ON CERTAIN CLASSES OF MULTIVALENT FUNCTIONS INVOLVING A GENERALIZED DIFFERENTIAL OPERATOR

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ABSTRACT. Making use of a generalized differential operator we introduce some new subclasses of multivalent analytic functions in the open unit disk and investigate their inclusion relationships. Some integral preserving properties of these subclasses are also discussed.

1. Introduction and preliminaries

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$(1) \qquad f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{p+n}, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

which are analytic and p-valent in the open unit disk $\mathbb{U} = \{z : z \in C \text{ and } |z| < 1\}$. For functions f given by (1) and g given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_n z^{p+n},$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_n b_n z^{p+n}.$$

Given two functions f and g, which are analytic in $\mathbb{U}$, the function f is said to be subordinate to g in $\mathbb{U}$ if there exists a function w analytic in $\mathbb{U}$ with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function g is univalent in $\mathbb{U}$, then $f(z) \prec g(z) \ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $\mathcal{P}$ denote the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in $\mathbb{U}$ and for which $\Re\{h(z)\} > 0 \ (z \in \mathbb{U})$.

Analogous to the operator defined recently by Selvaraj and Santhosh Moni [6], we define an operator $D_{\lambda,g}^{\delta} f$ on $\mathcal{A}_p$ as follows:

For a fixed function $g \in \mathcal{A}_p$ given by

$$(2) \qquad g(z) = z^p + \sum_{n=1}^{\infty} b_n z^{p+n}, \quad (b_n \geq 0; \ p \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

$D_{\lambda,g}^{\delta} f(z) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ is defined by

$$D_{\lambda,g}^{0} f(z) = (f * g)(z),$$

$$D_{\lambda,g}^{1} f(z) = (1 - \lambda)(f * g)(z) + \frac{\lambda}{p} z((f * g)(z))',$$

$$(3) \qquad D_{\lambda,g}^{\delta} f(z) = D_{\lambda,g}^{1}(D_{\lambda,g}^{\delta-1} f(z)).$$

If $f(z) \in \mathcal{A}_p$, then we have

$$(4) \qquad D_{\lambda,g}^{\delta} f(z) = z^p + \sum_{n=1}^{\infty} \left(1 + \frac{\lambda n}{p}\right)^{\delta} a_n b_n z^{p+n},$$

where $\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$. It easily follows from (3) that

$$(5) \qquad \frac{\lambda z}{p} (D_{\lambda,g}^{\delta} f(z))' = D_{\lambda,g}^{\delta+1} f(z) - (1 - \lambda) D_{\lambda,g}^{\delta} f(z).$$

Throughout this paper, we assume that $p, k \in \mathbb{N}, \varepsilon_k = \exp(\frac{2\pi i}{k})$, and

$$(6) \qquad f_{p,k}^{\delta}(\lambda; g; z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{-jp} (D_{\lambda,g}^{\delta} f(\varepsilon_k^j z)) = z^p + \cdots, \quad (f \in \mathcal{A}_p).$$

Clearly, for $k = 1$, we have

$$f_{p,1}^{\delta}(\lambda; g; z) = D_{\lambda,g}^{\delta} f(z).$$

Making use of the operator $D_{\lambda,g}^{\delta} f(z)$, we now introduce and study the following subclasses of $\mathcal{A}_p$ of p-valent analytic functions.

Definition. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$, if it satisfies

$$(7) \qquad \frac{z(D_{\lambda,g}^{\delta} f(z))'}{p f_{p,k}^{\delta}(\lambda; g; z)} \prec h(z) \quad (z \in \mathbb{U}),$$

where $h \in \mathcal{P}$ and $f_{p,k}^{\delta}(\lambda; g; z) \neq 0 \ (z \in \mathbb{U})$.

Remark 1.1. If we let

$$\delta = 0 \quad \text{and} \quad g(z) = z^p \, {}_l\mathcal{F}_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

then $\mathcal{S}_{p,k}^\delta(\lambda; g; h)$ reduces to the function class $\mathcal{S}_{p,k}^{l,m}(\alpha_1; h)$ introduced and investigated by Zhi-Gang, Wang Yue-Ping Jiang, and H. M. Srivastava [10].

Remark 1.2. If we let

$$\delta = 0 \quad \text{and} \quad g(z) = z^p + \sum_{n=1}^\infty \frac{(a)_n}{(c)_n} z^{p+n},$$

then $\mathcal{S}_{p,k}^\delta(\lambda; g; h)$ reduces to the function class $T_{p,k}(a, c; h)$ introduced and investigated by N-Eng Xu and Ding-Gong Yang [7].

Remark 1.3. Let

$$g(z) = h(z) = \frac{1+z}{1-z}.$$

Then $\mathcal{S}_{1,2}^0(\lambda; g; h) = S_s^*$. The class S_s^* of functions starlike with respect to symmetric points has been studied by several authors (see [3], [5], [9]).

Definition. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_{p,k}^\delta(\lambda; g; h)$, if it satisfies

$$(8) \qquad \frac{z(D_{\lambda,g}^\delta f(z))'}{p\varphi_{p,k}^\delta(\lambda; g; z)} \prec h(z) \quad (z \in \mathbb{U})$$

for some $\varphi(z) \in \mathcal{S}_{p,k}^\delta(\lambda; g; h)$, where $h \in \mathcal{P}$ and $\varphi_{p,k}^\delta(\lambda; g; z) \neq 0$ is defined as in (6).

Definition. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{C}_{p,k}^\delta(\alpha, \lambda; g; h)$, if it satisfies

$$(9) \qquad (1-\alpha)\frac{z(D_{\lambda,g}^\delta f(z))'}{p\varphi_{p,k}^\delta(\lambda; g; z)} + \alpha\frac{(z(D_{\lambda,g}^\delta f(z))')'}{p(\varphi_{p,k}^\delta(\lambda; g; z))'} \prec h(z) \qquad (z \in \mathbb{U})$$

for some α $(\alpha \geq 0)$ and $\varphi(z) \in \mathcal{S}_{p,k}^\delta(\lambda; g; h)$, where $h \in \mathcal{P}$ and $(\varphi_{p,k}^\delta(\lambda; g; z))' \neq 0$.

We need the following lemmas to derive our results.

Lemma 1.4 ([1])**.** *Let β $(\beta \neq 0)$ and γ be complex numbers and let $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $\Re\{\beta h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$. If $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = h(0)$, then the subordination*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z) \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 1.5 ([2])**.** *Let $h(z)$ be analytic and convex univalent in $\mathbb{U}$ and let $w(z)$ be analytic in $\mathbb{U}$ with $\Re\{w(z)\} \geq 0$ $(z \in \mathbb{U})$. If $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = h(0)$, then the subordination*

$$q(z) + w(z)zq'(z) \prec h(z) \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 1.6. *Let $f(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$. Then*

$$(10) \qquad \frac{z(f_{p,k}^{\delta}(\lambda; g; z))'}{pf_{p,k}^{\delta}(\lambda; g; z)} \prec h(z) \quad (z \in \mathbb{U}).$$

Proof. For $f(z) \in \mathcal{A}_p$, we have from (6) that

$$
\begin{aligned}
f_{p,k}^{\delta}(\lambda; g; \varepsilon_k^j z) &= \frac{1}{k} \sum_{m=0}^{k-1} \varepsilon_k^{-mp} D_{\lambda,g}^{\delta} f(\varepsilon_k^{m+j} z) \\
&= \frac{\varepsilon_k^{jp}}{k} \sum_{m=0}^{k-1} \varepsilon_k^{-(m+j)p} D_{\lambda,g}^{\delta} f(\varepsilon_k^{m+j} z) \\
&= \varepsilon_k^{jp} f_{p,k}^{\delta}(\lambda; g; z), \quad (j \in \{0, 1, \ldots, k-1\})
\end{aligned}
$$

and

$$(f_{p,k}^{\delta}(\lambda; g; z))' = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{j(1-p)} (D_{\lambda,g}^{\delta} f(\varepsilon_k^j z))'.$$

Hence

$$
\begin{aligned}
\frac{z(f_{p,k}^{\delta}(\lambda; g; z))'}{pf_{p,k}^{\delta}(\lambda; g; z)} &= \frac{1}{k} \sum_{j=0}^{k-1} \frac{\varepsilon_k^{j(1-p)} z(D_{\lambda,g}^{\delta} f(\varepsilon_k^j z))'}{pf_{p,k}^{\delta}(\lambda; g; z)} \\
(11) \qquad &= \frac{1}{k} \sum_{j=0}^{k-1} \frac{\varepsilon_k^j z(D_{\lambda,g}^{\delta} f(\varepsilon_k^j z))'}{pf_{p,k}^{\delta}(\lambda; g; \varepsilon_k^j z)} \quad (z \in \mathbb{U}).
\end{aligned}
$$

Since $f(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$, we have

$$(12) \qquad \frac{\varepsilon_k^j z(D_{\lambda,g}^{\delta} f(\varepsilon_k^j z))'}{pf_{p,k}^{\delta}(\lambda; g; \varepsilon_k^j z)} \prec h(z) \quad \text{for} \ \ j \in \{0, 1, \ldots, k-1\}.$$

Noting that $h(z)$ is convex univalent in $\mathbb{U}$, from (11) and (12) we conclude that (10) holds true. $\qquad\square$

2. A set of inclusion relationships

Theorem 2.1. *Let $h(z) \in \mathcal{P}$ with*

$$(13) \qquad \Re\{h(z)\} > 1 - \frac{1}{\lambda} \quad (z \in \mathbb{U}; \lambda > 1).$$

If $f(z) \in \mathcal{S}_{p,k}^{\delta+1}(\lambda; g; h)$, then $f(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$ provided $f_{p,k}^{\delta}(\lambda; g; z) \neq 0$ $(z \in \mathbb{U})$.

Proof. By using (5) and (6), we have
(14)

$$(1-\lambda)f_{p,k}^{\delta}(\lambda; g; z) + \frac{\lambda z}{p}(f_{p,k}^{\delta}(\lambda; g; z))' = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{-jp}(D_{\lambda,g}^{\delta+1}f(\varepsilon_k^j z)) = f_{p,k}^{\delta+1}(\lambda; g; z).$$

Let $f(z) \in \mathcal{S}_{p,k}^{\delta+1}(\lambda; g; h)$ and

$$(15) \qquad w(z) = \frac{z(f_{p,k}^{\delta}(\lambda; g; z))'}{pf_{p,k}^{\delta}(\lambda; g; z)}.$$

Then $w(z)$ is analytic in $\mathbb{U}$, with $w(0) = 1$, and from (14) and (15) we have

$$(16) \qquad 1 - \lambda + \lambda w(z) = \frac{f_{p,k}^{\delta+1}(\lambda; g; z)}{f_{p,k}^{\delta}(\lambda; g; z)}.$$

Differentiating (16) with respect to z and using (15), we get

$$(17) \qquad w(z) + \frac{zw'(z)}{\frac{p}{\lambda}(1-\lambda) + pw(z)} = \frac{z(f_{p,k}^{\delta+1}(\lambda; g; z))'}{pf_{p,k}^{\delta+1}(\lambda; g; z)}.$$

From (17) and Lemma 1.6 we note that

$$(18) \qquad w(z) + \frac{zw'(z)}{\frac{p}{\lambda}(1-\lambda) + pw(z)} \prec h(z) \quad (z \in \mathbb{U}).$$

In view of (13) and (18), we deduce from Lemma 1.4 that

$$(19) \qquad w(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Suppose that

$$q(z) = \frac{z(D_{\lambda,g}^{\delta}f(z))'}{pf_{p,k}^{\delta}(\lambda; g; z)}.$$

Then $q(z)$ is analytic in $\mathbb{U}$, with $q(0) = 1$, and we obtain from (5) that

$$(20) \qquad f_{p,k}^{\delta}(\lambda; g; z)q(z) = \frac{1}{\lambda}D_{\lambda,g}^{\delta+1}f(z) + \left(1 - \frac{1}{\lambda}\right)D_{\lambda,g}^{\delta}f(z).$$

Differentiating both sides of (20) with respect to z, we get

$$(21) \qquad zq'(z) + \left(p\left(\frac{1}{\lambda} - 1\right) + \frac{z(f_{p,k}^{\delta}(\lambda; g; z))'}{f_{p,k}^{\delta}(\lambda; g; z)}\right)q(z) = \frac{z(D_{\lambda,g}^{\delta+1}f(z))'}{\lambda f_{p,k}^{\delta}(\lambda; g; z)}.$$

Now, we find from (14), (15) and (21) that

$$(22) \qquad q(z) + \frac{zq'(z)}{\frac{p}{\lambda}(1-\lambda) + pw(z)} = \frac{z(D_{\lambda,g}^{\delta+1}f(z))'}{pf_{p,k}^{\delta+1}(\lambda;g;z)} \prec h(z) \quad (z \in \mathbb{U}),$$

since $f(z) \in \mathcal{S}_{p,k}^{\delta+1}(\lambda;g;h)$. From (13) and (19) we observe that

$$\Re\left\{\frac{p}{\lambda}(1-\lambda) + pw(z)\right\} > 0.$$

Therefore, from (22) and Lemma 1.5 we conclude that

$$q(z) \prec h(z) \quad (z \in \mathbb{U})$$

which shows that $f(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$. $\qquad\qquad\square$

Theorem 2.2. *Let* $h(z) \in \mathcal{P}$ *with*

$$(23) \qquad\qquad \Re\{h(z)\} > 1 - \frac{1}{\lambda} \quad (z \in \mathbb{U}; \lambda > 1).$$

If $f(z) \in \mathcal{K}_{p,k}^{\delta+1}(\lambda;g;h)$ *with respect to* $\varphi(z) \in \mathcal{S}_{p,k}^{\delta+1}(\lambda;g;h)$, *then* $f(z) \in \mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$ *provided* $\varphi_{p,k}^{\delta}(\lambda;g;z) \neq 0$ $(z \in \mathbb{U})$.

Proof. Let $f(z) \in \mathcal{K}_{p,k}^{\delta+1}(\lambda;g;h)$. Then there exists a function $\varphi(z) \in \mathcal{S}_{p,k}^{\delta+1}(\lambda;g;h)$ such that

$$(24) \qquad\qquad \frac{z(D_{\lambda,g}^{\delta+1}f(z))'}{p\varphi_{p,k}^{\delta+1}(\lambda;g;z)} \prec h(z) \quad (z \in \mathbb{U}).$$

An application of Theorem 2.1 yields $\varphi(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$ and Lemma 1.6 leads to

$$(25) \qquad\qquad \psi(z) = \frac{z(\varphi_{p,k}^{\delta}(\lambda;g;z))'}{p\varphi_{p,k}^{\delta}(\lambda;g;z)} \prec h(z) \quad (z \in \mathbb{U}).$$

Let

$$q(z) = \frac{z(D_{\lambda,g}^{\delta}f(z))'}{p\varphi_{p,k}^{\delta}(\lambda;g;z)}.$$

By using (5), $q(z)$ can be written as follows

$$(26) \qquad \varphi_{p,k}^{\delta}(\lambda;g;z)q(z) = \frac{1}{\lambda}D_{\lambda,g}^{\delta+1}f(z) + \left(1 - \frac{1}{\lambda}\right)D_{\lambda,g}^{\delta}f(z).$$

Differentiating both sides of (26) with respect to z and using (14) (with f replaced by φ), we get

$$(27) \qquad\qquad q(z) + \frac{zq'(z)}{\frac{p}{\lambda}(1-\lambda) + p\psi(z)} = \frac{z(D_{\lambda,g}^{\delta+1}f(z))'}{p\varphi_{p,k}^{\delta+1}(\lambda;g;z)}.$$

Now, from (24) and (27) we find that

$$(28) \qquad q(z) + \frac{zq'(z)}{\frac{p}{\lambda}(1-\lambda) + p\psi(z)} \prec h(z) \quad (z \in \mathbb{U}).$$

Combining (23), (25) and (28), we deduce from Lemma 1.5 that

$$q(z) \prec h(z) \quad (z \in \mathbb{U})$$

which shows that $f(z) \in \mathcal{K}_{p,k}^{\delta}(\lambda; g; h)$ with respect to $\varphi(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$. $\qquad \square$

Corollary 2.3. *Let* $0 < \alpha \le 1$, $-1 \le B < A \le 1$ *and*

$$(29) \qquad h(z) = \left(\frac{1 + Az}{1 + Bz} \right)^{\alpha} \quad (z \in \mathbb{U}).$$

If $\lambda \le \left[1 - \left(\frac{1-A}{1-B} \right)^{\alpha} \right]^{-1}$, *then* $\mathcal{S}_{p,k}^{\delta+1}(\lambda; g; h) \subset \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$ *and* $\mathcal{K}_{p,k}^{\delta+1}(\lambda; g; h) \subset \mathcal{K}_{p,k}^{\delta}(\lambda; g; h)$.

Proof. The analytic function $h(z)$ defined by (29) is convex univalent in $\mathbb{U}$ (see [8]), $h(0) = 1$ and $h(\mathbb{U})$ is symmetric with respect to real axis. Thus $h(z) \in \mathcal{P}$ and

$$0 \le \left(\frac{1-A}{1-B} \right)^{\alpha} < \Re\{h(z)\} < \left(\frac{1+A}{1+B} \right)^{\alpha} \quad (z \in \mathbb{U}; 0 < \alpha \le 1; -1 < B < A \le 1).$$

Hence, by using Theorems 2.1 and 2.2 we have the corollary. $\qquad \square$

Corollary 2.4. *Let* $0 < \alpha \le 1$ *and*

$$(30) \qquad h(z) = 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{\alpha z}}{1 - \sqrt{\alpha z}} \right) \right)^2 \quad (z \in \mathbb{U}).$$

If $\lambda \le \frac{\pi^2}{8} \left(\arctan \sqrt{\alpha} \right)^{-2}$, *then* $\mathcal{S}_{p,k}^{\delta+1}(\lambda; g; h) \subset \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$ *and* $\mathcal{K}_{p,k}^{\delta+1}(\lambda; g; h) \subset \mathcal{K}_{p,k}^{\delta}(\lambda; g; h)$ $(z \in \mathbb{U})$.

Proof. The function $h(z)$ defined by (30) is in the class $\mathcal{P}$ (cf. [4]) and satisfies $h(\bar{z}) = \overline{h(z)}$. Therefore,

$$\Re\{h(z)\} > h(-1) = 1 - \frac{8}{\pi^2} \left(\arctan \sqrt{\alpha} \right)^2 \ge \frac{1}{2} \quad (z \in \mathbb{U}; 0 < \alpha \le 1).$$

Hence, by Theorems 2.1 and 2.2 we have the desired result. $\qquad \square$

Theorem 2.5. *Let* $0 \le \alpha_1 < \alpha_2$. *Then*

$$\mathcal{C}_{p,k}^{\delta}(\alpha_2, \lambda; g; h) \subset \mathcal{C}_{p,k}^{\delta}(\alpha_1, \lambda; g; h).$$

Proof. Let $f(z) \in \mathcal{C}_{p,k}^{\delta}(\alpha_2, \lambda; g; h)$. Then there exists a function $\varphi(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$ such that

$$(31) \qquad (1 - \alpha_2) \frac{z(D_{\lambda,g}^{\delta} f(z))'}{p\varphi_{p,k}^{\delta}(\lambda; g; z)} + \alpha_2 \frac{(z(D_{\lambda,g}^{\delta} f(z))')'}{p(\varphi_{p,k}^{\delta}(\lambda; g; z))'} \prec h(z) \quad (z \in \mathbb{U}).$$

Suppose that

$$(32) \qquad q(z) = \frac{z(D_{\lambda,g}^\delta f(z))'}{p\varphi_{p,k}^\delta(\lambda;g;z)}.$$

Then $q(z)$ is analytic in $\mathbb{U}$, with $q(0) = 1$. Differentiating both sides of (32) we get

$$(33) \qquad q(z) + \frac{\varphi_{p,k}^\delta(\lambda;g;z)}{(\varphi_{p,k}^\delta(\lambda;g;z))'}q'(z) = \frac{(z(D_{\lambda,g}^\delta f(z))')'}{p(\varphi_{p,k}^\delta(\lambda;g;z))'}.$$

Now, using (31), (32) and (33) we deduce that

$$(34) \qquad q(z) + w(z)zq'(z) \prec h(z),$$

where

$$w(z) = \alpha_2 \left(\frac{z(\varphi_{p,k}^\delta(\lambda;g;z))'}{\varphi_{p,k}^\delta(\lambda;g;z)} \right)^{-1}.$$

In view of Lemma 1.6 and $\alpha_2 > 0$, we observe that $w(z)$ is analytic in $\mathbb{U}$ and $\Re\{w(z)\} > 0$. Consequently, in view of (34), we deduce from Lemma 1.5 that

$$(35) \qquad q(z) \prec h(z).$$

Since $0 \le \frac{\alpha_1}{\alpha_2} < 1$ and since $h(z)$ is convex univalent in $\mathbb{U}$, we deduce from (31) and (35) that

$$\begin{aligned}
(1 &- \alpha_1)\frac{z(D_{\lambda,g}^\delta f(z))'}{p\varphi_{p,k}^\delta(\lambda;g;z)} + \alpha_1\frac{(z(D_{\lambda,g}^\delta f(z))')'}{p(\varphi_{p,k}^\delta(\lambda;g;z))'} \\
&= \frac{\alpha_1}{\alpha_2}\left((1-\alpha_2)\frac{z(D_{\lambda,g}^\delta f(z))'}{p\varphi_{p,k}^\delta(\lambda;g;z)} + \alpha_2\frac{(z(D_{\lambda,g}^\delta f(z))')'}{p(\varphi_{p,k}^\delta(\lambda;g;z))'} \right) + \left(1 - \frac{\alpha_1}{\alpha_2} \right)q(z) \\
&\prec h(z).
\end{aligned}$$

Thus $f(z) \in \mathcal{C}_{p,k}^\delta(\alpha_1,\lambda;g;h)$ which completes the proof of Theorem 2.5. $\qquad \square$

3. Integral operator

Theorem 3.1. *Let $h(z) \in \mathcal{P}$ and*

$$\Re\{h(z)\} > \max\left\{ 0, -\frac{\Re(c)}{p} \right\} \quad (z \in \mathbb{U}),$$

where c is a complex number such that $\Re(c) > -p$. If $f(z) \in \mathcal{S}_{p,k}^\delta(\lambda;g;h)$, then the function

$$(36) \qquad F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1}f(t)dt$$

is also in the class $\mathcal{S}_{p,k}^\delta(\lambda;g;h)$, provided that $F_{p,k}^\delta(\lambda;g;z) \ne 0$ $(0 < |z| < 1)$ where $F_{p,k}^\delta(\lambda;g;z)$ is defined as in (6).

Proof. Let $f(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$. Then from (36) and $\Re(c) > -p$, we note that $F(z) \in \mathcal{A}_p$ and

$$(37) \qquad (c+p)D_{\lambda,g}^{\delta}f(z) = cD_{\lambda,g}^{\delta}F(z) + z(D_{\lambda,g}^{\delta}F(z))'.$$

Also, from the above, we have

$$(c+p)f_{p,k}^{\delta}(\lambda; g; z) = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{-jp}\left(cD_{\lambda,g}^{\delta}F(\varepsilon_k^j z) + \varepsilon_k^j z(D_{\lambda,g}^{\delta}F(\varepsilon_k^j z))'\right)$$
$$(38) \qquad\qquad = cF_{p,k}^{\delta}(\lambda; g; z) + z(F_{p,k}^{\delta}(\lambda; g; z))'.$$

Let

$$w(z) = \frac{z(F_{p,k}^{\delta}(\lambda; g; z))'}{pF_{p,k}^{\delta}(\lambda; g; z)}.$$

Then $w(z)$ is analytic in $\mathbb{U}$, with $w(0) = 1$, and from (38) we observe that

$$(39) \qquad pw(z) + c = (c+p)\frac{f_{p,k}^{\delta}(\lambda; g; z)}{F_{p,k}^{\delta}(\lambda; g; z)}.$$

Differentiating both sides of (39) with respect to z and using Lemma 1.6, we obtain

$$(40) \qquad w(z) + \frac{zw'(z)}{pw(z) + c} = \frac{z(f_{p,k}^{\delta}(\lambda; g; z))'}{pf_{p,k}^{\delta}(\lambda; g; z)} \prec h(z).$$

In view of (40), Lemma 1.5 leads to $w(z) \prec h(z)$. If we let

$$q(z) = \frac{z(D_{\lambda,g}^{\delta}F(z))'}{pF_{p,k}^{\delta}(\lambda; g; z)},$$

then $q(z)$ is analytic in $\mathbb{U}$, with $q(0) = 1$, and it follows from (37) that

$$(41) \qquad F_{p,k}^{\delta}(\lambda; g; z)q(z) = \frac{c+p}{p}D_{\lambda,g}^{\delta}f(z) - \frac{c}{p}D_{\lambda,g}^{\delta}F(z).$$

Differentiating both sides of (41), we get

$$zq'(z) + \frac{z(F_{p,k}^{\delta}(\lambda; g; z))'}{F_{p,k}^{\delta}(\lambda; g; z)}q(z) = (c+p)\frac{z(D_{\lambda,g}^{\delta}f(z))'}{pF_{p,k}^{\delta}(\lambda; g; z)} - c\frac{z(D_{\lambda,g}^{\delta}F(z))'}{pF_{p,k}^{\delta}(\lambda; g; z)}$$

or equivalently,

$$(42) \qquad zq'(z) + \big(pw(z) + c\big)q(z) = (c+p)\frac{z(D_{\lambda,g}^{\delta}f(z))'}{pF_{p,k}^{\delta}(\lambda; g; z)}.$$

Now, from (39) and (42) we deduce that

$$q(z) + \frac{zq'(z)}{pw(z)+c} = \frac{c+p}{pw(z)+c}\,\frac{z(D_{\lambda,g}^{\delta}f(z))'}{pF_{p,k}^{\delta}(\lambda;g;z)}$$

$$= \frac{z(D_{\lambda,g}^{\delta}f(z))'}{pf_{p,k}^{\delta}(\lambda;g;z)}$$

(43) $$\prec h(z), \quad \text{because } f(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h).$$

Combining, $\Re\{h(z)\} > \max\{0, -\frac{\Re(c)}{p}\}$ and $w(z) \prec h(z)$ we have $\Re\{pw(z) + c\} > 0$ $(z \in \mathbb{U})$.

Therefore, from (43) and Lemma 1.5 we find that $q(z) \prec h(z)$, which shows that $F(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$. $\square$

By applying similar method as in Theorem 3.1, we have:

Theorem 3.2. *Let* $h(z) \in \mathcal{P}$ *and*

$$\Re\{h(z)\} > \max\left\{0, -\frac{\Re(c)}{p}\right\} \quad (z \in \mathbb{U}; \quad \Re(c) > -p).$$

If $f(z) \in \mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$ *with respect to* $\varphi(z) \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$, *then the function*

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt$$

belongs to the class $\mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$ *with respect to*

$$G(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}g(t)dt,$$

provided that $G_{p,k}^{\delta}(\lambda;g;z) \neq 0$ $(0 < |z| < 1)$.

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