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CONVERGENCE THEOREMS FOR INVERSE-STRONGLY MONOTONE MAPPINGS AND QUASI-ϕ-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we consider a hybrid projection algorithm for a pair of inverse-strongly monotone mappings and a quasi-ϕ-nonexpansive mapping. Strong convergence theorems are established in the framework of Banach spaces.

1. Introduction and preliminaries

Let E be a real Banach space with the norm $\|\cdot\|$ and let C be a nonempty closed convex subset of E. Let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ the generalized duality pairing between E and E^*. The modulus of convexity of E is the function $\delta : (0, 2] \to [0, 1]$ defined by

$$\delta(\epsilon) = \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\Big\}.$$

E is said to *uniformly convex* if and only if $\delta(\epsilon) > 0$ for all $0 < \epsilon \le 2$. Let $p > 1$. Then E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\epsilon) \ge c\epsilon^p$ for all $\epsilon \in [0, 2]$. Let $U = \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.1}$$

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Alber [1] recently introduced a generalized projection operator Π_C in a real Banach space which is an analogue of the metric projection in Hilbert spaces. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem: $\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x)$. The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1], [2], [7], [9], [18]).

Let C be a nonempty closed convex subset of a Banach space E and T a mapping from C into itself. In this paper, we use $F(T)$ to denote the fixed point set of the mapping T. A point p in C is said to be an *asymptotic fixed point* of T [14] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F(T)}$. A point p in C is said to be an *strong asymptotic fixed point* of T [19] if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$.

Definition 1.1. A mapping T from C into itself is said to be *relatively nonexpansive* [4]-[6] if $\widehat{F(T)} = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

The asymptotic behavior of a relatively nonexpansive mapping was studied in [4]-[6].

Recently, Zegeye and Shahzad [19] introduced the following definition.

Definition 1.2. A mapping T from C into itself is said to be *relatively weak nonexpansive* if $\widetilde{F(T)} = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Since, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F(T)} \subset \widehat{F(T)}$. It is obvious that the class of relatively weak nonexpansive mappings includes the class of relatively nonexpansive mappings (see [19] for more details).

In [13], the authors introduced the following definition.

Definition 1.3. A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

We remark that the class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings and relatively weak nonexpansive mappings. To be more precise, we relaxed the strong restriction: $F(T) = \widehat{F(T)}$ or $F(T) = \widetilde{F(T)}$.

Recall that a mapping $A : C \to E^*$ is said to be *monotone* if

$$\langle x - y, Ax - Ay \rangle \geq 0, \quad \forall x, y \in C.$$

$A : C \to E^*$ is said to be α-*inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Recall also that a monotone mapping A is said to be *maximal* if its graph $G(A) = \{(x, f) : f \in Ax\}$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping A is maximal if and only if for any $(x, f) \in E \times E^*$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(A)$ implies $f \in Ax$. An operator A from C into E is said to be *hemi-continuous* if, for all $x, y \in C$, the mapping f of $[0, 1]$ into E defined by $f(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*.

Next, we consider the following variational inequality problem for a monotone and hemi-continuous mapping $A : C \to E^*$. To find an $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

We denoted by $VI(C, A)$ the set of solutions of the problem (1.2).

Recently, many authors studied the hybrid projection algorithm for monotone mappings and relatively nonexpansive mappings, see, for instance, [8], [10]-[12], [16], [17], [19]. Zegeye and Shzhzad [19] proved the following theorem.

Theorem ZS. *Let E be a uniformly smooth and 2-uniformly convex Banach space with dual E^*. Let K be a nonempty closed convex subset of E. Let $A : K \to E^*$ be a γ-inverse strongly monotone mapping and let $T : K \to K$ be a relatively weak nonexpansive mapping with $VI(K, A) \cap F(T) \neq \emptyset$. Assume that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in K$ and $p \in VI(K, A)$. Let $0 < \alpha_n \leq b_0 := \frac{c^2\gamma}{2}$, where c is a constant. Then sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in K \quad \text{chosen arbitrarily}, \\ y_n = \Pi_K[J^{-1}(Jx_n - \alpha_n Ax_n)], \\ z_n = Ty_n, \\ H_0 = \{v \in K : \phi(v, z_0) \leq \phi(v, y_0) \leq \phi(v, x_0)\}, \\ H_n = \{v \in H_{n-1} \cap W_{n-1} : \phi(v, z_n) \leq \phi(v, y_n) \leq \phi(v, x_n)\}, \\ W_0 = K, \\ W_n = \{v \in W_{n-1} \cap H_{n-1} : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Then $\{x_n\}$ converges strongly to $p = \Pi_{F(T)\cap VI(K,A)}x_0$, where $\Pi_{F(T)\cap VI(K,A)}$ is the generalized projection form E onto $F(T) \cap VI(K, A)$.

In this paper, motivated and inspired by Zegeye and Shahzad [19], we introduce a more general hybrid projection algorithm for a pair of inverse-strongly monotone mappings and a single quasi-ϕ-nonexpansive mapping. strong convergence theorems are established in the framework of Banach spaces. The results presented in this paper mainly improve the corresponding results in [8] and [19].

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 ([3]). *Let E be a 2-uniformly convex Banach space. Then we have*

$$\|x-y\| \le \frac{2}{c^2}\|Jx-Jy\|, \quad \forall x,y \in E, \tag{1.3}$$

where J is the normalized duality mapping on E and $0 < c \le 1$.

Lemma 1.2 ([9]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 1.3 ([1]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \ge 0, \quad \forall y \in C.$$

Lemma 1.4 ([1]). *Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y,x), \quad \forall y \in C.$$

Lemma 1.5 ([13]). *Let E be a uniformly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let T be a closed and quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

We denote by $N_C(x)$ the normal cone for C at a point $x \in C$, that is $N_C(x) := \{x^* \in E^* : \langle x-y, x^*\rangle \ge 0 \text{ for all } y \in C\}$. The following lemma is important for our main results.

Lemma 1.6 ([15]). *Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone and hemi-continuous operator of C into E. Let $Q \subset E \times E^*$ be an operator defined as follows*:

$$Qx := \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then Q is maximal monotone and $Q^{-1}(0) = VI(C,A)$.

Albert [1] studied the following functional $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x,x^*) = \|x\|^* - 2\langle x, x^*\rangle + \|x^*\|^2, \quad \forall x \in E,\ x^* \in E^*.$$

From the definition of the functional V, we see that $V(x,x^*) = \phi(x, J^{-1}x^*)$.

Lemma 1.7 ([1]). *Let E be a reflexive, strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^* + y^*), \quad \forall x \in E,\ x^*, y^* \in E^*.$$

2. Main results

Now, we are ready to give our main results in this paper.

Theorem 2.1. *Let C be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let $A : C \to E^*$ be an α-inverse strongly monotone mapping, let $B : C \to E^*$ be a β-inverse strongly monotone mapping and let $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping. Assume that $F = F(T) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ u_n = \Pi_C[J^{-1}(Jx_n - \eta_n Bx_n)], \\ z_n = \Pi_C[J^{-1}(Ju_n - \lambda_n Au_n)], \\ y_n = Tz_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{2.1}$$

where J is the duality mapping on E. Assume that $\|Ax\| \leq \|Ax - Aq\|$ and $\|Bx\| \leq \|Bx - Bq\|$ for all $x \in C$ and $q \in VI(C, A) \cap VI(C, B)$. Let $\{\lambda_n\}$ and $\{\eta_n\}$ be positive number sequences such that $0 < d \leq \lambda_n$ and $\eta_n \leq \frac{c^2\gamma}{2}$ for all $n \geq 1$, where c is the constant defined by (1.3) and $\gamma = \min\{\alpha, \beta\}$. Then $\{x_n\}$ converges strongly to $p = \Pi_F x_0$.

Proof. By mathematical induction, it is not hard to see that C_n is closed and convex for each $n \geq 1$. Next, we prove that $F \subset C_n$ for all $n \geq 1$. $F \subset C_1 = C$ is obvious. Suppose $F \subset C_k$ for some $k \in \mathbb{N}$. Then, for all $v \in F \subset C_k$, from Lemma 1.4, Lemma 1.7 and the assumption $0 < \eta_n \leq \frac{c^2\gamma}{2}$ for all $n \geq 1$, one see that

$$\begin{aligned} &\phi(v, u_k) \\ &\leq \phi(v, J^{-1}(Jx_k - \eta_k Bx_k)) \\ &= V(v, Jx_k - \eta_k Bx_k) \\ &\leq V(v, Jx_k - \eta_k Bx_k + \eta_k Bx_k) - 2\langle J^{-1}(Jx_k - \eta_k Bx_k) - v, \eta_k Bx_k\rangle \\ &\leq \phi(v, x_k) - 2\eta_k \langle J^{-1}(Jx_k - \eta_k Bx_k) - J^{-1}Jx_k, Bx_k\rangle \\ &\quad - 2\eta_k\beta\|Bx_k - Bv\|^2 \\ &\leq \phi(v, x_k) + 2\eta_k\|J^{-1}(Jx_k - \eta_k Bx_k) - J^{-1}Jx_k\|\|Bx_k\| \\ &\quad - 2\eta_k\beta\|Bx_k - Bv\|^2 \\ &\leq \phi(v, x_k) + \frac{4}{c^2}\eta_k^2\|Bx_k - Bv\|^2 - 2\eta_k\beta\|Bx_k - Bv\|^2 \\ &\leq \phi(v, x_k). \end{aligned} \tag{2.2}$$

In similar way, we can obtain that

$$\phi(v, z_k) \le \phi(v, u_k). \tag{2.3}$$

It follows that

$$\phi(v, y_k) = \phi(v, Tz_k) \le \phi(v, z_k) \le \phi(v, u_k) \le \phi(v, x_k), \tag{2.4}$$

which implies that $v \in C_{k+1}$. This shows that $F \subset C_n$ for all $n \ge 1$.

On the other hand, from $x_n = \Pi_{C_n} x_0$, one sees that

$$\langle x_n - z, Jx_0 - Jx_n \rangle \ge 0, \quad \forall z \in C_n. \tag{2.5}$$

Since $F \subset C_n$ for all $n \ge 1$, we arrive at

$$\langle x_n - v, Jx_0 - Jx_n \rangle \ge 0, \quad \forall v \in F. \tag{2.6}$$

It follows from Lemma 1.4 that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(v, x_0) - \phi(v, x_n) \le \phi(v, x_0)$$

for all $v \in F \subset C_n$ and $n \ge 1$. Therefore, the sequence $\phi(x_n, x_0)$ is bounded. Noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, one obtains that

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \ \forall n \ge 1.$$

This shows that the sequence $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\le \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{2.7}$$

Letting $m, n \to \infty$ in (2.7), one has $\phi(x_m, x_n) \to 0$. It follows from Lemma 1.2 that $x_m - x_n \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that

$$x_n \to p \in C \quad (n \to \infty). \tag{2.8}$$

By taking $m = n + 1$ in (2.7), we arrive at

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.9}$$

From Lemma 1.2, we see that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.10}$$

Since $x_{n+1} \in C_{n+1}$, we obtain that

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, z_n) \le \phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n). \tag{2.11}$$

It follows from (2.9) and (2.11) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0, \qquad \lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0.$$

In virtue of Lemma 1.2, we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0,\qquad \lim_{n\to\infty}\|x_{n+1}-z_n\|=0. \tag{2.12}$$

On the other hand, we have

$$\|Tz_n-z_n\|=\|y_n-z_n\|\le\|x_{n+1}-y_n\|+\|x_{n+1}-z_n\|.$$

It follows from (2.12) that

$$\lim_{n\to\infty}\|Tz_n-z_n\|=0. \tag{2.13}$$

Notice that

$$\|x_n-z_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-z_n\|.$$

Combining (2.10) with (2.12), we assert that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{2.14}$$

From (2.8), we arrive at

$$z_n\to p\in C\quad (n\to\infty). \tag{2.15}$$

From the closed-ness of the mapping T, we obtain that $p\in F(T)$.

Next, we show that $p\in VI(C,A)$. Let Q be the maximal monotone operator defined by Lemma 1.6:

$$Qx:=\begin{cases}Ax+N_Cx, & x\in C,\\ \emptyset, & x\notin C.\end{cases}$$

For any given $(x,y)\in G(Q)$, we see that $y-Ax\in N_Cx$. Since $z_n\in C$, by the definition of N_Cx, we have

$$\langle x-z_n, y-Ax\rangle\ge 0.$$

On the other hand, from $z_n=\Pi_C[J^{-1}(Ju_n-\lambda_nAu_n)]$ and Lemma 1.3, we obtain that

$$\Big\langle x-z_n, \frac{Jz_n-Ju_n}{\lambda_n}+Au_n\Big\rangle\ge 0.$$

Therefore, we have

$$\begin{aligned}&\langle x-z_n, y\rangle\\ &\ge\langle x-z_n, Ax\rangle\\ &\le\langle x-z_n, Ax\rangle-\Big\langle x-z_n, \frac{Jz_n-Ju_n}{\lambda_n}+Au_n\Big\rangle\\ &=\langle x-z_n, Ax-Az_n\rangle+\langle x-z_n, Az_n-Au_n\rangle-\Big\langle x-z_n, \frac{Jz_n-Ju_n}{\lambda_n}\Big\rangle\\ &\ge\langle x-z_n, Az_n-Au_n\rangle-\Big\langle x-z_n, \frac{Jz_n-Ju_n}{\lambda_n}\Big\rangle.\end{aligned} \tag{2.16}$$

From (2.11) and Lemma 1.2, we see that

$$\lim_{n\to\infty}\|x_{n+1}-u_n\|=0. \tag{2.17}$$

Notice that

$$\|z_n - u_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - u_n\|.$$

It follows from (2.12) and (2.17), we arrive at

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{2.18}$$

Since A is α-inverse strongly monotone, we obtain that

$$\lim_{n\to\infty} \|Az_n - Au_n\| = 0.$$

From (2.16), we arrive at $\langle x - p, y\rangle \geq 0$. Since Q is maximal monotone, we obtain that $p \in A^{-1}(0)$ and hence $p \in VI(C, A)$. It follows from (2.18) that

$$u_n \to p \quad (n \to \infty).$$

Similarly, we can show that $p \in VI(C, B)$. That is, $p \in F = F(T) \cap VI(C, A) \cap VI(C, B)$.

Finally, we prove that $p = \Pi_F x_0$. From (2.6), we see that

$$\langle p - v, Jx_0 - Jp\rangle \geq 0, \quad \forall v \in F.$$

Thanks to Lemma 1.3, we obtain that $p = \Pi_F x_0$. This completes the proof. □

As some applications of Theorem 2.1, we have the following results immediately.

Corollary 2.1. *Let C be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let $A : C \to E^*$ be an α-inverse strongly monotone mapping and let $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping. Assume that $F = F(T) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner*:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = \Pi_C[J^{-1}(Jx_n - \lambda_n Ax_n)], \\ y_n = Tz_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where J is the duality mapping on E. Assume that $\|Ax\| \leq \|Ax - Aq\|$ for all $x \in C$ and $q \in VI(C, A)$. Let $\{\lambda_n\}$ be a positive number sequence such that $0 < d \leq \lambda_n \leq \frac{c^2\alpha}{2}$ for all $n \geq 1$, where c is the constant defined by (1.3)*. Then $\{x_n\}$ converges strongly to $p = \Pi_F x_0$.*

Remark 2.1. Corollary 2.1 improves the corresponding results announced by Zegeye and Shahzad [19] in the following sense.

(1) from relatively weak nonexpansive mappings to quasi-ϕ-nonexpansive mappings. To be more precise, we relax the strict: $\widetilde{F(T)} = F(T)$, where $\widetilde{F(T)}$ denote the set of strong asymptotic fixed point of T, see [19] for more details;

(2) from the computational point of view, we relax the iterative step of "W_n" in the algorithm of Zegeye and Shazad [19].

If $T = I$, the identity mapping, in Theorem 2.1, we have the following result.

Corollary 2.2. *Let C be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let $A : C \to E^*$ be an α-inverse strongly monotone mapping and let $B : C \to E^*$ be a β-inverse strongly monotone mapping. Assume that $F = VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ u_n = \Pi_C[J^{-1}(Jx_n - \eta_n Bx_n)], \\ z_n = \Pi_C[J^{-1}(Ju_n - \lambda_n Au_n)], \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \leq \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where J is the duality mapping on E. Assume that $\|Ax\| \leq \|Ax - Aq\|$ and $\|Bx\| \leq \|Bx - Bq\|$ for all $x \in C$ and $q \in VI(C, A) \cap VI(C, B)$. Let $\{\lambda_n\}$ and $\{\eta_n\}$ be positive number sequences such that $0 < d \leq \lambda_n$ and $\eta_n \leq \frac{c^2\gamma}{2}$ for all $n \geq 1$, where c is the constant defined by (1.3) *and $\gamma = \min\{\alpha, \beta\}$. Then $\{x_n\}$ converges strongly to $p = \Pi_F x_0$.*

Remark 2.2. Corollary 2.2 can be viewed as an improvement of the corresponding results in Iiduka and Takahashi [8].

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