \widetilde{M} .

(b) The equality case of (9) is satisfied by $X \in T_p^1 M$ if and only if

$$(10) \quad \sigma(X, X) = \frac{n}{2} H(p) \quad \text{and} \quad \sigma(X, Y) = 0$$

for all $Y \in T_p M$ such that $\langle X, Y \rangle = 0$.

(c) The equality case of (9) holds for all $X \in T_p^1 M$ if and only if either (1) p is a totally geodesic point or (2) $n = 2$ and p is a totally umbilical point.

From Theorem 2.1, we immediately have the following:

Corollary 2.2. *Let M be an n -dimensional submanifold of a Riemannian manifold. For $X \in T_p^1 M$ any two of the following three statements imply the remaining one.*

(a) The mean curvature vector $H(p)$ vanishes.

(b) The unit vector X belongs to the relative null space \mathcal{N}_p .

(c) The unit vector X satisfies the equality case of (9), namely

$$(11) \quad \text{Ric}(X) = \frac{1}{4} n^2 \|H\|^2 + \widetilde{\text{Ric}}_{(T_p M)}(X).$$

3. \mathcal{S} -manifolds and its submanifolds

Let \widetilde{M} be a $(2m+s)$ -dimensional framed metric manifold [36] (also known as framed f -manifold [27] or almost r -contact metric manifold [35]) with a framed metric structure $(f, \xi_\alpha, \eta^\alpha, \widetilde{g})$, $\alpha \in \{1, \dots, s\}$, that is, f is a $(1, 1)$ tensor field defining an f -structure of rank $2m$; ξ_1, \dots, ξ_s are vector fields; η^1, \dots, η^s are 1-forms and \widetilde{g} is a Riemannian metric on \widetilde{M} such that for all $X, Y \in T\widetilde{M}$ and $\alpha, \beta \in \{1, \dots, s\}$

$$(12) \quad f^2 = -I + \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta_\beta^\alpha, \quad f(\xi_\alpha) = 0, \quad \eta^\alpha \circ f = 0,$$

$$(13) \quad \langle fX, fY \rangle = \langle X, Y \rangle - \sum_{\alpha=1}^s \eta^\alpha(X) \eta^\alpha(Y),$$

$$(14) \quad \Omega(X, Y) \equiv \langle X, fY \rangle = -\Omega(Y, X), \quad \langle X, \xi_\alpha \rangle = \eta^\alpha(X),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of the metric \widetilde{g} . A framed metric structure is an \mathcal{S} -structure [2] if the Nijenhuis tensor of f equals $-2d\eta^\alpha \otimes \xi_\alpha$ and $\Omega = d\eta^\alpha$ for all $\alpha \in \{1, \dots, s\}$.

When $s = 1$, a framed metric structure is an almost contact metric structure, while an \mathcal{S} -structure is a Sasakian structure. If a framed metric structure on \widetilde{M} is an \mathcal{S} -structure, then it is known [2] that

$$(15) \quad (\widetilde{\nabla}_X f)Y = \sum_{\alpha=1}^s (\langle fX, fY \rangle \xi_\alpha + \eta^\alpha(Y) f^2 X),$$

$$(16) \quad \widetilde{\nabla} \xi_\alpha = -f, \quad \alpha \in \{1, \dots, s\}.$$

The converse may also be proved. In case of Sasakian structure (that is, $s = 1$), (15) implies (16). For $s > 1$, examples of \mathcal{S} -structures are given in [2], [3] and [5]. Thus, the bundle space of a principal toroidal bundles over a Kaehler manifold with certain conditions is an \mathcal{S} -manifold. Thus, a generalization of the Hopf fibration $\pi' : S^{2m+1} \rightarrow PC^m$ is a canonical example of an \mathcal{S} -manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. Also, every n -dimensional Lie group G admits a framed f -structure of rank $2k$, where k is any positive integer less than $(n + 1)/2$ (cf. [21]).

A plane section in $T_p \widetilde{M}$ is a f -section if there exists a vector $X \in T_p \widetilde{M}$ orthogonal to ξ_1, \dots, ξ_s such that $\{X, fX\}$ span the section. The sectional curvature of a f -section is called a f -sectional curvature. It is known that [7] in an \mathcal{S} -manifold of constant f -sectional curvature c

$$(17) \quad \begin{aligned} \widetilde{R}(X, Y)Z &= \sum_{\alpha, \beta} \{ \eta^\alpha(X) \eta^\beta(Z) f^2 Y - \eta^\alpha(Y) \eta^\beta(Z) f^2 X \\ &\quad - \langle fX, fZ \rangle \eta^\alpha(Y) \xi_\beta + \langle fY, fZ \rangle \eta^\alpha(X) \xi_\beta \} \\ &\quad + \frac{c+3s}{4} \{ -\langle fY, fZ \rangle f^2 X + \langle fX, fZ \rangle f^2 Y \} \\ &\quad + \frac{c-s}{4} \{ \langle X, fZ \rangle fY - \langle Y, fZ \rangle fX + 2 \langle X, fY \rangle fZ \} \end{aligned}$$

for all $X, Y, Z \in T\widetilde{M}$, where \widetilde{R} is the curvature tensor of \widetilde{M} . An \mathcal{S} -manifold of constant f -sectional curvature c is called an \mathcal{S} -space form $\widetilde{M}(c)$.

When $s = 1$, an \mathcal{S} -space form $\widetilde{M}(c)$ reduces to a Sasakian space form $\widetilde{M}(c)$ [4] and (17) reduces to

$$\begin{aligned} \widetilde{R}(X, Y)Z &= \frac{c+3}{4} \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \} \\ &\quad + \frac{c-1}{4} \{ \langle X, fZ \rangle fY - \langle Y, fZ \rangle fX + 2 \langle X, fY \rangle fZ \\ &\quad + \eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X \\ &\quad + \langle X, Z \rangle \eta(Y) \xi - \langle Y, Z \rangle \eta(X) \xi \}, \end{aligned}$$

where $\xi_1 \equiv \xi$ and $\eta^1 \equiv \eta$.

For a submanifold M of a framed metric manifold \widetilde{M} , we put

$$fX \equiv PX + FX, \quad X, PX \in TM, \quad FX \in T^\perp M,$$

$$fN \equiv tN + vN, \quad (N, vN \in T^\perp M, \quad tN \in TM),$$

and we say that M is

- (1) *f*-invariant [24] if $F = 0$,
- (2) *invariant* [24] if it is *f*-invariant and $\xi_1, \dots, \xi_s \in TM$,
- (3) *anti-f*-invariant [34] if $P = 0$, and
- (4) *anti-invariant* [36] if it is anti-*f*-invariant and $\xi_1, \dots, \xi_s \in TM$.

Let M be a submanifold of a framed metric manifold such that ξ_α 's $\in TM = \mathcal{E} \oplus \mathcal{E}^\perp$, where \mathcal{E} denotes the distribution spanned by ξ_1, \dots, ξ_r and \mathcal{E}^\perp is the complementary orthogonal distribution to \mathcal{E} in M .

Now, we recall the definition of almost semi-invariant submanifold [29, 33] of a framed metric manifold.

Definition. A submanifold M of a framed metric manifold \widetilde{M} with all ξ_α 's $\in TM$, is called an almost semi-invariant submanifold of \widetilde{M} if there are k distinct functions $\lambda_1, \dots, \lambda_k$ defined on M with values in the open interval $(0, 1)$ such that TM is decomposed as P -invariant mutually orthogonal differentiable distributions given by

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \dots \oplus \mathcal{D}^{\lambda_k} \oplus \mathcal{E},$$

where $\mathcal{D}_p^1 = \ker(F|_{\mathcal{E}^\perp})_p$, $\mathcal{D}_p^0 = \ker(P|_{\mathcal{E}^\perp})_p$ and

$$\mathcal{D}_p^{\lambda_i} = \ker(P^2|_{\mathcal{E}^\perp} + \lambda_i^2(p)I)_p, \quad i \in \{1, \dots, k\}.$$

If in addition, each λ_i is constant, then M is called an *almost semi-invariant* submanifold*.

An almost semi-invariant submanifold becomes

- (1) a *semi-invariant submanifold* [23] or *CR submanifold* [25, 9] if $k = 0$.
- (2) an *invariant submanifold* [24] if $k = 0$ and $\mathcal{D}^0 = \{0\}$.
- (3) an *anti-invariant submanifold* [8] if $k = 0$ and $\mathcal{D}_p^1 = \{0\}$.
- (4) a *generic CR submanifold* [1] if $k = 0$ and $J\mathcal{D}^0 = T^\perp M$.
- (5) a (*proper*) *θ -slant submanifold* [11] if $\mathcal{D}^1 = \{0\} = \mathcal{D}^0$, $k = 1$ and λ_1 is constant. In this case, we have $TM = \mathcal{D}^{\lambda_1} \oplus \{\xi\}$ and the slant angle θ is given by $\lambda_1 = \cos \theta$.
- (6) a *semi-slant submanifold* if $\mathcal{D}^1 \neq \{0\}$, $\mathcal{D}^0 = \{0\}$, $k = 1$ and λ_1 is constant. In this case, we have $TM = \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_1} \oplus \{\xi\}$, and the slant angle θ of the distribution \mathcal{D}^{λ_1} is given by $\lambda_1 = \cos \theta$.
- (7) a *bi-slant submanifold* if $\mathcal{D}^1 = \{0\} = \mathcal{D}^0$, $k = 2$ and λ_1, λ_2 are constants. In this case, we have $TM = \mathcal{D}^{\lambda_1} \oplus \mathcal{D}^{\lambda_2} \oplus \{\xi\}$, and the slant angles θ_i of the distributions \mathcal{D}^{λ_i} are given by $\lambda_i = \cos \theta_i$.

Thus, the definition of almost semi-invariant submanifold seems to be the most logical generalized definition. If M is an almost semi-invariant submanifold of a framed metric manifold \widetilde{M} , then for $X \in TM$ we may write

$$X = U^1 X + U^0 X + U^{\lambda_1} X + \dots + U^{\lambda_k} X + \eta^\alpha(X) \xi_\alpha,$$

where $U^1, U^0, U^{\lambda_1}, \dots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}$ respectively. Then, it follows that

$$\|X\|^2 = \|U^1 X\|^2 + \|U^0 X\|^2 + \|U^{\lambda_1} X\|^2 + \dots + \|U^{\lambda_k} X\|^2 + \sum_{\alpha=1}^s \eta^\alpha(X)^2.$$

We also have

$$P^2X = -U^1X - \lambda_1^2(U^{\lambda_1}X) - \cdots - \lambda_k^2(U^{\lambda_k}X),$$

which implies that

$$(18) \quad \|PX\|^2 = \langle PX, PX \rangle = -\langle P^2X, X \rangle = \sum_{\lambda \in \{1, \lambda_1, \dots, \lambda_k\}} \lambda^2 \|U^\lambda X\|^2.$$

In particular, if M is an n -dimensional θ -slant submanifold, then $\lambda_1^2 = \cos^2 \theta$ and we have

$$(19) \quad \|PX\|^2 = \cos^2 \theta \|U^{\lambda_1}X\|^2 = \cos^2 \theta \left(\|X\|^2 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right).$$

If $X \in T_p^1M$, then (19) becomes

$$(20) \quad \|PX\|^2 = \cos^2 \theta \left(1 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right).$$

Moreover, if the unit vector $X \in T_p^1M$ is orthogonal to the structure vector field ξ , then

$$(21) \quad \|PX\|^2 = \cos^2 \theta.$$

4. Ricci curvature of submanifolds of \mathcal{S} -space forms

We begin with the following:

Lemma 4.1. *Let M be a submanifold of a framed metric manifold satisfying (16). If at least one structure vector field ξ_α is tangent to the submanifold and $p \in M$ is a totally umbilical point, then p must be a totally geodesic point, and hence the tangent space T_pM is f -invariant, that is, $f(T_pM) \subset T_pM$.*

Proof. In view of (16) we have

$$(22) \quad \nabla_X \xi_\alpha = -PX \quad \text{and} \quad \sigma(X, \xi_\alpha) = -FX$$

for all $X \in TM$, where PX and FX are the tangential and the normal parts of fX respectively. Let $p \in M$ be a totally umbilical point. Then, we get

$$H = \langle \xi_\alpha, \xi_\alpha \rangle H = \sigma(\xi_\alpha, \xi_\alpha) = -F\xi_\alpha = 0,$$

which shows that $\sigma(X, Y) = 0$ for all $X, Y \in T_pM$, that is, p is a totally geodesic point. Since p is a totally geodesic point, therefore we have

$$0 = \sigma(X, \xi_\alpha) = -FX$$

for all $X \in TM$, which shows that $f(T_pM) \subset T_pM$. □

Consequently, we have the following:

Proposition 4.2. *If M is a totally umbilical submanifold of a framed metric manifold satisfying (16) such that at least one structure vector field ξ_α is tangent to the submanifold, then M is a totally geodesic f -invariant submanifold. In particular, if all structure vector fields are tangent to M , then it is invariant.*

Now, we have the following theorem.

Theorem 4.3. *Let M be an n -dimensional submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$ such that ξ_1, \dots, ξ_s are tangent to the submanifold. Then the following statements are true.*

(a) *For every unit vector X in $T_p M$, it follows that*

$$(23) \quad \text{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + \frac{3(c-s)}{4} \|PX\|^2 + (n-s) \left(\sum_{\alpha=1}^s \eta^\alpha(X) \right)^2 + \frac{1}{4} (4s(n-s) + (c-s)(n-1-s)) \left(1 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right).$$

(b) *The equality case of (23) is satisfied by a unit vector X in $T_p M$ if and only if (10) is true. If $H(p) = 0$, then a unit vector $X \in T_p M$ satisfies equality in (23) if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of (23) holds for all unit vectors $X \in T_p M$ if and only if $f(T_p M) \subset T_p M$ and p is a totally geodesic point.*

Proof. Let M be an n -dimensional submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$ such that ξ_1, \dots, ξ_s are tangent to the submanifold. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_p M$. Then from (17) and (14) it follows that

$$\begin{aligned} \widetilde{K}_{ij} &= \frac{3(c-s)}{4} \langle Pe_i, e_j \rangle^2 - 2 \langle fe_j, fe_i \rangle \sum_{\alpha=1}^s \eta^\alpha(e_i) \sum_{\beta=1}^s \eta^\beta(e_j) \\ &\quad + \langle fe_i, fe_i \rangle \left(\sum_{\gamma=1}^s \eta^\gamma(e_j) \right)^2 + \langle fe_j, fe_j \rangle \left(\sum_{\gamma=1}^s \eta^\gamma(e_i) \right)^2 \\ &\quad + \frac{c+3s}{4} \left\{ \langle fe_i, fe_i \rangle \langle fe_j, fe_j \rangle - \langle fe_i, fe_j \rangle^2 \right\}. \end{aligned}$$

Using (13) in the above equation, we get

$$(24) \quad \begin{aligned} \widetilde{K}_{ij} &= \frac{3(c-s)}{4} \langle Pe_i, e_j \rangle^2 \\ &\quad + 2 \sum_{\gamma=1}^s \eta^\gamma(e_i) \eta^\gamma(e_j) \sum_{\alpha=1}^s \eta^\alpha(e_i) \sum_{\beta=1}^s \eta^\beta(e_j) \\ &\quad + \left(1 - \sum_{\gamma=1}^s \eta^\gamma(e_i)^2 \right) \left(\sum_{\gamma=1}^s \eta^\gamma(e_j) \right)^2 \end{aligned}$$

$$\begin{aligned}
& + \left(1 - \sum_{\gamma=1}^s \eta^\gamma(e_j)^2\right) \left(\sum_{\gamma=1}^s \eta^\gamma(e_i)\right)^2 \\
& + \frac{c+3s}{4} \left\{ \left(1 - \sum_{\gamma=1}^s \eta^\gamma(e_i)^2\right) \left(1 - \sum_{\gamma=1}^s \eta^\gamma(e_j)^2\right) \right. \\
& \quad \left. - \left(\sum_{\gamma=1}^s \eta^\gamma(e_i) \eta^\gamma(e_j)\right)^2 \right\}.
\end{aligned}$$

Using (24) in $\widetilde{\text{Ric}}_{(T_p M)}(e_i) = \sum_{j \neq i}^n \widetilde{K}_{ij}$, we obtain

$$\begin{aligned}
\widetilde{\text{Ric}}_{(T_p M)}(e_i) &= \frac{3(c-s)}{4} \sum_{j \neq i} \langle P e_i, e_j \rangle^2 \\
&+ 2 \sum_{\alpha=1}^s \eta^\alpha(e_i) \sum_{j \neq i} \left(\sum_{\gamma=1}^s \eta^\gamma(e_i) \eta^\gamma(e_j) \sum_{\beta=1}^s \eta^\beta(e_j) \right) \\
&+ \left(1 - \sum_{\gamma=1}^s \eta^\gamma(e_i)^2\right) \sum_{j \neq i} \left(\sum_{\gamma=1}^s \eta^\gamma(e_j) \right)^2 \\
&+ \left(\sum_{\gamma=1}^s \eta^\gamma(e_i) \right)^2 \sum_{j \neq i} \left(1 - \sum_{\gamma=1}^s \eta^\gamma(e_j)^2\right) \\
&+ \frac{c+3s}{4} \left\{ \left(1 - \sum_{\gamma=1}^s \eta^\gamma(e_i)^2\right) \sum_{j \neq i} \left(1 - \sum_{\gamma=1}^s \eta^\gamma(e_j)^2\right) \right. \\
&\quad \left. - \sum_{j \neq i} \left(\sum_{\gamma=1}^s \eta^\gamma(e_i) \eta^\gamma(e_j) \right)^2 \right\}.
\end{aligned}$$

After simplifying the above equation, we get

$$\begin{aligned}
\widetilde{\text{Ric}}_{(T_p M)}(e_i) &= \frac{3(c-s)}{4} \|P e_i\|^2 \\
&+ 2 \sum_{\alpha=1}^s \eta^\alpha(e_i) \sum_{j=1}^n \sum_{\beta=1}^s \eta^\beta(e_j) \sum_{\gamma=1}^s \eta^\gamma(e_j) \eta^\gamma(e_i) \\
&+ \left(1 - \sum_{\gamma=1}^s \eta^\gamma(e_i)^2\right) \sum_{j=1}^n \left(\sum_{\gamma=1}^s \eta^\gamma(e_j) \right)^2 \\
&+ (n-2-s) \left(\sum_{\gamma=1}^s \eta^\gamma(e_i) \right)^2
\end{aligned}$$

$$+ \frac{c+3s}{4} \left\{ (n-1-s) - (n-2-s) \sum_{\gamma=1}^s \eta^\gamma(e_i)^2 - \sum_{j=1}^n \left(\sum_{\gamma=1}^s \eta^\gamma(e_i) \eta^\gamma(e_j) \right)^2 \right\}.$$

Now, for any unit vector $X \in T_p M$, from the above equation we obtain

$$\begin{aligned} \widetilde{\text{Ric}}_{(T_p M)}(X) &= \frac{3(c-s)}{4} \|PX\|^2 \\ &+ 2 \sum_{\alpha=1}^s \eta^\alpha(X) \sum_{j=1}^n \sum_{\beta=1}^s \eta^\beta(e_j) \sum_{\gamma=1}^s \eta^\gamma(e_j) \eta^\gamma(X) \\ &+ \left(1 - \sum_{\gamma=1}^s \eta^\gamma(X)^2 \right) \sum_{j=1}^n \left(\sum_{\gamma=1}^s \eta^\gamma(e_j) \right)^2 \\ &+ (n-2-s) \left(\sum_{\gamma=1}^s \eta^\gamma(X) \right)^2 \\ &+ \frac{c+3s}{4} \left\{ (n-1-s) - (n-2-s) \sum_{\gamma=1}^s \eta^\gamma(X)^2 - \sum_{j=1}^n \left(\sum_{\gamma=1}^s \eta^\gamma(X) \eta^\gamma(e_j) \right)^2 \right\}. \end{aligned}$$

Since, the above expression is independent of the choice of orthonormal bases for $T_p M$, therefore choosing an orthonormal basis $\{e_1, \dots, e_{n-s}, \xi_1, \dots, \xi_s\}$ for $T_p M$, from the above expression, we get

$$\begin{aligned} &\widetilde{\text{Ric}}_{(T_p M)}(X) \\ &= \frac{3(c-s)}{4} \|PX\|^2 + 2 \left(\sum_{\alpha=1}^s \eta^\alpha(X) \right)^2 \\ &+ s \left(1 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right) + (n-2-s) \left(\sum_{\alpha=1}^s \eta^\alpha(X) \right)^2 \\ &+ \frac{c+3s}{4} \left\{ (n-1-s) - (n-2-s) \sum_{\alpha=1}^s \eta^\alpha(X)^2 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right\}, \end{aligned}$$

which leads to

$$\begin{aligned}
(25) \quad & \widetilde{\text{Ric}}_{(T_p M)}(X) \\
&= \frac{3(c-s)}{4} \|PX\|^2 + (n-s) \left(\sum_{\alpha=1}^s \eta^\alpha(X) \right)^2 \\
&\quad + \frac{1}{4} (4s(n-s) + (c-s)(n-1-s)) \left(1 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right).
\end{aligned}$$

Now, the inequality (23) follows from (9) and (25). Proof of **(b)** is as usual. Next, in view of the statement **(c)** of Theorem 2.1 and Lemma 4.1, the statement **(c)** follows easily. \square

We immediately have the following:

Corollary 4.4. *Let M be an n -dimensional submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$ such that ξ_1, \dots, ξ_s are tangent to the submanifold. Then for every unit vector X in $T_p M$, which is perpendicular to all the structure vectors, it follows that*

$$(26) \quad 4 \text{Ric}(X) \leq n^2 \|H\|^2 + (c-s) (3 \|PX\|^2 - 1) + (c+3s)(n-s).$$

When $s = 1$, from Theorem 4.3, we get the following:

Corollary 4.5 (Theorem 4.3, [20]). *Let M be an n -dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ such that the structure vector field ξ is tangent to the submanifold M . Then, the following statements are true.*

(a) *For each $X \in T_p^1 M$ we have*

$$(27) \quad 4 \text{Ric}(X) \leq n^2 \|H\|^2 + 4(n-1) + (c-1) \{3 \|PX\|^2 + (n-2)(1 - \eta(X)^2)\}.$$

(b) *A vector $X \in T_p^1 M$ satisfies the equality case of (27) if and only if (10) is true. If $H(p) = 0$, then $X \in T_p^1 M$ satisfies the equality case of (27) if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of (27) holds for all $X \in T_p^1 M$ if and only if $\varphi(T_p M) \subset T_p M$ and p is a totally geodesic point.*

The above result is an improvement of Theorem 3.2 of [31].

One can see that the inequality (26) is the inequality (3.8) of [16] with a change in the dimension of the submanifold. In fact, the dimension of the submanifold in [16] is taken to be $(n+s)$ instead of n . The inequality (27) is same as the inequality (9) of Theorem 3.2 of [31] except a change in dimension of the submanifold. In fact, the dimension of the submanifold in [31] is taken as $(n+1)$ instead of n .

Now, from Theorem 4.3 we immediately have the following corollary.

Corollary 4.6. *Let M be an n -dimensional submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$ such that all the structure vector fields ξ 's are tangent to M . Then, the following statements are true.*

(a) If M is an almost semi-invariant submanifold, then for $X \in T_p^1 M$ we have

$$(28) \quad \text{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + \frac{3(c-s)}{4} \sum_{\lambda \in \{1, \lambda_1, \dots, \lambda_k\}} \lambda^2 \|U^\lambda X\|^2 \\ + (n-s) \left(\sum_{\alpha=1}^s \eta^\alpha(X) \right)^2 \\ + \frac{1}{4} (4s(n-s) + (c-s)(n-1-s)) \left(1 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right),$$

where $U^1, U^{\lambda_1}, \dots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^1, \mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}$ respectively.

(b) If M is a θ -slant submanifold, then for $X \in T_p^1 M$ we have

$$(29) \quad \text{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + \frac{3(c-s)}{4} \cos^2 \theta + (n-s) \left(\sum_{\alpha=1}^s \eta^\alpha(X) \right)^2 \\ + \frac{1}{4} (4s(n-s) + (c-s)(n-1-s)) \left(1 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right).$$

(c) If M is an anti-invariant submanifold, then for $X \in T_p^1 M$, we have

$$(30) \quad \text{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + (n-s) \left(\sum_{\alpha=1}^s \eta^\alpha(X) \right)^2 \\ + \frac{1}{4} (4s(n-s) + (c-s)(n-1-s)) \left(1 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right).$$

(d) The equality cases of (28), (29) and (30) are satisfied by $X \in T_p^1 M$ if and only if (10) is true. If $H(p) = 0$, then $X \in T_p^1 M$ satisfies the equality cases of (28), (29) and (30) if and only if $X \in \mathcal{N}_p$.

Proof. Using (18) in the inequality (23) we get (28). Next, using (20) in the inequality (23) we get the inequality (29). Putting $\theta = \pi/2$ in (29) we get (30). Rest of the proof is straightforward. \square

Now, we prove the following:

Theorem 4.7. Let M be an n -dimensional non-invariant almost semi-invariant submanifold of an \mathcal{S} -space form $\tilde{M}(c)$. Then, for any unit vector $X \in \mathcal{E}_p^\perp$

$$(31) \quad \text{Ric}(X) < \frac{1}{4} \left\{ n^2 \|H\|^2 + (c-s) \left(3 \sum_{i=1}^k \lambda_i^2 \|U^{\lambda_i} X\|^2 - 1 \right) + (c+3s)(n-s) \right\},$$

where $U^{\lambda_1}, \dots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}$ respectively. In particular, if M is a non-invariant θ -slant submanifold, then

$$(32) \quad \text{Ric}(X) < \frac{1}{4} \{n^2 \|H\|^2 + (3 \cos^2 \theta - 1)(c - s) + (c + 3s)(n - s)\}.$$

In particular, if M is anti-invariant, then

$$(33) \quad \text{Ric}(X) < \frac{1}{4} \{n^2 \|H\|^2 - (c - s) + (c + 3s)(n - s)\}.$$

Proof. By using $\eta^\alpha(X) = 0$ in (28), for a unit vector $X \in \mathcal{E}_p^\perp$ we get

$$(34) \quad \text{Ric}(X) \leq \frac{1}{4} \left\{ n^2 \|H\|^2 + (c - s) \left(3 \sum_{i=1}^k \lambda_i^2 \|U^{\lambda_i} X\|^2 - 1 \right) + (c + 3s)(n - s) \right\}.$$

If possible, let equality case of (34) be satisfied by a unit vector $X \in \mathcal{E}_p^\perp$. Then, it follows that $\sigma(X, \xi) = 0$, which in view of (22), gives $FX = 0$, a contradiction. Thus, (31) is proved. Other two inequalities are straightforward. \square

Since the right hand side of the inequality (31) is not attained by any unit vector in \mathcal{E}_p^\perp , $p \in M$, therefore we recall the following result, which is a combination of Theorems 3.1 and 3.3 of [16]). But first we need the notion of totally f -geodesic and totally f -umbilical submanifolds. An n -dimensional submanifold M of an \mathcal{S} -space form $\widetilde{M}(c)$, tangent to the structure vector fields, is known to be (i) a totally f -geodesic submanifold if $\sigma(X, Y) = 0$ for all vector fields $X, Y \in \mathcal{E}^\perp$, and (ii) a totally f -umbilical submanifold if $\sigma(X, Y) = \frac{n-s}{n} \langle X, Y \rangle H$ for all vector fields $X, Y \in \mathcal{E}^\perp$.

Theorem 4.8. *Let M be an n -dimensional submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$, tangent to the structure vector fields. Then, for any unit vector field $X \in \mathcal{E}^\perp$*

$$(35) \quad \text{Ric}(X) \leq \frac{1}{4} \{n^2 \|H\|^2 + (3c + s) \|PX\|^2 + (c + 3s)(n - 1 - s)\},$$

where $U^{\lambda_1}, \dots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}$ respectively. The equality case of (35) holds for all unit vector fields in \mathcal{E}^\perp if and only if either M is a totally f -geodesic submanifold or $n = 2 + s$ and M is a totally f -umbilical submanifold.

Now, we apply Theorem 4.8 to find corresponding results for non-invariant almost semi-invariant submanifolds of an \mathcal{S} -space form.

Theorem 4.9. *Let M be an n -dimensional submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$, tangent to the structure vector fields and X be any unit vector field belonging to \mathcal{E}^\perp . Then, the following statements are true.*

(a) If M is a non-invariant almost semi-invariant submanifold, then

$$(36) \quad \text{Ric}(X) \leq \frac{1}{4} \left\{ n^2 \|H\|^2 + (3c + s) \sum_{i=1}^k \lambda_i^2 \|U^{\lambda_i} X\|^2 + (c + 3s)(n - 1 - s) \right\},$$

where $U^{\lambda_1}, \dots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}$ respectively.

(b) If M is a non-invariant θ -slant submanifold, then

$$(37) \quad \text{Ric}(X) \leq \frac{1}{4} \{ n^2 \|H\|^2 + (3c + s) \cos^2 \theta + (c + 3s)(n - 1 - s) \}.$$

(c) If M is an anti-invariant submanifold, then

$$(38) \quad \text{Ric}(X) \leq \frac{1}{4} \{ n^2 \|H\|^2 + (c + 3s)(n - 1 - s) \}.$$

(d) The equality cases of (36), (37), and (38) holds for all unit vector fields in \mathcal{E}^\perp if and only if either M is a totally f -geodesic submanifold or $n = 2 + s$ and M is a totally f -umbilical submanifold.

Proof. Using (18) and $\mathcal{D}^1 = \{0\}$ in (35) we get (36). Similarly, using (21) in (35) we get (37). Putting $\theta = 0$ in (37) we get (38). Rest of the proof is straightforward. \square

In fact, the equations (37) and (38) are identical with the equations (3.9) and (3.11) in [16] respectively, except a change in the dimension of the submanifold.

5. Totally umbilical invariant submanifold

In this section, we find a necessary and sufficient condition for a totally umbilical invariant submanifold to be an Einstein manifold. In fact, we prove the following:

Theorem 5.1. *A totally umbilical invariant submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$ is Einstein if and only if the ambient \mathcal{S} -space form reduces to Sasakian space form $\widetilde{M}(1)$ (that is, $s = 1 = c$).*

Proof. Let M be an n -dimensional invariant submanifold of an \mathcal{S} -manifold \widetilde{M} . Then all structure vector fields ξ_1, \dots, ξ_s are tangent to the submanifold and M is a minimal \mathcal{S} -manifold ([24, Proposition 2.4], [6, Proposition 2.2(i)]). Since M is invariant, for any unit vector X in the submanifold, we get

$$(39) \quad \|PX\|^2 = 1 - \sum_{\gamma=1}^s \eta^\gamma(X)^2.$$

If M is assumed to be totally umbilical also, by minimality it becomes totally geodesic. Now, if the ambient \mathcal{S} -manifold \widetilde{M} is an \mathcal{S} -space form $\widetilde{M}(c)$, then

in view of Theorem 4.3, the submanifold M satisfies the equality case of (23). Thus, using $H = 0$ and (39) in the equality case of (23) we get

$$(40) \quad \begin{aligned} \text{Ric}(X) = & \frac{3(c-s)}{4} \|PX\|^2 + (n-s) \left(\sum_{\alpha=1}^s \eta^\alpha(X) \right)^2 \\ & + \frac{1}{4} (4s(n-s) + (c-s)(n-1-s)) \left(1 - \sum_{\alpha=1}^s \eta^\alpha(X)^2 \right), \end{aligned}$$

where $\{e_1, \dots, e_n\}$ is any orthonormal basis of the tangent space $T_p M$, $p \in M$. Now suppose that M is Einstein also. Since there is no Einstein \mathcal{S} -manifold if the number of structure vector fields is greater than one [24, Proposition 1.3], therefore $s = 1$ and the ambient \mathcal{S} -space form reduces to Sasakian space form $\widetilde{M}(c)$. Using $s = 1$ in (40), we get

$$(41) \quad 4\text{Ric}(X) = 4(n-1) + (c-1)(n+1)(1 - \eta^1(X)^2)$$

for all unit vectors $X \in T_p M$, $p \in M$. Since M is assumed to be Einstein, then for any unit vector $X \in T_p M$, $p \in M$, orthogonal to ξ_1 , from (41) it follows that

$$0 = \text{Ric}(X) - \text{Ric}(\xi_1) = \frac{1}{4}(c-1)(n+1),$$

which shows that $c = 1$.

Conversely, it is easy to see that a totally umbilical invariant submanifold of a Sasakian space form $\widetilde{M}(1)$ is always Einstein. \square

6. Scalar curvature

We begin with the following:

Theorem 6.1 (Theorem 4.2, [17]). *For an n -dimensional submanifold M in an m -dimensional Riemannian manifold, at each point $p \in M$, we have*

$$(42) \quad \tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \widetilde{\tau}(T_p M)$$

with equality if and only if p is a totally umbilical point.

Remark 6.2. Using an inequality for roots of a polynomial, B. Suceava proved Theorem 6.1 for a hypersurface (see Proposition 1, [28]). Then in general codimension case, he proved Theorem 6.1 with out any information about equality case (see Proposition 2, [28]). The proof of Theorem 6.1 given in [17, Theorem 4.2] is very short and also includes the necessary and sufficient condition for the equality case.

For each integer k , $2 \leq k \leq n$, the Riemannian invariant Θ_k on an n -dimensional Riemannian manifold M is defined by [15]

$$(43) \quad \Theta_k(p) = \left(\frac{1}{k-1} \right) \inf_{\Pi_k, X} \text{Ric}_{\Pi_k}(X), \quad p \in M,$$

where Π_k runs over all k -plane sections in $T_p M$ and X runs over all unit vectors in Π_k . We denote by $\Pi_{i_1 \dots i_k}$ the k -plane section spanned by e_{i_1}, \dots, e_{i_k} . From (2) and (3), it follows that

$$(44) \quad \tau(\Pi_{i_1 \dots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} \text{Ric}_{\Pi_{i_1 \dots i_k}}(e_i),$$

and

$$(45) \quad \tau(p) = \frac{1}{C_{k-2}^{n-2}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \tau(\Pi_{i_1 \dots i_k}).$$

Combining (43), (44), and (45), one obtains

$$(46) \quad \tau(p) \geq \frac{n(n-1)}{2} \Theta_k(p).$$

In view of the equations (42) and (46), we have the following relationship between the Riemannian invariant Θ_k and the squared mean curvature for submanifolds of a Riemannian manifold.

Theorem 6.3 (Theorem 6.4, [20]). *Let M be an n -dimensional submanifold of a Riemannian manifold. Then, for each integer k , $2 \leq k \leq n$, and every point $p \in M$, we have*

$$(47) \quad \Theta_k(p) \leq \|H\|^2 + \tilde{\tau}_N(T_p M),$$

where $\tilde{\tau}_N(T_p M)$ is the normalized scalar curvature of the n -plane section $T_p M$ in the ambient submanifold.

Now, we study scalar curvature of submanifolds of \mathcal{S} -space forms. In fact, we have the following:

Theorem 6.4. *Let M be an n -dimensional submanifold of a \mathcal{S} -space form $\widetilde{M}(c)$, such that the structure vector field ξ is tangent to M . Then at each point $p \in M$, we have*

$$(48) \quad \tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \frac{1}{8} \left\{ 3(c-s) \|P\|^2 + (n-s)(8s + (c+3s)(n-1-s)) \right\}.$$

with equality if and only if p is a totally umbilical point.

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_p M$. The squared norm of P at $p \in M$ is defined to be $\|P\|^2 = \sum_{i,j=1}^n \langle P e_i, e_j \rangle^2$. Then, using $2\tilde{\tau}(T_p M) = \sum_{i=1}^n \widetilde{\text{Ric}}_{(T_p M)}(e_i)$ in (25), we get

$$(49) \quad \tilde{\tau}(T_p M) = \frac{1}{8} \left\{ 3(c-s) \|P\|^2 + (n-s)(8s + (c+3s)(n-1-s)) \right\}.$$

Using (49) in (42) gives (48). \square

Next, using (49) in (47) gives the following

Theorem 6.5. *Let M be an n -dimensional submanifold of a \mathcal{S} -space form $\widetilde{M}(c)$ such that $\xi \in TM$. Then, for each integer k , $2 \leq k \leq n$, and every point $p \in M$, we have*

$$(50) \quad \Theta_k(p) \leq \|H\|^2 + \frac{1}{4n(n-1)} \left\{ 3(c-s)\|P\|^2 + (n-s)(8s + (c+3s)(n-1-s)) \right\}.$$

Now, we have the following:

Corollary 6.6. *Let M be an n -dimensional θ -slant submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$. Then the following statements are true.*

(a) *At each point $p \in M$ it follows that*

$$(51) \quad \tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \frac{(n-s)}{8} \left\{ 3(c-s)\cos^2\theta + (8s + (c+3s)(n-1-s)) \right\}$$

with equality if and only if p is a totally umbilical point.

(b) *For each integer k , $2 \leq k \leq n$, and every point $p \in M$, we have*

$$(52) \quad \Theta_k(p) \leq \|H\|^2 + \frac{(n-s)}{4n(n-1)} \left\{ 3(c-s)\cos^2\theta + (8s + (c+3s)(n-1-s)) \right\}.$$

Proof. Using $\|P\|^2 = (n-s)\cos^2\theta$ in (48) and (50) gives (51) and (52) respectively. \square

Using $\theta = 0$ and $\|H\|^2 = 0$ in (51) and (52), we get similar results for invariant submanifolds. Similarly, using $\theta = 0$ in (51) and (52), we get corresponding results for anti-invariant submanifolds.

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