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HYPERSURFACES OF ALMOST r-PARACONTACT RIEMANNIAN MANIFOLD ENDOWED WITH SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. We define a semi-symmetric metric connection in an almost r-paracontact Riemannian manifold and we consider invariant, non-invariant and anti-invariant hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric metric connection.

1. Introduction

Let ∇^* be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T of ∇^* is given by

$$T(X,Y) = \nabla^*_X Y - \nabla^*_Y X - [X,Y]$$

for all vector fields X and Y in M and is of type $(1,2)$. The connection ∇^* is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇^* is metric connection if there is a Riemannian metric g in M such that $\nabla^* g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric if it is the Levi-Civita connection.

In [7], [8], A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be *semi-symmetric* if its torsion tensor T is of the form

$$T(X,Y) = u(Y)X - u(X)Y,$$

where u is a 1-forms. In [9], K. Yano considered a semi-symmetric metric connection and studied some of its properties. Almost r-paracontact structures were defined by A. Bucki and Miernowski in [5]. In [4], A. Bucki introduced r-paracontact structures of P-Sasakian type. Properties of hypersurface of almost r-paracontact Riemannian manifold were studied by A. Bucki in [3]. M. Ahmad, J.-B. Jun, and A. Haseeb studied some properties of hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric

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metric connection in [1]. Also M. Ahmad and C. Ozgur studied those of them endowed with a semi-symmetric non-metric connection in [2].

In this paper we study properties of hypersurfaces of almost r-paracontact Riemannian manifold endowed with a semi-symmetric metric connection.

The paper is organized as follows: In Section 2, we give a brief introduction about an almost r-paracontact Riemannian manifold. In Section 3, we show that the induced connection on a hypersurface of an almost r-paracontact Riemannian manifold with semi-symmetric metric connection with respect to the normal is also a semi-symmetric metric connection. We find the characteristic properties of invariant, non-invariant and anti-invariant hypersurfaces of almost r-paracontact Riemannian manifold endowed with a semi-symmetric metric connection.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold with a positive definite metric g. If on M there exist a tensor field ϕ of type (1,1), r vector fields $\xi_1, \xi_2, \ldots, \xi_r$ $(n > r)$, r 1-forms $\eta^1_, \eta^2, \ldots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \alpha, \beta \in (r) = \{1, 2, 3, \ldots, r\}, \tag{2.1}$$

$$\phi^2(X) = X - \eta^\alpha(X)\xi_\alpha, \tag{2.2}$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \Sigma_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{2.4}$$

where X and Y are vector fields on M and $a^\alpha b_\alpha \overset{\text{def}}{=} \Sigma_\alpha a^\alpha b_\alpha$, then the structure $\Sigma = (\phi,\ \xi_\alpha,\ \eta^\alpha,\ \mathrm{g})_{\alpha\in(r)}$ is said to be an *almost r-paracontact Riemannian structure* on M and M is an *almost r-paracontact Riemannian manifold* [5]. From (2.1) through (2.4), we also have:

$$\phi(\xi_\alpha) = 0, \quad \alpha \in (r), \tag{2.5}$$

$$\eta^\alpha \circ \phi = 0, \quad \alpha \in (r),$$

$$\Phi(X, Y) \overset{\text{def}}{=} g(\phi X, Y) = g(X, \phi Y).$$

An almost r-paracontact Riemannian manifold M equipped with the Riemannian connection ∇^* with a structure $\Sigma = (\phi,\ \xi_\alpha,\ \eta^\alpha,\ \mathrm{g})_{\alpha\in(r)}$ is said to be of *S-paracontact type* if

$$\Phi(X, Y) = (\nabla^*_Y \eta^\alpha)(X) \tag{2.6}$$

for all $\alpha \in (r)$. An almost r-paracontact Riemannian manifold M with a structure $\Sigma = (\phi,\ \xi_\alpha,\ \eta^\alpha,\ \mathrm{g})_{\alpha\in(r)}$ is said to be of *P-Sasakian type* if it satisfies

(2.6) and

$$(\nabla^*_Z\Phi)(X,Y) = -\Sigma_\alpha\eta^\alpha(X)[g(Y,Z) - \Sigma_\beta\eta^\beta(Y)\eta^\beta(Z)] -\Sigma_\alpha\eta^\alpha(Y)[g(X,Z) - \Sigma_\beta\eta^\beta(X)\eta^\beta(Z)] \tag{2.7}$$

for all vector fields X, Y and Z on M [4]. The conditions (2.6) and (2.7) are equivalent respectively to

$$\phi X = \nabla^*_X\xi_\alpha \tag{2.8}$$

and

$$(\nabla^*_Y\phi)(X) = -\Sigma_\alpha\eta^\alpha(X)[Y - \eta^\alpha(Y)\xi_\alpha] -[g(X,Y) - \Sigma_\alpha\eta^\alpha(X)\eta^\alpha(Y)]\Sigma_\beta\xi_\beta \tag{2.9}$$

for all $\alpha \in (r)$. On the other hand, a *semi-symmetric metric connection* ∇ on M is defined as

$$\nabla_XY = \nabla^*_XY + \eta^\alpha(Y)X - g(X,Y)\xi_\alpha \tag{2.10}$$

for any $\alpha \in (r)$. Using (2.5) and (2.10) in (2.8) and (2.9), we get

$$\phi X = \nabla_X\xi_\alpha - X + \eta^\alpha(X)\xi_\alpha, \tag{2.11}$$

$$(\nabla_Y\phi)(X) = -\Sigma_\alpha\eta^\alpha(X)[Y - \eta^\alpha(Y)\xi_\alpha] -[g(X,Y) - \Sigma_\alpha\eta^\alpha(X)\eta^\alpha(Y)]\Sigma_\beta\xi_\beta - g(Y,\phi X)\xi_\alpha. \tag{2.12}$$

3. Hypersurfaces of almost r-paracontact Riemannian manifold endowed with a semi-symmetric metric connection

Let M^{n+1} be an almost r-paracontact Riemannian manifold with a positive definite metric g and M^n be a hypersurface immersed in M^{n+1} by the immersion $\tau : M^n \to M^{n+1}$. If τ_* denotes the differential of the immersion τ and $\bar{X}$ is a vector field on M^n, then we shall identify $\bar{X}$ and $\tau_*\bar{X}$. We denote the objects belonging to M^n by the mark of hyphen placed over them, e.g., $\bar{\phi}, \bar{X}, \bar{\eta}, \bar{\xi}$ etc.

Let N be the unit normal vector field to M^n. The induced metric $\bar{g}$ on M^n is defined by

$$\bar{g}(\bar{X},\bar{Y}) = g(\bar{X},\bar{Y}). \tag{3.1}$$

Then we have [6]

$$g(\bar{X},N) = 0 \text{ and } g(N,N) = 1. \tag{3.2}$$

If $\bar{\nabla}^*$ be the induced connection on the hypersurface from the Riemannian connection ∇^* in M^{n+1} with respect to the unit normal N, then the Gauss and Weingarten formulae are given respectively by

$$\nabla^*_{\bar{X}}\bar{Y} = \bar{\nabla}^*_{\bar{X}}\bar{Y} + h(\bar{X},\bar{Y})N, \tag{3.3}$$

$$\nabla^*_{\bar{X}}N = -H(\bar{X}),$$

where h is the second fundamental tensor and H is a tensor field of type $(1,1)$ called the shape operator of M^n in M^{n+1} which satisfying

$$h(\bar{Y},\bar{X}) = h(\bar{X},\bar{Y}) = \bar{g}(H(\bar{X}),\bar{Y}).$$

If $\bar{\nabla}$ is the induced connection on the hypersurface from the semi-symmetric metric connection ∇ in M^{n+1} with respect to the unit normal N, then we have

$$\nabla_{\bar{X}}\bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + m(\bar{X},\bar{Y})N, \tag{3.4}$$

where m is a tensor field of type $(0,2)$ of the hypersurface. From (2.10), we obtain

$$\nabla_{\bar{X}}\bar{Y} = \nabla^*_{\bar{X}}\bar{Y} + \eta^\alpha(\bar{Y})\bar{X} - g(\bar{X},\bar{Y})\xi_\alpha, \quad \alpha \in (r). \tag{3.5}$$

From equations (3.3), (3.4) and (3.5), we get for each $\alpha \in (r)$

$$\bar{\nabla}_{\bar{X}}\bar{Y} + m(\bar{X},\bar{Y})N = \bar{\nabla}^*_{\bar{X}}\bar{Y} + h(\bar{X},\bar{Y})N + \eta^\alpha(\bar{Y})\bar{X} - g(\bar{X},\bar{Y})(\xi_\alpha + a_\alpha N),$$

where we put

$$\xi_\alpha = \bar{\xi}_\alpha + a_\alpha N. \tag{3.6}$$

By taking the tangential and normal parts respectively from the both sides, we get

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \bar{\nabla}^*_{\bar{X}}\bar{Y} + \eta^\alpha(\bar{Y})\bar{X} - g(\bar{X},\bar{Y})\xi_\alpha, \quad \alpha \in (r)$$

and

$$m(\bar{X},\bar{Y}) = h(\bar{X},\bar{Y}) - a_\alpha g(\bar{X},\bar{Y}). \tag{3.7}$$

Thus we get the following theorem:

Theorem 3.1. *The connection induced on a hypersurface of an almost r-paracontact Riemannian manifold with semi-symmetric metric connection with respect to the unit normal is also a semi-symmetric metric connection.*

From (3.4) and (3.7), we have

$$\nabla_{\bar{X}}\bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + \{h(\bar{X},\bar{Y}) - a_\alpha g(\bar{X},\bar{Y})\}N, \tag{3.8}$$

which is the Gauss formula for a semi-symmetric metric connection. From equation (3.5), we have

$$\nabla_{\bar{X}}N = \nabla^*_{\bar{X}}N + a_\alpha\bar{X}, \tag{3.9}$$

where

$$a_\alpha = \eta^\alpha(N). \tag{3.10}$$

From $(3.3)_2$ and (3.9), we have

$$\nabla_{\bar{X}}N = -H\bar{X} + a_\alpha\bar{X}, \tag{3.11}$$

which is the Weingarten formula with respect to semi-symmetric metric connection.

Now, suppose that $\Sigma = (\phi,\ \xi_\alpha,\ \eta^\alpha,\ \mathrm{g})_{\alpha\in(r)}$ is an almost r-paracontact Riemannian structure on M^{n+1}. Then every vector field X on M^{n+1} is decomposed as

$$X = \bar{X} + \lambda(X)N,$$

where λ is a 1-forms on M^{n+1} and $\bar{X}$ is a vector field and N is a normal field on M^n. Then we have

$$\phi\bar{X} = \bar{\phi}\bar{X} + b(\bar{X})N, \tag{3.12}$$

$$\phi N = \bar{N} + KN, \tag{3.13}$$

where $\bar{\phi}$ is a tensor field of type (1,1), b is a 1-forms and K is a scalar function on the hypersurface M^n. Now, we define $\bar{\eta}^\alpha$ as

$$\bar{\eta}^\alpha(\bar{X}) = \eta^\alpha(\bar{X}), \quad \alpha \in (r). \tag{3.14}$$

Making use of (3.6), (3.10), (3.12) and (3.13), we obtain from (2.1) through (2.5)

$$b(\bar{N}) + K^2 = 1 - \Sigma_\alpha(a_\alpha)^2, \tag{3.15}$$

$$Ka_\alpha + b(\bar{\xi}_\alpha) = 0, \quad \alpha \in (r), \tag{3.16}$$

$$\Phi(\bar{X},\bar{Y}) = \bar{g}(\bar{\phi}\bar{X},\bar{Y}) = \bar{g}(\bar{X},\bar{\phi}\bar{Y}) = \bar{\Phi}(\bar{X},\bar{Y}). \tag{3.17}$$

Making use of (3.1), (3.2), (3.5), (3.12) and (3.13), we have

$$0 = g(\bar{\phi}\bar{X},N) = g(\phi\bar{X},N) - b(\bar{X}) = g(\bar{X},\phi N) - b(\bar{X}).$$

Hence we get

$$\bar{g}(\bar{X},\bar{N}) = b(\bar{X}). \tag{3.18}$$

Differentiating covariantly (3.12) and (3.13) along M^n and making use of (3.8) and (3.11), we get

$$\begin{aligned}(\nabla_{\bar{Y}}\phi)(\bar{X}) \;=\; & (\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}) - b(\bar{X})H(\bar{Y}) + b(\bar{X})a_\alpha\bar{Y} \\ & -(h(\bar{X},\bar{Y}) - a_\alpha g(\bar{X},\bar{Y}))\bar{N} \\ & +[(\bar{\nabla}_{\bar{Y}}b)(\bar{X}) + h(\bar{\phi}\bar{X},\bar{Y}) - K(h(\bar{X},\bar{Y}) - a_\alpha g(\bar{X},\bar{Y}))]N,\end{aligned} \tag{3.19}$$

$$\begin{aligned}(\nabla_{\bar{Y}}\phi)N \;=\; & \bar{\nabla}_{\bar{Y}}\bar{N} + \bar{\phi}(H(\bar{Y})) - KH(\bar{Y}) - a_\alpha(\bar{\phi}\bar{Y} + K\bar{Y}) \\ & +[\bar{Y}(K) - 2a_\alpha b(\bar{Y}) + 2h(\bar{Y},\bar{N})]N.\end{aligned} \tag{3.20}$$

From (3.6) and (3.10), we have

$$\nabla_{\bar{Y}}\xi_\alpha = \bar{\nabla}_{\bar{Y}}\bar{\xi}_\alpha - a_\alpha H(\bar{Y}) + (a_\alpha)^2\bar{Y} + [\bar{Y}(a_\alpha) + h(\bar{Y},\bar{\xi}_\alpha) - a_\alpha\bar{\eta}^\alpha(\bar{Y})]N, \tag{3.21}$$

$$(\nabla_{\bar{Y}}\eta^\alpha)(\bar{X}) = (\bar{\nabla}_{\bar{Y}}\bar{\eta}^\alpha)(\bar{X}) - a_\alpha h(\bar{X},\bar{Y}) + (a_\alpha)^2\bar{g}(\bar{X},\bar{Y}). \tag{3.22}$$

From the identity $(\nabla_Z\Phi)(X,Y) = g((\nabla_Z\phi)(X),Y)$, making use of (3.17), (3.18) and (3.19), we have

$$\begin{aligned}(\nabla_{\bar{Z}}\Phi)(\bar{X},\bar{Y}) &= (\bar{\nabla}_{\bar{Z}}\bar{\Phi})(\bar{X},\bar{Y}) - b(\bar{X})h(\bar{Z},\bar{Y}) - b(\bar{Y})h(\bar{Z},\bar{X}) \\ &\quad + a_\alpha b(\bar{X})\bar{g}(\bar{Z},\bar{Y}) + a_\alpha b(\bar{Y})\bar{g}(\bar{Z},\bar{X}).\end{aligned} \tag{3.23}$$

Theorem 3.2 ([4]). *If M^n is an invariant hypersurface immersed in an almost r-paracontact Riemannian manifold M^{n+1} endowed with semi-symmetric metric connection with structure $\Sigma = (\phi,\xi_\alpha,\eta^\alpha,g)_{\alpha\in(r)}$, then either*

(i) *All ξ_α are tangent to M^n and M^n admits an almost r-paracontact Riemannian structure $\Sigma_1 = (\bar{\phi},\bar{\xi}_\alpha,\bar{\eta}^\alpha,\bar{g})_{\alpha\in(r)}, (n-r>2)$ or*

(ii) *One of ξ_α (say, ξ_r) is normal to M^n and remaining ξ_α are tangent to M^n and M^n admits an almost $(r-1)$-paracontact Riemannian structure $\Sigma_2 = (\bar{\phi},\bar{\xi}_i,\bar{\eta}^i,\bar{g})_{i\in(r)}, (n-r>1)$.*

Proof. From (3.15) and (3.16) after computations similar to the computations in the proof of theorem 3.1 in [3] we obtain our theorem. □

Corollary 3.1. *If M^n is a hypersurface immersed in an almost r-paracontact Riemannian manifold M^{n+1} with a structure $\Sigma = (\phi,\xi_\alpha,\eta^\alpha,g)_{\alpha\in(r)}$ endowed with a semi-symmetric metric connection, then the following statements are equivalent:*

(1) *M^n is invariant.*

(2) *The normal vector field N is an eigenvector of ϕ.*

(3) *All ξ_α are tangent to M^n if and only if M^n admits an almost r-paracontact Riemannian structure Σ_1, or one of ξ_α is normal and $(r-1)$ remaining ξ_i are tangent to M^n if and only if M^n admits an almost $(r-1)$-paracontact Riemannian structure Σ_2.*

Theorem 3.3. *If M^n is an invariant hypersurface immersed in an almost r-paracontact Riemannian manifold of P-Sasakian type endowed with semi-symmetric metric connection, then the induced almost r-paracontact Riemannian structure Σ_1 or $(r-1)$-paracontact Riemannian structure Σ_2 are also of P-Sasakian type.*

Proof. Making use of (3.1), (3.14), (3.17), (3.22) and (3.23), we can observe that the conditions (2.11) and (2.12) are satisfied for both $\Sigma_1$and Σ_2. □

On the other hand, we have the following.

Lemma 3.1.

$$\bar{\nabla}_{\bar{X}}(\mathrm{trace}\bar{\phi}) = \mathrm{trace}(\bar{\nabla}_{\bar{X}}\bar{\phi}).$$

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal basis of TM^n and

$$\mathrm{trace}\bar{\phi} \overset{\mathrm{def}}{=} \Sigma_a(\bar{\phi}(e_a), e_a),$$

where $a \in (n-1)$, then after computations similar to the computations in the proof of Lemma 4.1 in [3] we easily obtain our lemma. □

Theorem 3.4. *Let M^n be a non-invariant hypersurface of an almost r-paracontact Riemannian manifold M^{n+1} endowed with the semi-symmetric metric connection with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ satisfying $\nabla\phi = 0$ along M^n. Then M^n is totally geodesic if and only if $(\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}) + a_\alpha \bar{g}(\bar{X}, \bar{Y})\bar{N} + a_\alpha b(\bar{X})\bar{Y} = 0$.*

Proof. From (3.19) we have

$$(\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}) - (h(\bar{X}, \bar{Y}) - a_\alpha \bar{g}(\bar{X}, \bar{Y}))\bar{N} - b(\bar{X})H(\bar{Y}) + a_\alpha b(\bar{X})\bar{Y} = 0, \tag{3.24}$$

$$(\bar{\nabla}_{\bar{Y}} b)(\bar{X}) + a_\alpha \bar{g}(\bar{X}, \bar{Y}) + h(\bar{Y}, \bar{\phi}\bar{X}) - Kh(\bar{X}, \bar{Y}) = 0.$$

If M^n is totally geodesic, then $h = 0$ and $H = 0$, so we get from (3.24),

$$(\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}) + a_\alpha \bar{g}(\bar{X}, \bar{Y})\bar{N} + a_\alpha b(\bar{X})\bar{Y} = 0.$$

Conversely, if $(\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}) + a_\alpha \bar{g}(\bar{X}, \bar{Y})\bar{N} + a_\alpha b(\bar{X})\bar{Y} = 0$, then

$$h(\bar{X}, \bar{Y})\bar{N} + b(\bar{X})H(\bar{Y}) = 0. \tag{3.25}$$

Making use of (3.18), we have

$$b(\bar{Z})h(\bar{X}, \bar{Y}) + b(\bar{X})h(\bar{Y}, \bar{Z}) = 0. \tag{3.26}$$

Using (3.25), we get

$$b(\bar{X})h(\bar{Y}, \bar{Z}) = b(\bar{Y})h(\bar{X}, \bar{Z}). \tag{3.27}$$

From (3.26) and (3.27), we get $b(\bar{Z})h(\bar{X}, \bar{Y}) = 0$ which gives that $h = 0$ as $b \neq 0$. Using $h = 0$ in (3.25), we get $H = 0$. Thus, $h = 0$ and $H = 0$. Hence M^n is totally geodesic. □

Also we have the following:

Theorem 3.5. *Let M^n be a non-invariant hypersurface of an almost r-paracontact Riemannian manifold M^{n+1} with semi-symmetric metric connection satisfying $\nabla\phi = 0$ along M^n and* trace$\bar{\phi}$ *= constant, then M^n is totally umbilical.*

Proof. From (3.24) we have

$$\bar{g}((\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}), \bar{X}) = 2h(\bar{X}, \bar{Y})b(\bar{X}) - 2a_\alpha(\bar{X})g(\bar{X}, \bar{Y})$$

and

$$\bar{\nabla}_{\bar{X}}(\text{trace}\bar{\phi}) = \Sigma_a \bar{g}(\bar{\nabla}_{\bar{X}}\bar{\phi}(e_a), e_a).$$

Using Lemma 3.1, we get

$$h(\bar{X}, \bar{N}) = a_\alpha \Sigma_a b(e_a)\bar{g}(\bar{X}, e_a),$$

where $\bar{N} = \Sigma_a b(e_a)e_a$, which implies that M^n is totally umbilical. □

Now, let M^{n+1} be an almost r-paracontact Riemannian manifold of S-paracontact type. Then from (2.8), (3.12) and (3.21), we get

$$\bar{\phi}\bar{X} = \bar{\nabla}_{\bar{X}}\bar{\xi}_\alpha - a_\alpha H(\bar{X}) + (a_\alpha)^2(\bar{X}) - \bar{X} + \bar{\eta}^\alpha(\bar{X})\bar{\xi}_\alpha, \quad \alpha \in (r), \tag{3.28}$$

$$b(\bar{X}) = \bar{X}(a_\alpha) + h(\bar{X}, \bar{\xi}_\alpha), \quad \alpha \in (r). \tag{3.29}$$

Making use of (3.29), we have that if M^n is totally geodesic, then $a_\alpha = 0$ and $h = 0$. Hence $b = 0$, that is, M^n is invariant. Thus we have:

Proposition 3.1. *If M^n is totally geodesic hypersurface of an almost r-paracontact Riemannian manifold M^{n+1} endowed with the semi-symmetric metric connection of S-paracontact type with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ and all ξ_α are tangent to M^n, then M^n is invariant.*

Theorem 3.6. *If M^n is an anti-invariant hypersurface of an almost r-paracontact Riemannian manifold M^{n+1} endowed with the semi-symmetric metric connection of S-paracontact type with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, then $\bar{\nabla}_{\bar{X}}\xi_\alpha = \phi^2\bar{X}$.*

Proof. If M^n is anti-invariant, then $\bar{\phi} = 0$ and $a_\alpha = 0$ and also from (3.28) we have

$$\bar{\nabla}_{\bar{X}}\bar{\xi}_\alpha = \bar{X} - \bar{\eta}^\alpha(\bar{X})\bar{\xi}_\alpha, \quad \alpha \in (r).$$

That is,

$$\bar{\nabla}_{\bar{X}}\xi_\alpha = \phi^2\bar{X}.$$

□

Now, let M^{n+1} be an almost r-paracontact Riemannian manifold endowed with the semi-symmetric metric connection of P-Sasakian type. Then from (2.11) and (3.19), we have

$$\begin{aligned} &(\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}) - [h(\bar{X}, \bar{Y}) - a_\alpha\bar{g}(\bar{X}, \bar{Y})]\bar{N} - b(\bar{X})H(\bar{Y}) + a_\alpha b(\bar{X})\bar{Y} \\ = &- \Sigma_\alpha\bar{\eta}^\alpha(\bar{X})[\bar{Y} - \bar{\eta}^\alpha(\bar{Y})\bar{\xi}_\alpha] - [\bar{g}(\bar{X}, \bar{Y}) - \Sigma_\alpha\bar{\eta}^\alpha(\bar{X})\bar{\eta}^\alpha(\bar{Y})]\Sigma_\beta\bar{\xi}_\beta. \end{aligned} \tag{3.30}$$

Theorem 3.7. *Let M^{n+1} be an almost r-paracontact Riemannian manifold of P-Sasakian type with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ endowed with the semi-symmetric metric connection and let M^n be a hypersurface immersed in M^{n+1} such that none of ξ_α is tangent to M^n. Then M^n is totally geodesic if and only if*

$$\begin{aligned} &(\bar{\nabla}_{\bar{Y}}\bar{\phi})(\bar{X}) + a_\alpha b(\bar{X})\bar{Y} - a_\alpha\bar{g}(\bar{X}, \bar{Y})\bar{N} \\ = &- \Sigma_\alpha\bar{\eta}^\alpha(\bar{X})[\bar{Y} - \bar{\eta}^\alpha(\bar{Y})\bar{\xi}_\alpha] - [\bar{g}(\bar{X}, \bar{Y}) - \Sigma_\alpha\bar{\eta}^\alpha(\bar{X})\bar{\eta}^\alpha(\bar{Y})]\Sigma_\beta\bar{\xi}_\beta. \end{aligned} \tag{3.31}$$

Proof. If (3.31) is satisfied, then from (3.30), we get $h(\bar{X}, \bar{Y})\bar{N} + b(\bar{X})H(\bar{Y}) = 0$. Since $b \neq 0$ so that by use of Theorem 3.4, $h(\bar{X}, \bar{Y}) = 0$. Hence M^n is totally geodesic. Conversely, Let M^n is totally geodesic, that is $h(\bar{X}, \bar{Y}) = 0$, $H = 0$, then from (3.29) we have $b = 0$, which is a contradiction. Hence ξ_α are not tangent to M^n. □

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